Online Algorithms
Tutorial 4 — Self-Organizing Datastructures

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Consider a set- or dictionary-datastructure:

- C++ map, set, etc.
- Java HashMap, TreeMap, etc.
- Python dict, set
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Operations:

- Find given key
- Add key
- Delete key

Classic vs. Self-organizing

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- Insert, Delete change the set and internal representation
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Pro's & Con's:

- Possibly faster if some keys are requested more often than others
- Can be used in data compression
- More complicated
- Bad for multithreading

Homework: Linked lists

Linked List:

- Not a very practical set implementation
- Can be used for compression
- Instead of encoding bytes or byte pairs, encode list positions
- Length depends on position in list
- Common byte pairs become shorter, uncommon ones longer

Results:

- MoveToFront 2-competitive
- FrequencyCount, Transpose not competitive

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MoveToFront does not work well for arrays.

Double-ended queues or ring-buffers can work.

Hash tables

Have not been studied much; not much potential as self-organizing datastructures. Collisions should be rare.

Binary trees

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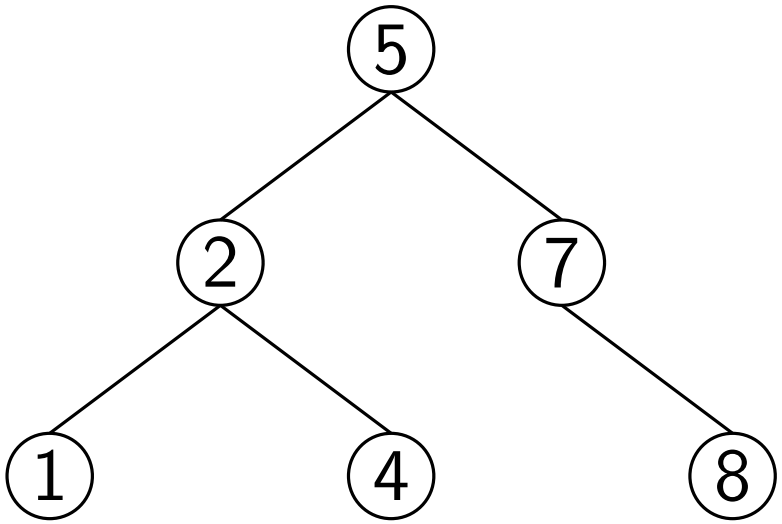

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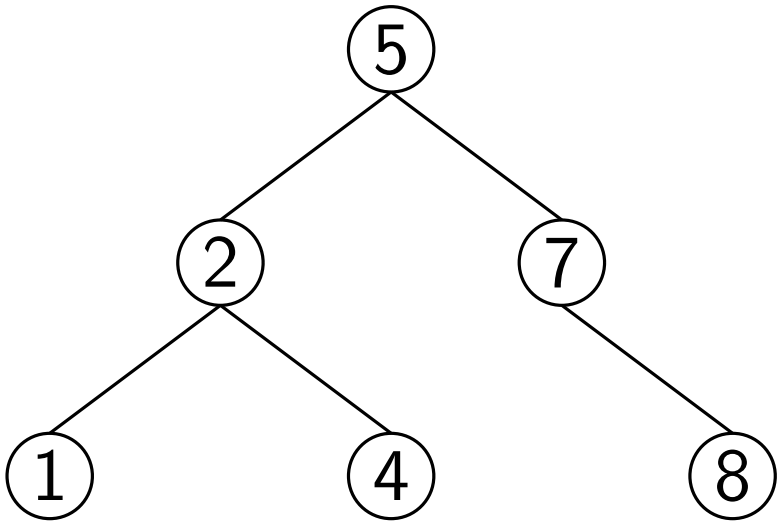

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Search 4: Costs 3
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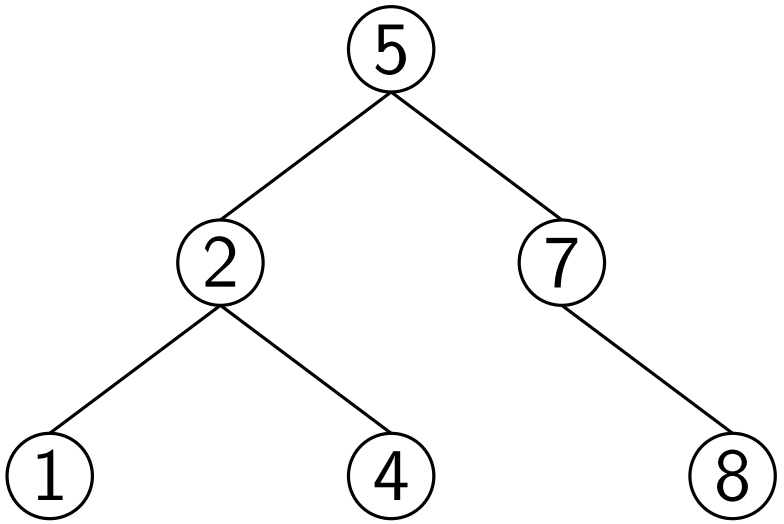

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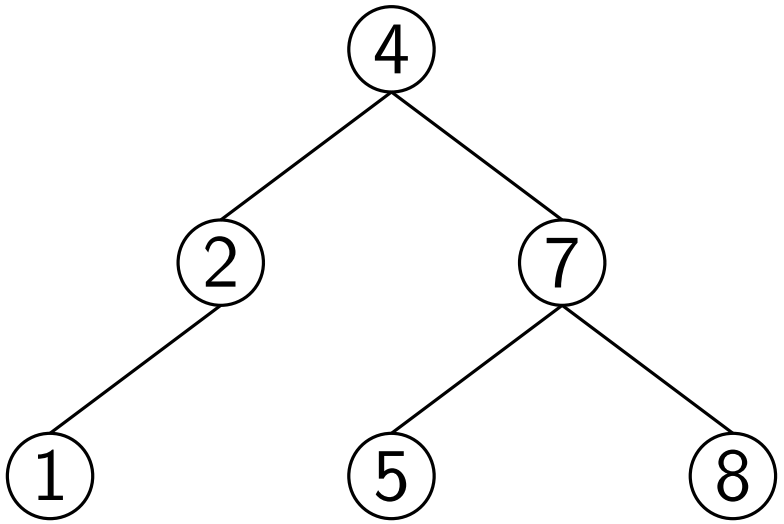

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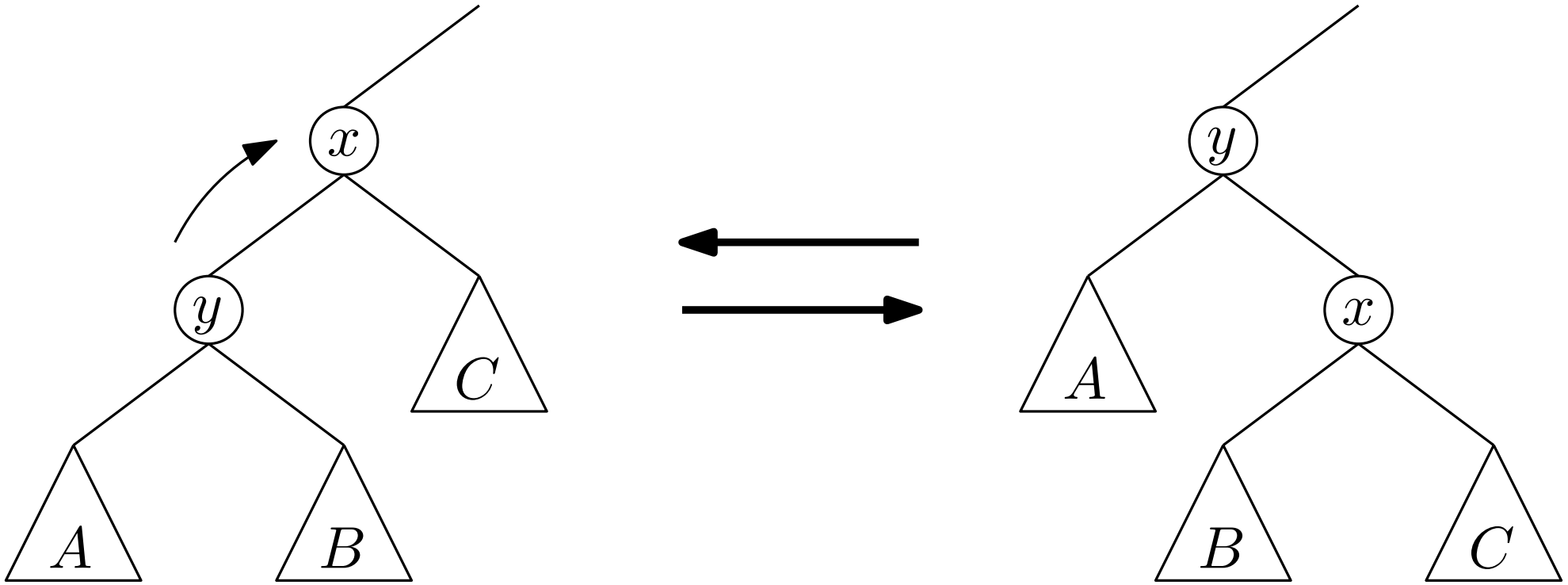
Operations on trees

Rotations: Transform one binary tree into the other.



Cost: 1 unit. Can also be used to realize AVL-trees. Can be applied anywhere in the tree.

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Actually, $2n - 6$ are sufficient.

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$f(n)$-static-competitive:
Compare A not to OPT, but to optimal static tree.

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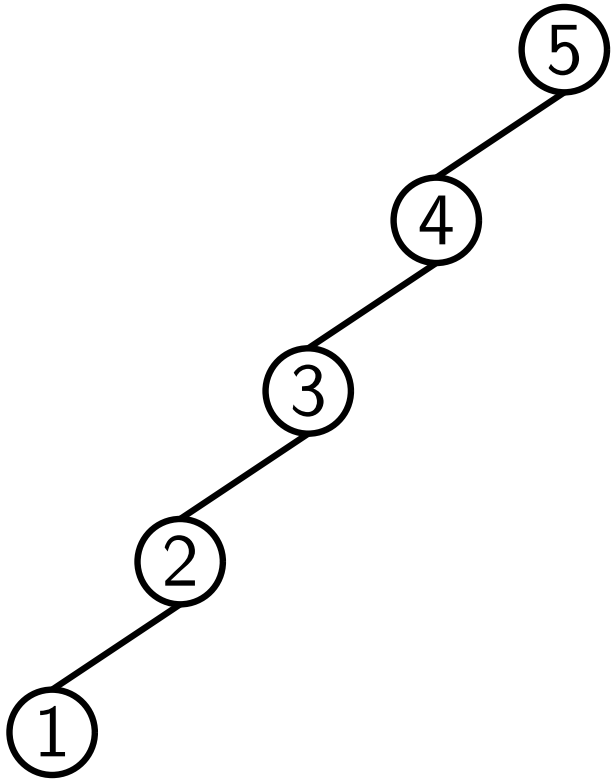

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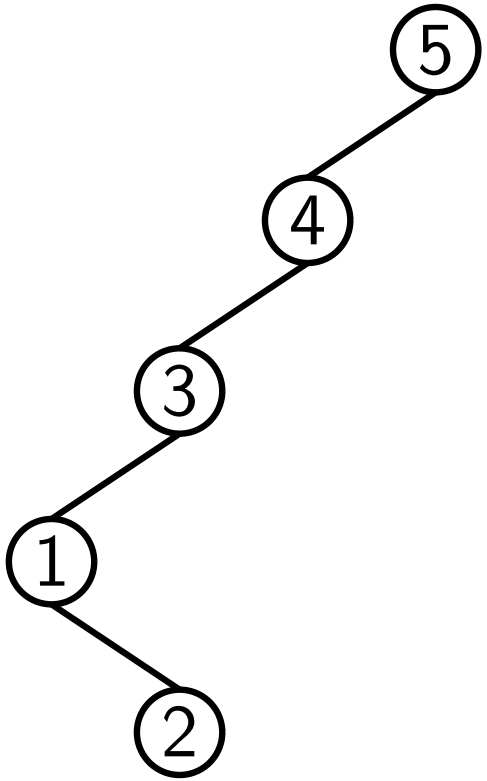

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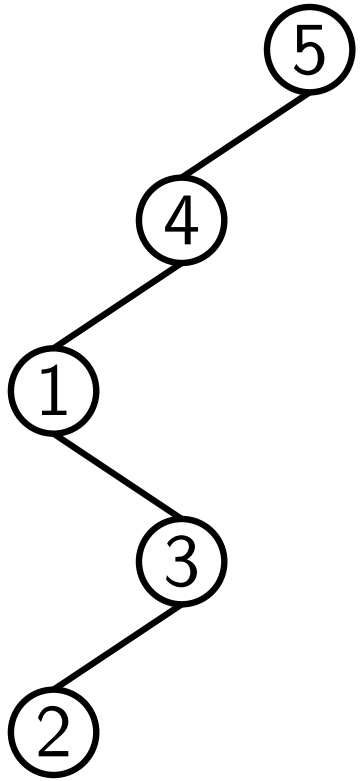

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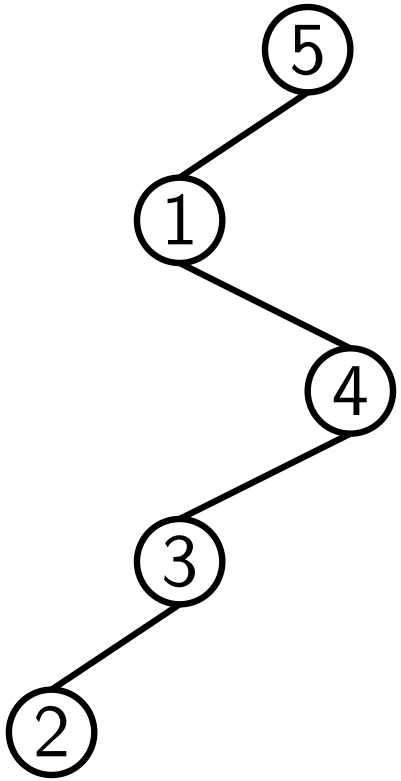

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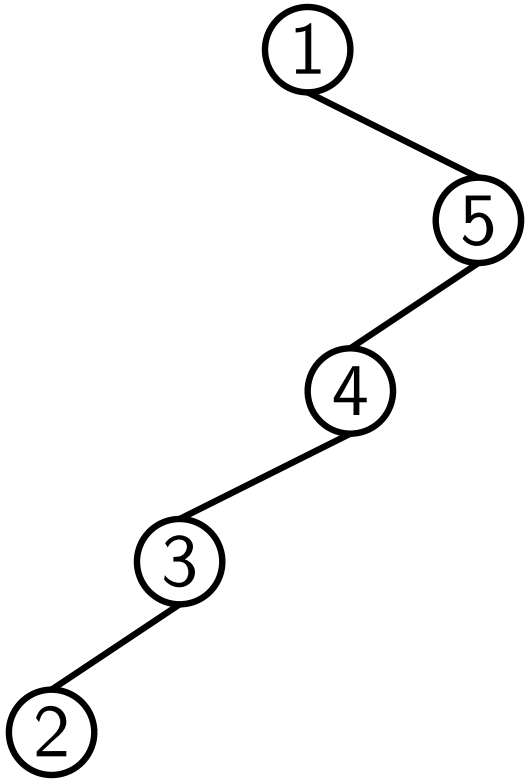

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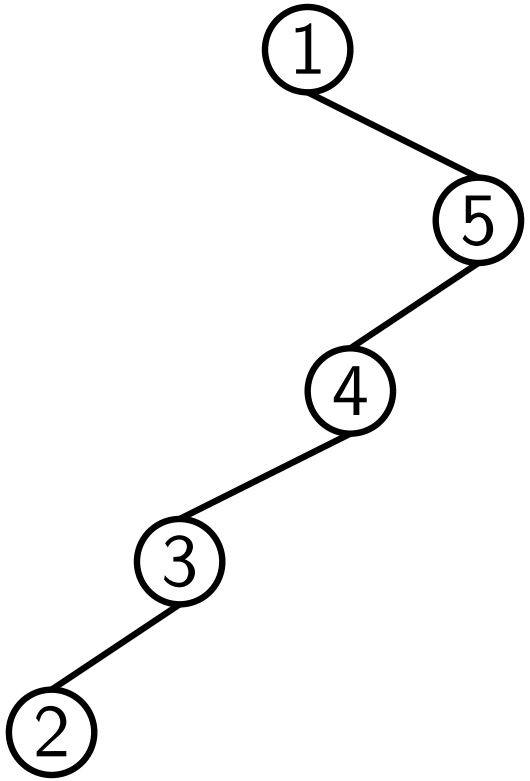

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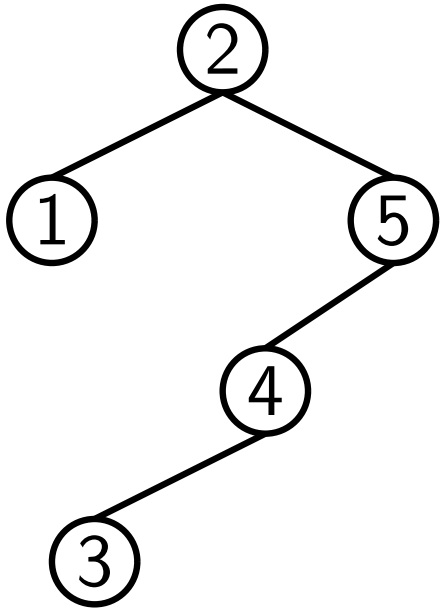
After request for 2:



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We can achieve $\mathcal{O}(n)$ per repetition:
First rotate into right path; costs $\mathcal{O}(n)$.
Then $\mathcal{O}(1)$ per request.
$\Rightarrow$ RotateToTop is $\Omega(n)$-competitive.

Better idea?

Can we do better?

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- If they are in the same direction, rotate upper edge first
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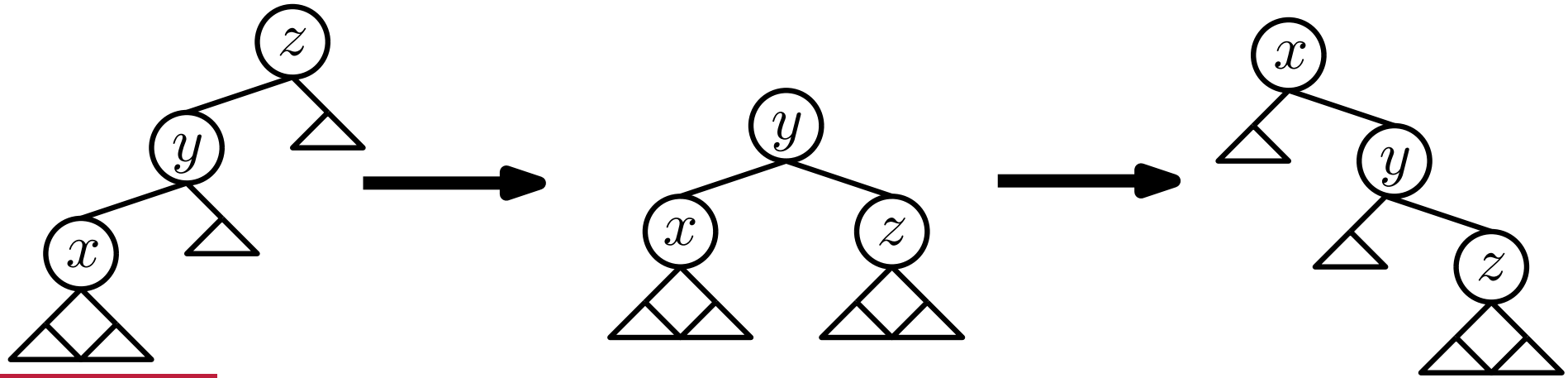

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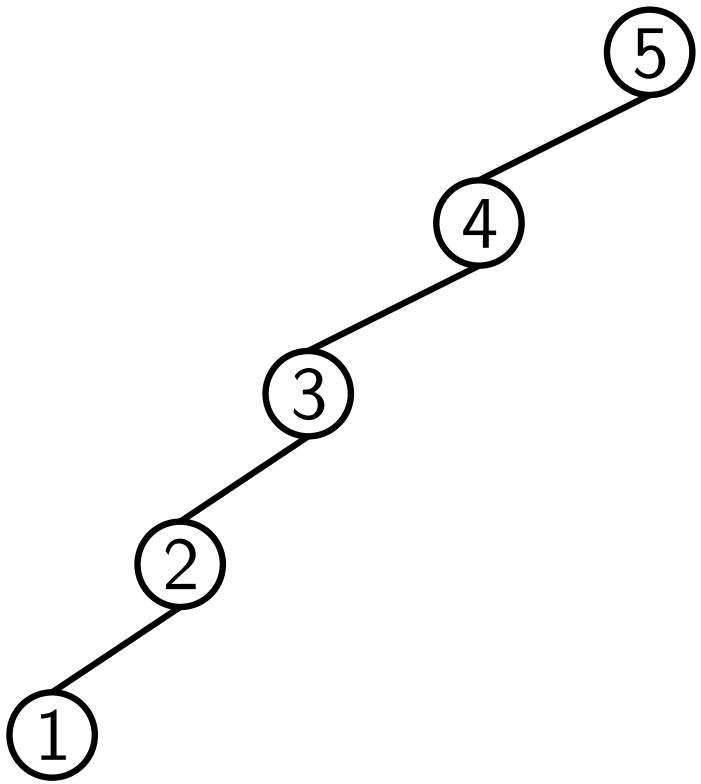

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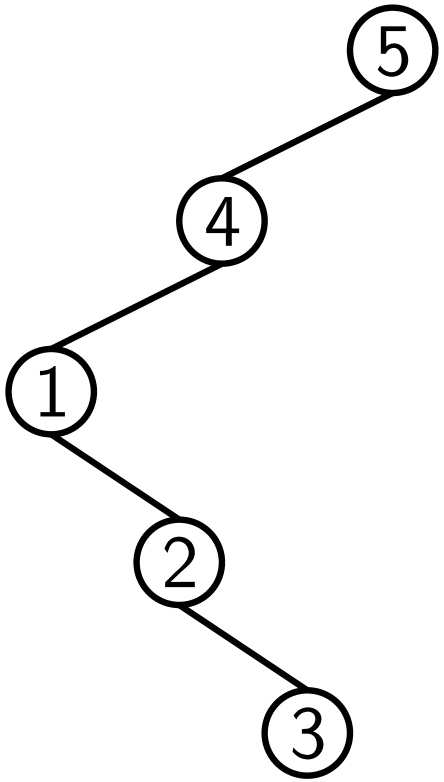

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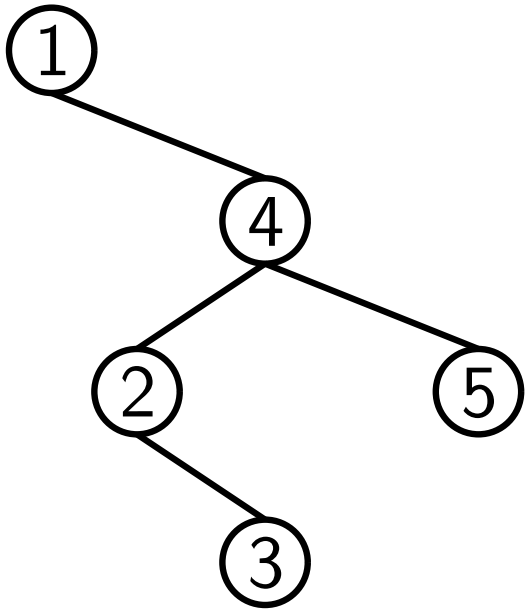

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Static Finger Theorem

In a sequence σ with many accesses close to a *finger* f:

$$\mathcal{O}\left(n \log n + \sum_{x \in \sigma} \log(|x - f| + 2)\right).$$

Dynamic Finger Theorem

Let x, y be consecutive elements in the sequence σ. Total cost:

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Further Properties

Dynamic Finger Theorem

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Working Set Theorem

Let $t(x)$ be the number of distinct elements accessed since last access to x. Total cost:

$$\mathcal{O}\left(n \log n + \sum_{x} \log(t(x) + 2)\right).$$

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There are specialized, better algorithms for that case though.