Algorithms Group **Summer 2022**
Department of Computer Science - IBR
TU Braunschweig

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Online Algorithms
Homework Assignment 3
June 13, 2022

Solutions are due on Monday, the 20th of June, 2022, until 15:00 in the homework cupboard. You can also hand in your solution in person before the big tutorial begins or via e-mail to `kramer@ibr.cs.tu-bs.de` with CC to `mperk@ibr.cs.tu-bs.de`.

Exercise 1 (Bin Packing):
Recall that the asymptotic competitive ratio of an online algorithm A over input sequences σ is defined as

$$\limsup_{n\to\infty}\left(\sup\left\{\frac{A(\sigma)}{n} \mid \mathrm{OPT}(\sigma)=n\right\}\right).$$

Furhermore, recall that the Next Fit bin packing algorithm works as follows. At any time, there is exactly one *open* bin. When the next item arrives and fits into that bin, it is placed there. Otherwise, the open bin is closed and never considered again. A new bin is opened and the item is packed into it. In this exercise, we consider the case where we are given a bound $\alpha \in (0,1]$ on the size of the items, i.e., all items a_i satisfy $a_i \leq \alpha$. For every $\alpha \in (0,1]$, determine (with proof) the asymptotic competitive ratio $c^{\infty}_{NF}(\alpha)$ of Next Fit depending on α.

(15 pts.)

Exercise 2 (Online Bin Covering):
In this exercise, we consider the problem of Bin Covering in an online scenario. Analogous to the situation for online bin packing, we are given a sequence of items of unknown weights $a_1,\dots,a_n \in [0,1]$ and want to assign these items to bins in an online fashion; however, the bins do not have limited capacity. In the Bin Covering problem, we want to *maximize* the number of *covered* bins, i.e., the number of bins that receive items of total weight at least 1.

a) Find an online algorithm for Bin Covering with an absolute competitive ratio of 2 and prove its competitive ratio. Prove that no deterministic online algorithm can have an absolute competitive ratio $c < 2$.

b) Prove that no deterministic online algorithm for Bin Covering can have an asymptotic competitive ratio $c < 3/2$.

(15+15 pts.)

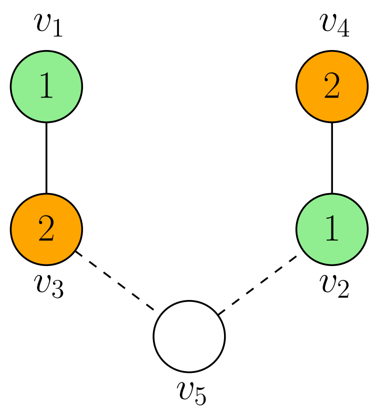

Figure 1: A snapshot from the execution of an online graph coloring algorithm. The algorithm has colored the vertices $v_1, \dots, v_4$ using colors 1 and 2 and is now given a new vertex v_5 connected to v_2 and v_3. It cannot use color 1 or 2 for v_5 and therefore has to introduce a new color for v_5.

Exercise 3 (Online Coloring):
In this exercise, we consider the problem of Graph Coloring in an online scenario. Our input sequence consists of the vertices $v_1, \dots, v_n$ of an undirected graph

$$G = (\{v_1, \dots, v_n\}, E).$$

Together with each vertex v_i, we are given the list E_i of all edges connecting v_i to previously given vertices $v_1, \dots, v_{i-1}$. Edges from v_i to vertices v_j with $j > i$ are only revealed once vertex v_j is given. The number of edges and vertices in the graph is not known in advance.
When we are given v_i, we have to choose a color $c(v_i) \in \mathbb{N}$ for v_i in such a way that no vertex adjacent to v_i has color $c(v_i)$. As usual, this choice is final; we cannot change the color later on. We want to minimize the number of colors used. For an example, see Figure 1.
We want to show that there is no deterministic c-competitive algorithm for this problem for any constant c. For any constant number $2 \leq k \in \mathbb{N}$ of colors and any deterministic online algorithm $\mathcal{A}$, devise a strategy for an adversary that satisfies the following requirements.

- The strategy always produces a forest $T_{\mathcal{A},k}$.
- The online algorithm $\mathcal{A}$ uses at least k colors on $T_{\mathcal{A},k}$.

Offline, every forest can be colored with 2 colors; therefore, this implies that $\mathcal{A}$'s competitive ratio is at least $k/2$. **(20 pts.)**