# Online Algorithms
# Tutorial 2 — Scheduling

Book chapter:
`https://link.springer.com/chapter/10.1007/BFb0029570`

# Finalizing Bin Packing

Remarks regarding Bin Packing:

- First Fit 1.7-competitive
- Best possible for Any Fit algorithms
- But there are better algorithms!
- Lower bound for any algorithm: 1.5401 (LP technique)
- Idea: Categorize items by size
- Harmonic algorithm uses categories $(\frac{1}{2}, 1], (\frac{1}{3}, \frac{1}{2}], \ldots$
- Next Fit within categories
- With sufficient categories, better than 1.7
- Better algorithms: More categories, more complex packing
- Currently: $1.57829\ldots$ (Advanced Harmonic) (2018)
- Current best lower bound: $1.54278\ldots$ (2020)

# Randomized Online Algorithm Adversaries

**Oblivious adversary**

- Adversary knows $A$
- Adversary generates $\sigma$ and optimal offline solution $\mathsf{OPT}(\sigma)$
- $A$ runs on $\sigma$, generating $A(\sigma)$

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**Adaptive online adversary**

- Adversary knows $A$
- While not done:
  - Adversary generates request $\sigma_i$
  - $A$ is given $\sigma_i$
  - Adversary learns response and state of $A$
  - Adversary responds to $\sigma_i$
  - Next input request or end

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Did we already see adaptive online adversaries in the lecture?

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Any (deterministic or randomized) online file migration algorithm has a competitive ratio of at least $3$.

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No! Can an adaptive online algorithm do that?

**Claim:**
Against an adaptive online adversary, any randomized online file migration algorithm has a competitive ratio of at least $3$.

# Online Scheduling

Another classic problem: Distribute jobs on machines

- $m$ machines $M_1, \ldots, M_m$, $m$ known
- $n$ jobs $J_1, \ldots, J_n$, $n$ unknown
- Running time $t(J_i) > 0$

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Many different versions:

- Precedence constraints
- Release times
- Preemption
- Machine faults
- Unsure job running time
- Different machines (speed, possible jobs)
- Parallel jobs
- Minimum makespan
- Minimum waiting time, equal load, ...

# Online Scheduling

Natural online problem: nearly all variants NP-hard ($m \geq 2$)

Our variant:

- $m$ identical machines, $n$ jobs
- Running times $t(J_i)$
- Minimize makespan
- Assign job $J_i$ to some machine before getting $J_{i+1}$
- Somewhat similar to bin packing

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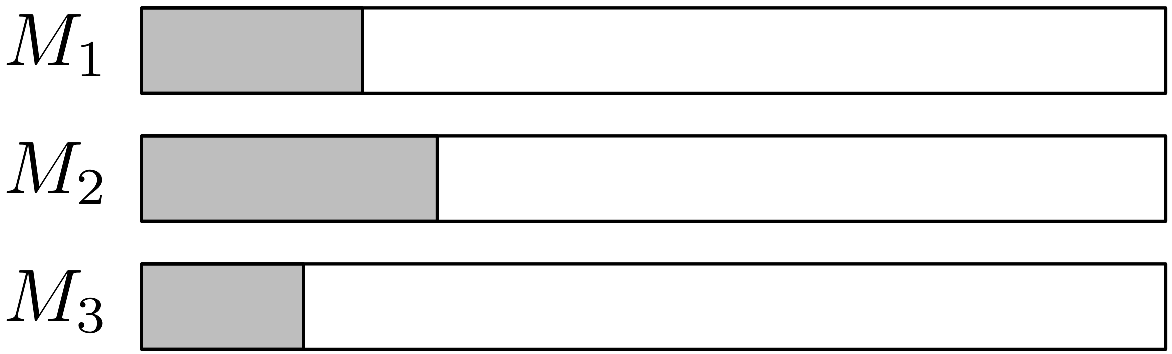

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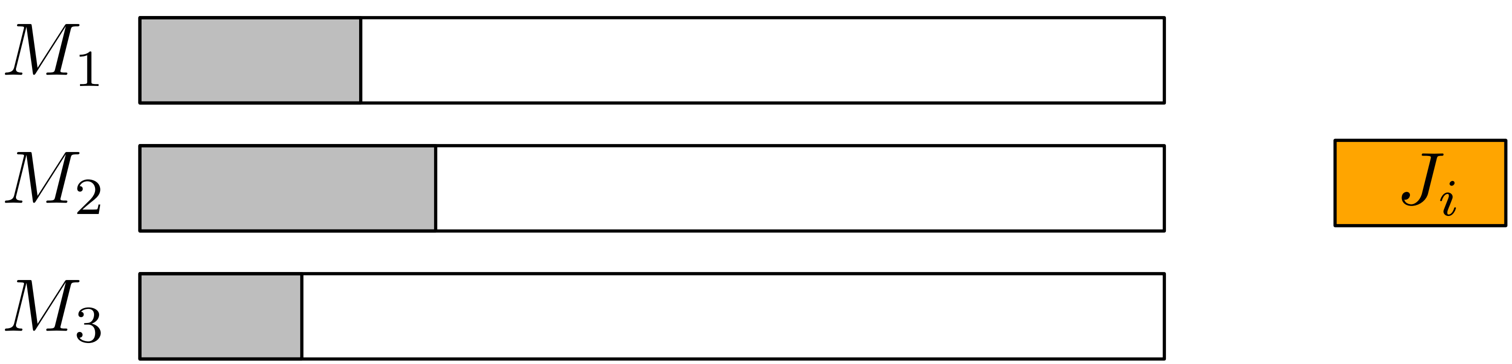

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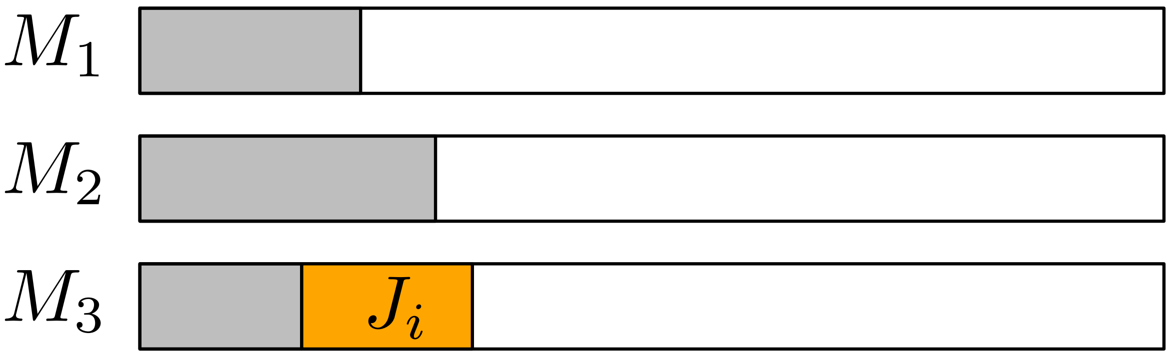

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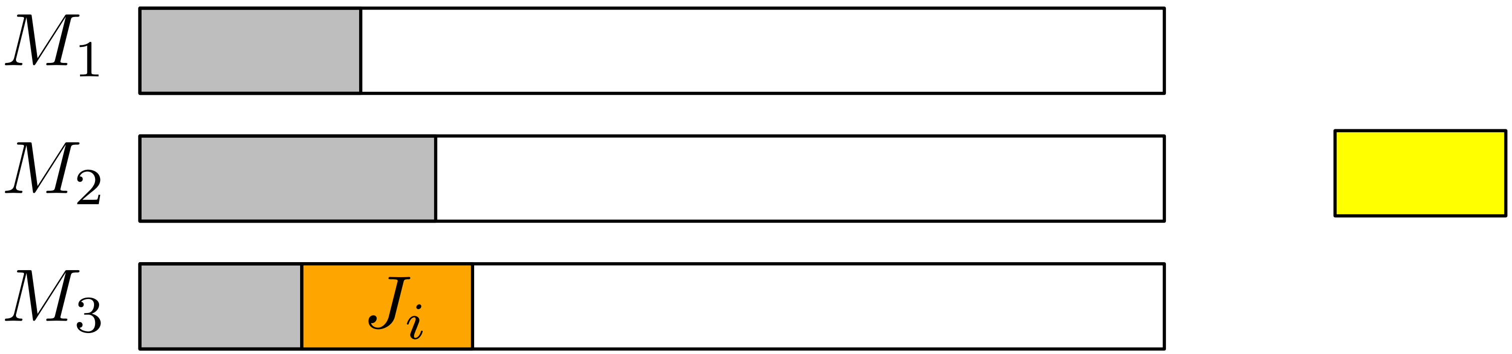

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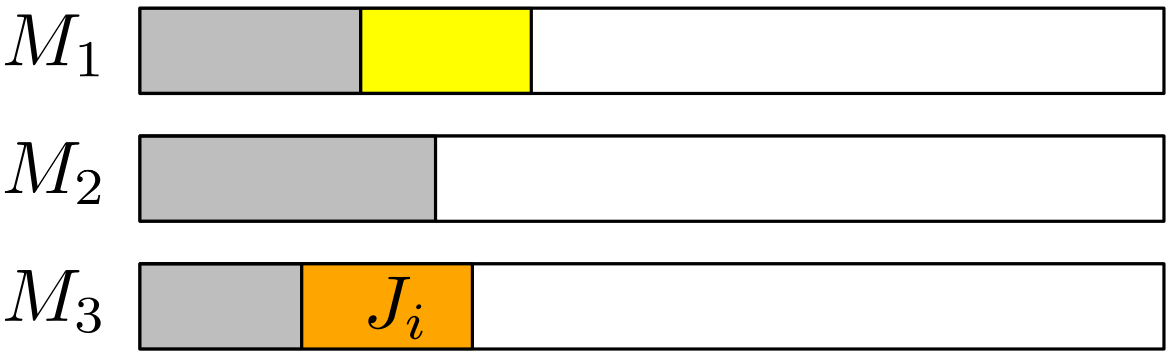

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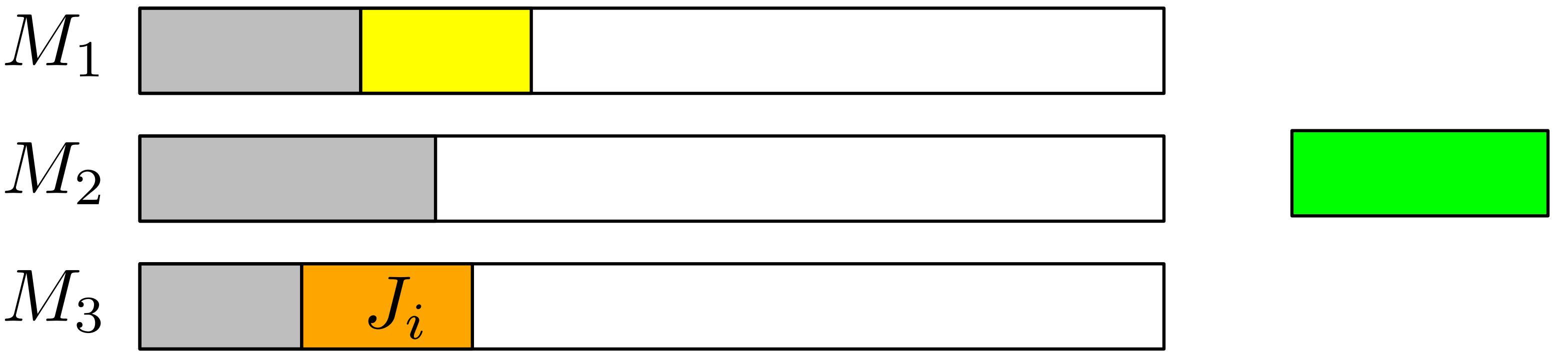

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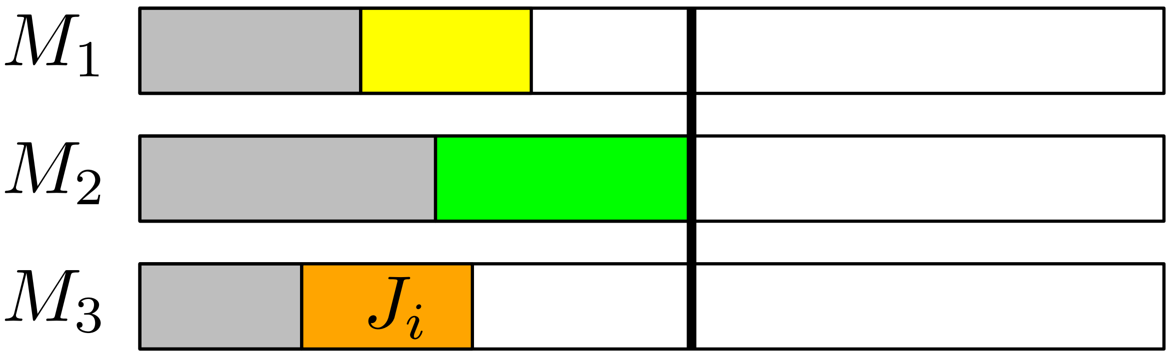

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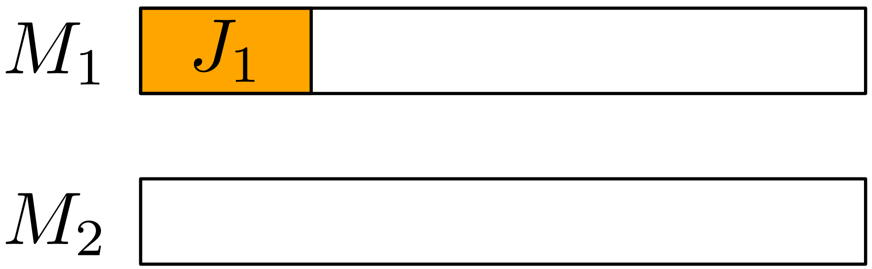

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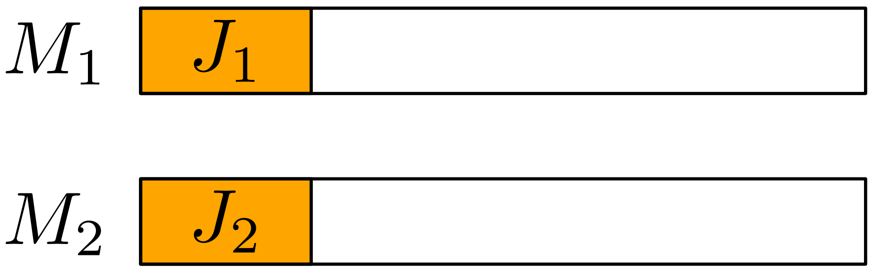

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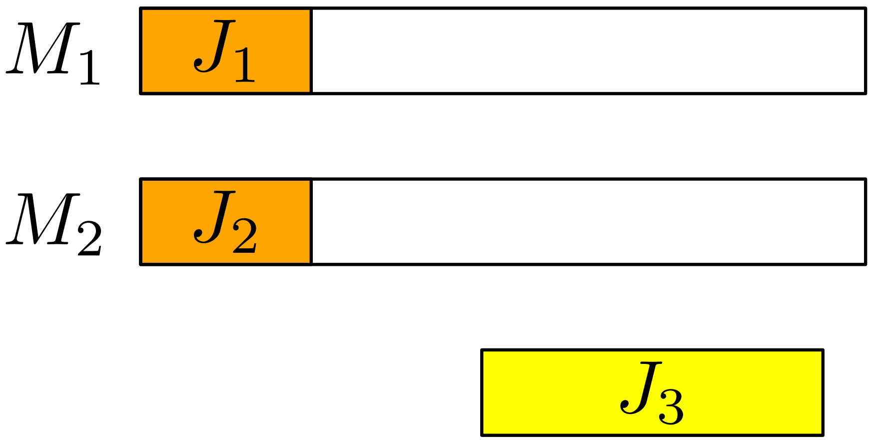

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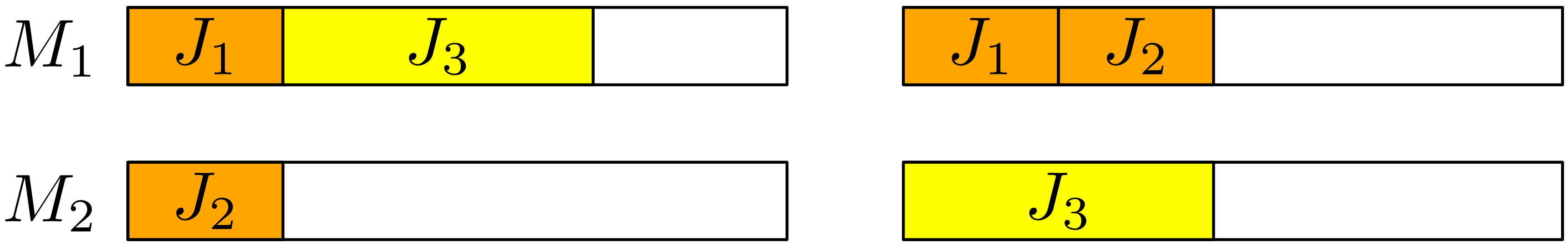


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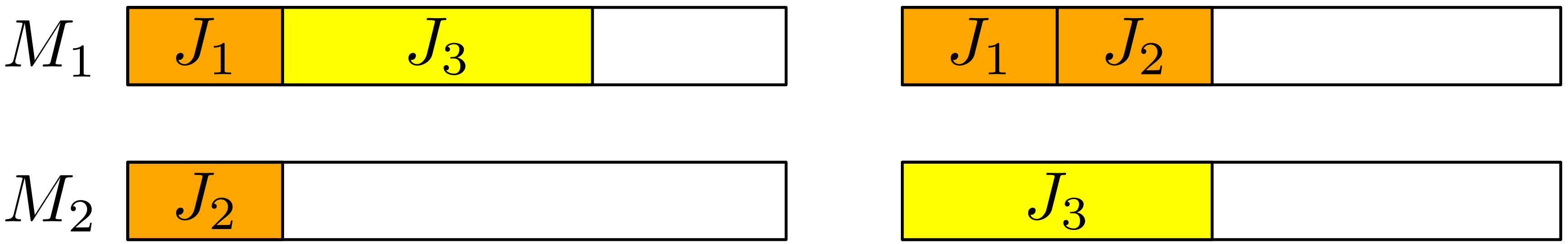


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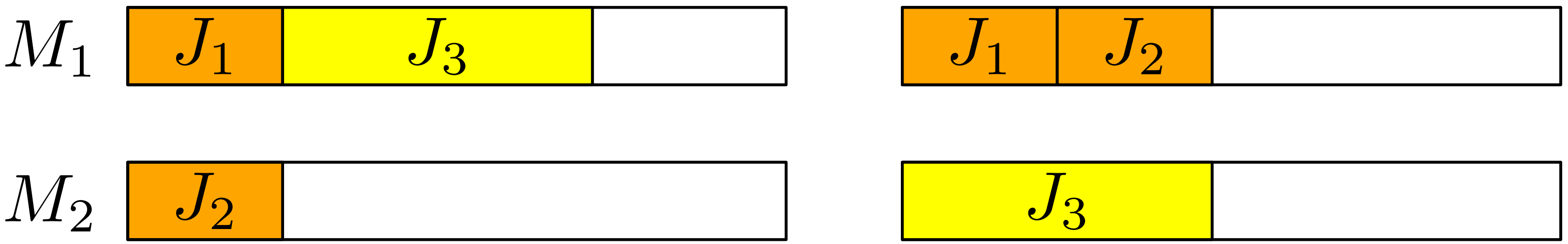


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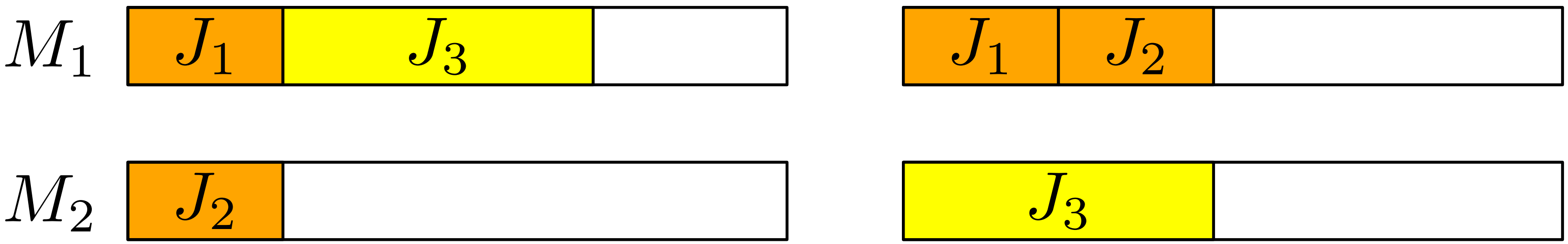


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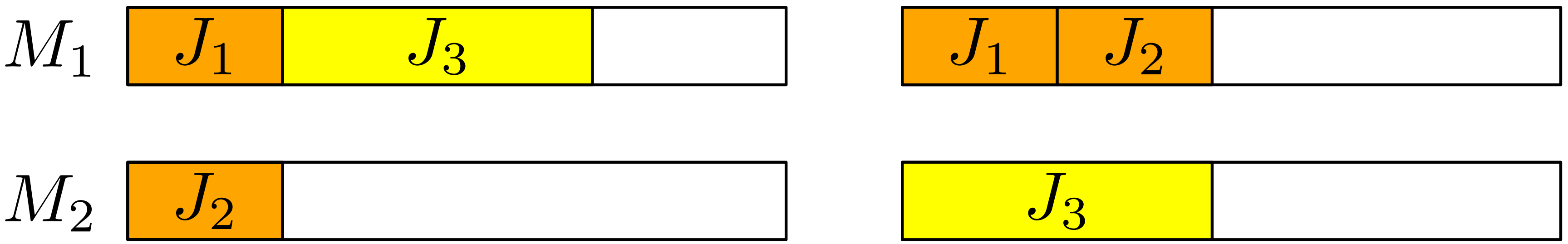


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Our makespan: $(m-1)+m$, OPT: $m$, $c \geq 2 - {}^1/_m$

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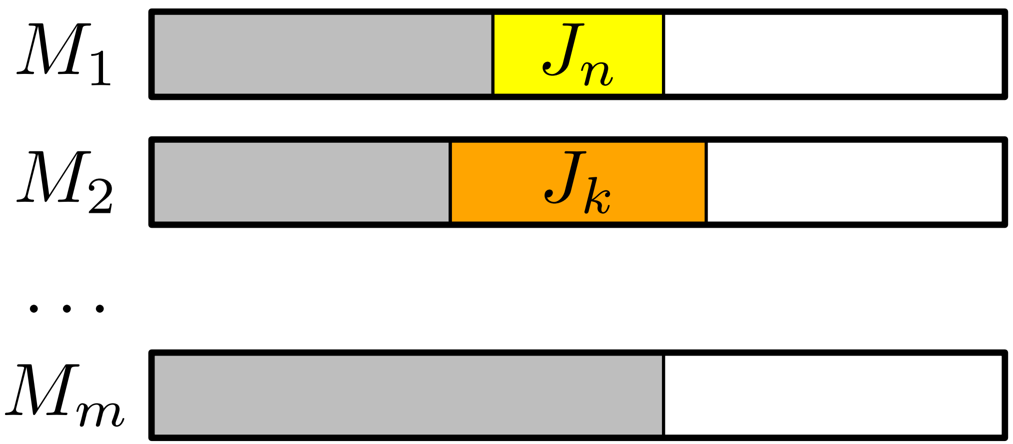

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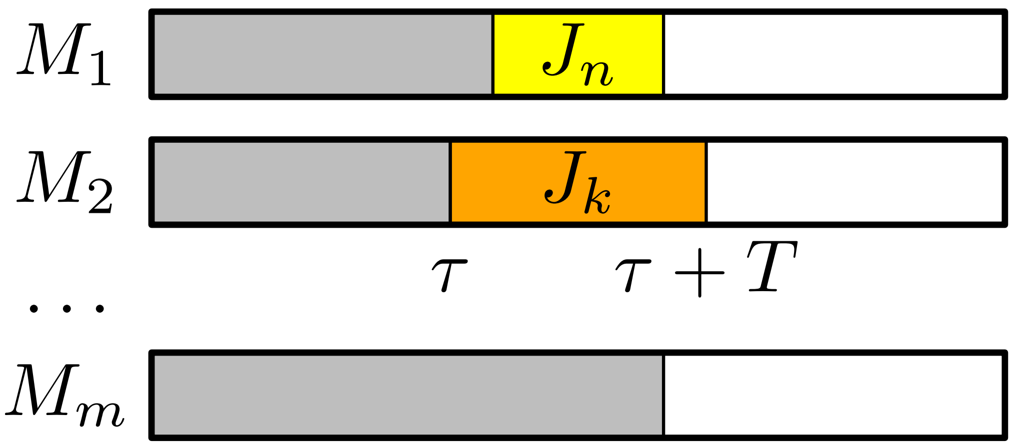


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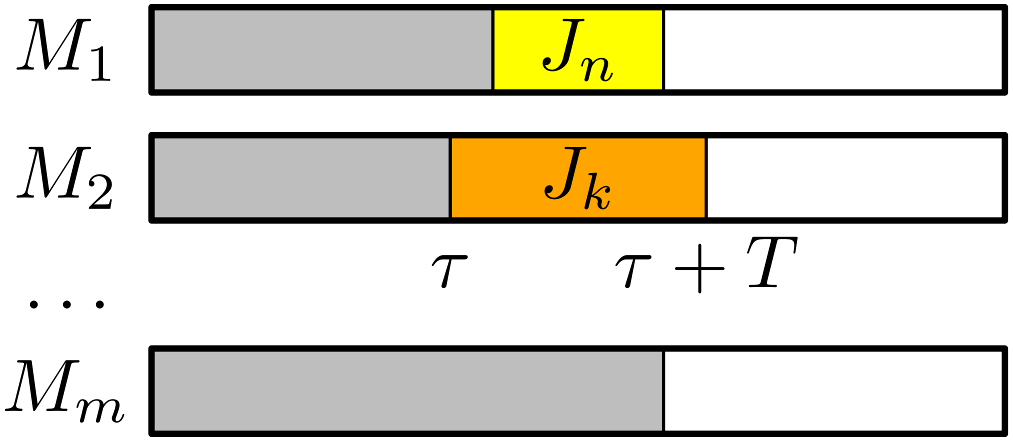


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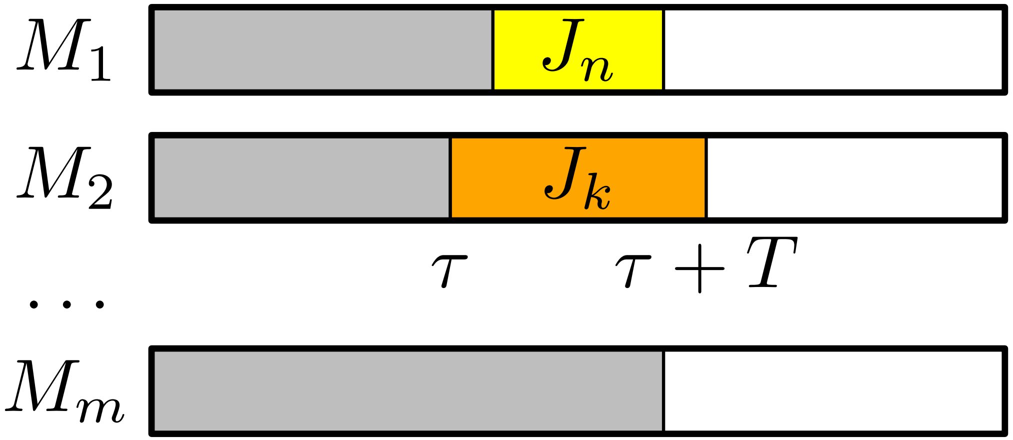


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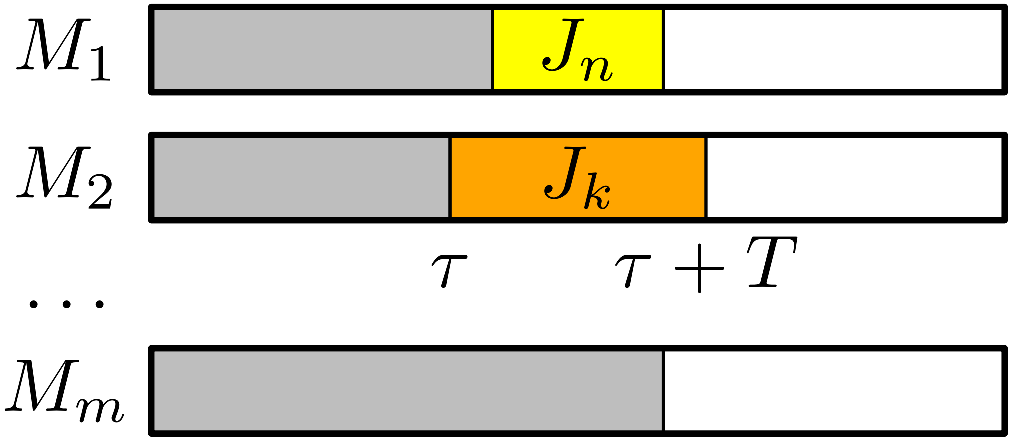


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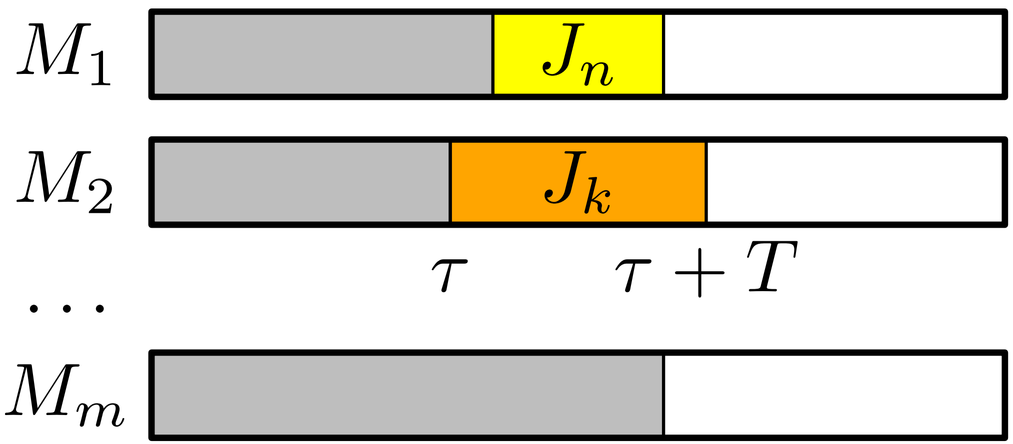


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Note: Both lower bounds on OPT tight in worst case!

# More Complex Models

We can adapt this analysis to:

- Unknown running times
- Precedence constraints (analysis technical)
- Jobs with release times

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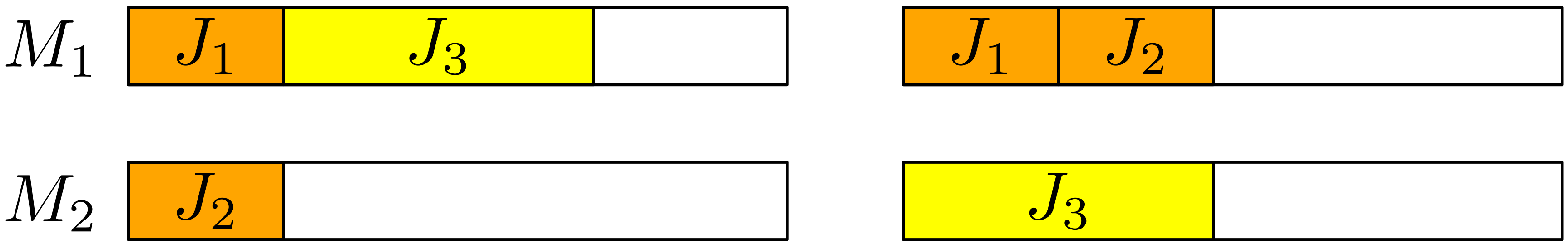

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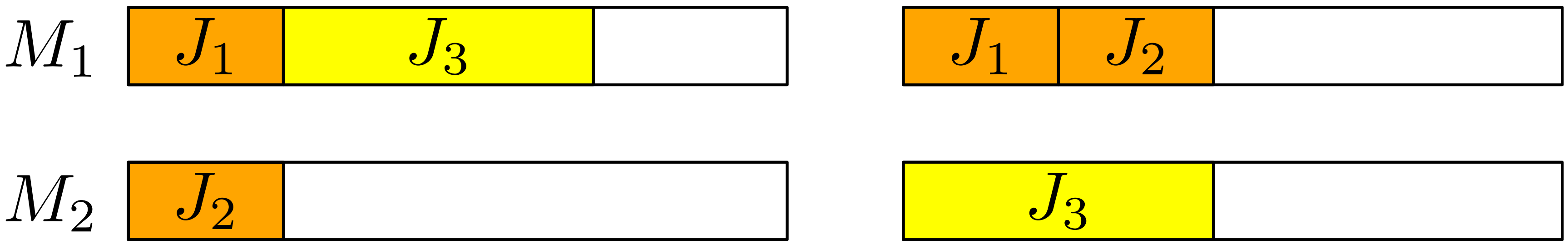


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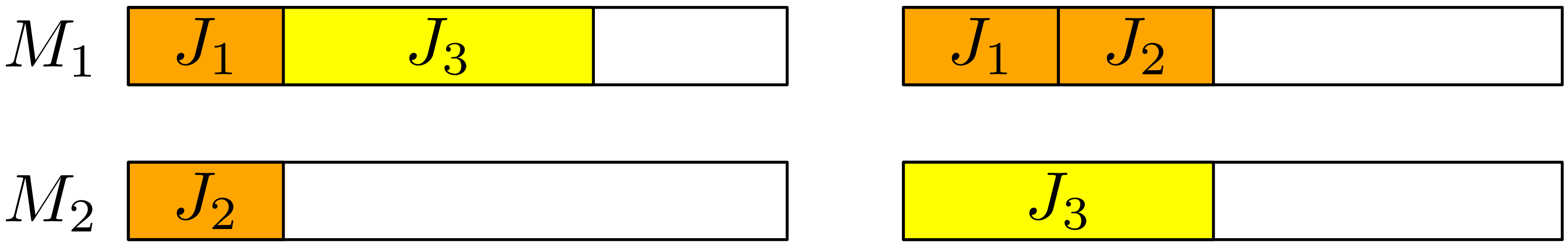


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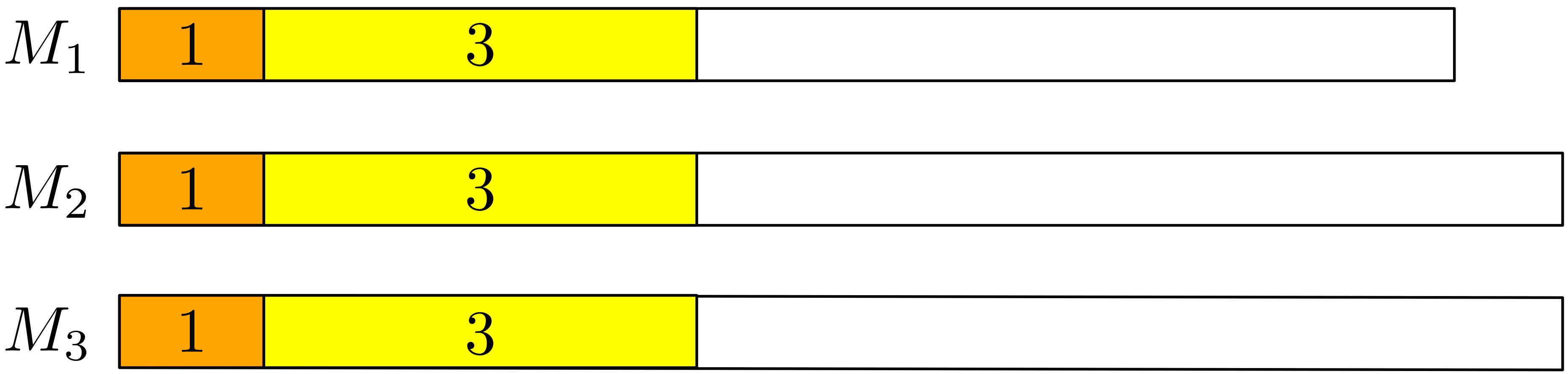


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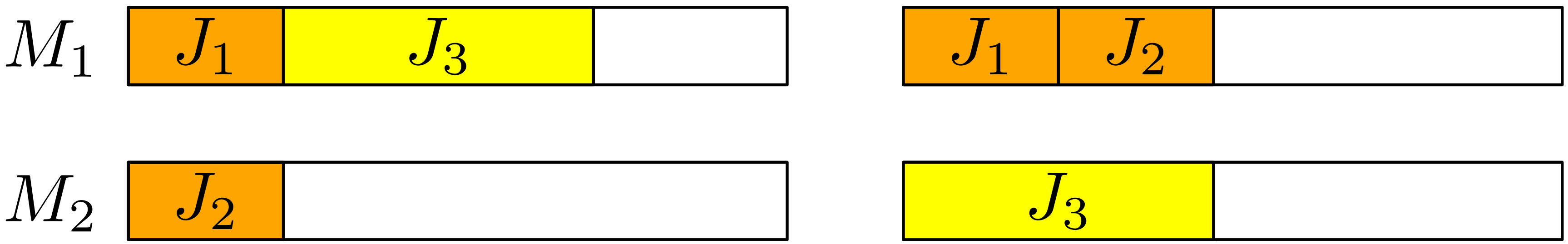


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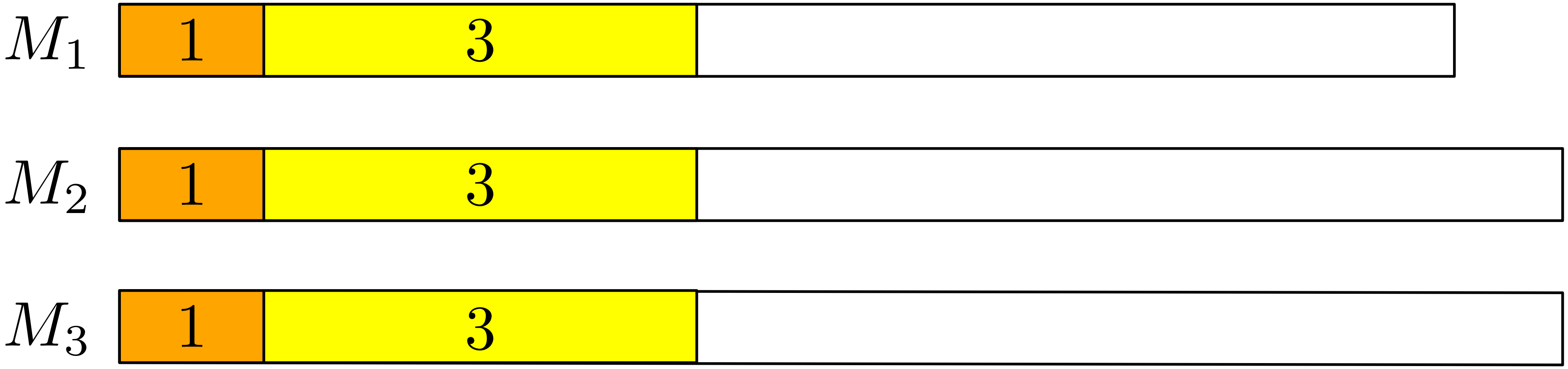


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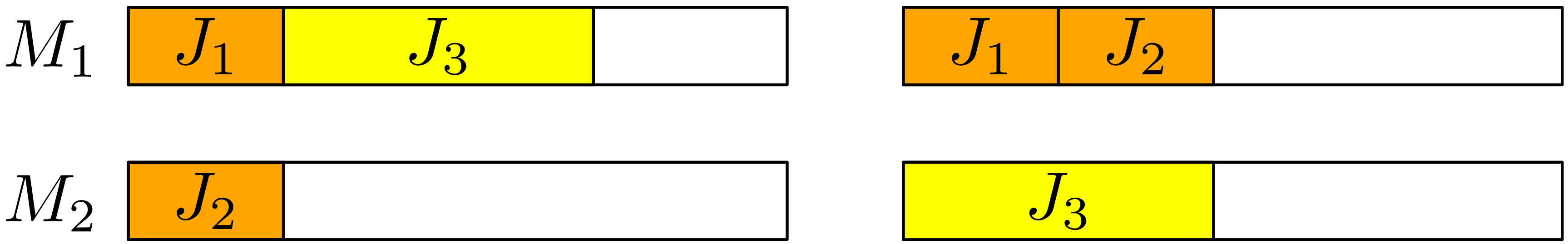


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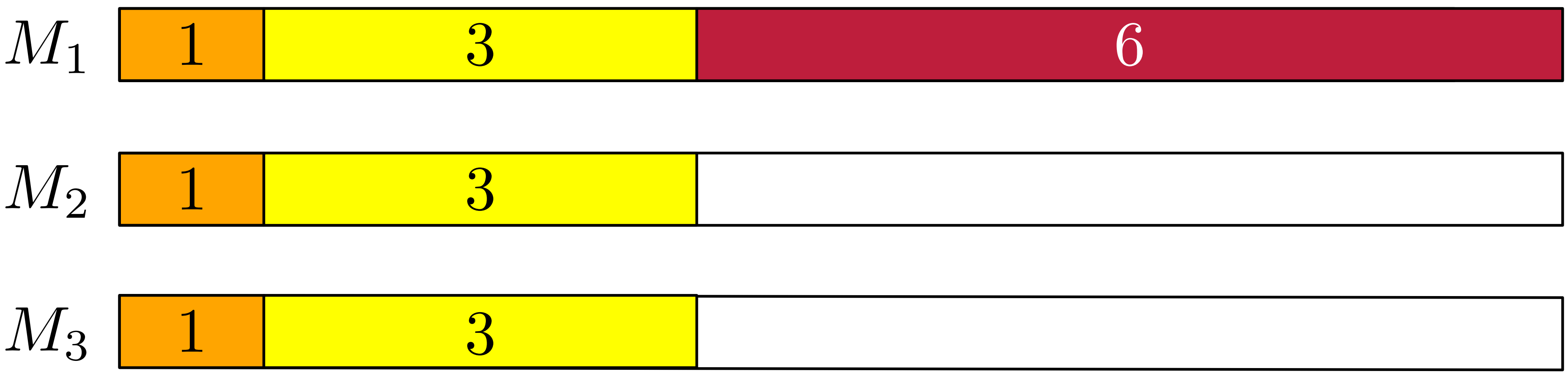


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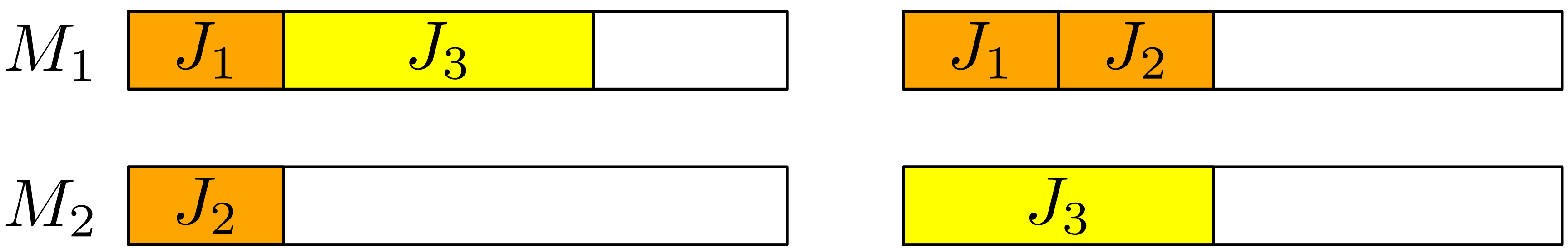


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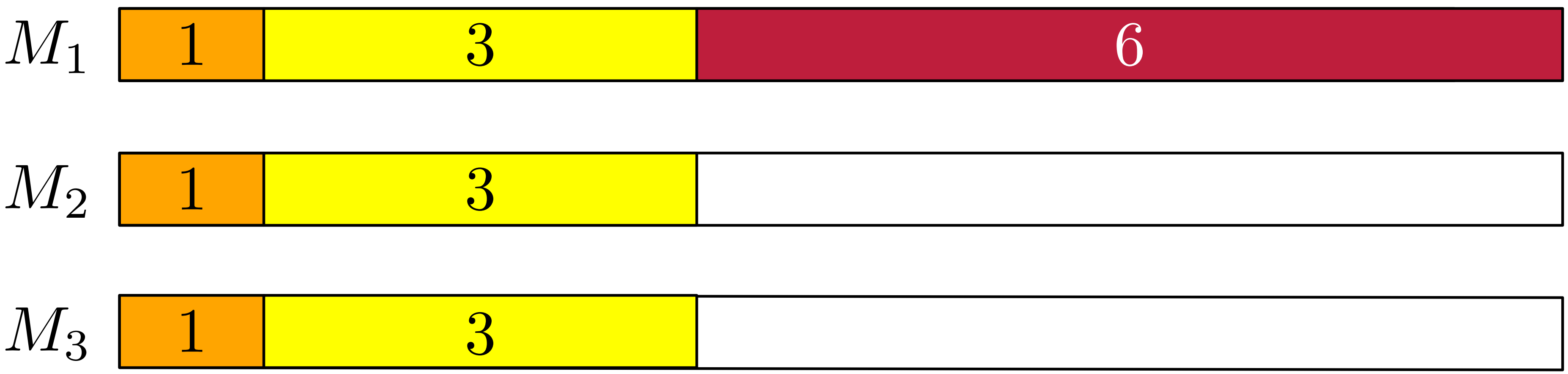


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$m > 3$: Yes, but not much, and it gets difficult.

# Randomization

Competitive ratio of $A$ against an oblivious adversary:

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First, a lower bound: For $m \geq 2$, we cannot be better than $4/3$.

- $m$ jobs of length $1$, possibly followed by a single $2$
- $p$ probability of makespan 1 after the $1$s
- After the $1$s: OPT $=$ 1, $\mathbb{E}[A(\sigma)] = p + (1-p) \cdot 2 = 2 - p$
- After the $2$: OPT $=$ 2, $\mathbb{E}[A(\sigma)] = 3p + 2(1-p) = 2 + p$

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- After the $2$: OPT $= 2$, $\mathbb{E}[A(\sigma)] = 3p + 2(1-p) = 2 + p$

$$c \geq \max\left\{2 - p, \frac{2+p}{2}\right\} \geq \frac{4}{3} \quad (p = 2/3)$$

# Bounds for Scheduling

| | deterministic | | | randomized | |
|---|---|---|---|---|---|
| $m$ | lower bound | upper bound | LS | lower bound | upper bound |
| 2 | 1.5000 | 1.5000 | 1.5000 | 1.3333 | 1.3334 |
| 3 | 1.6666 | 1.6667 | 1.6667 | 1.4210 | 1.5567 |
| 4 | 1.7310 | 1.7333 | 1.7500 | 1.4628 | 1.6589 |
| 5 | 1.7462 | 1.7708 | 1.8000 | 1.4873 | 1.7338 |
| 6 | 1.7730 | 1.8000 | 1.8333 | 1.5035 | 1.7829 |
| 7 | 1.7910 | 1.8229 | 1.8571 | 1.5149 | 1.8169 |
| $\infty$ | 1.8520 | 1.9230 | 2.0000 | 1.5819 | – |