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Due: 10.06.2024
Discussion: 17.06.2024

Sheet 3

Please submit your individual solutions using the boxes in front of IZ338, before the exercise timeslot on the due date above. Your homework submission may be handwritten using proper ink (no pencil, no red ink) or printed.

Exercise 1 (ONLINE BIN PACKING: NEXT FIT): **(15 points)**

Recall that the asymptotic competitive ratio of an online algorithm A over input sequences σ is

$$\limsup_{n\to\infty}\left(\sup\left\{\frac{A(\sigma)}{n} \mid \mathrm{OPT}(\sigma)=n\right\}\right).$$

Furthermore, recall that the NEXT FIT bin packing algorithm works as follows. At any time, there is exactly one *open* bin. When the next item arrives and fits into that bin, it is placed there. Otherwise, the open bin is closed and never considered again. A new bin is opened and the item is packed into it. In this exercise, we consider the case where we are given a bound $\alpha \in (0,1]$ on the size of the items, i.e., all items a_i satisfy $a_i \leq \alpha$. For every $\alpha \in (0,1]$, determine (with proof) the asymptotic competitive ratio $c^{\infty}_{NF}(\alpha)$ of NEXT FIT depending on α.

Exercise 2 (ONLINE BIN COVERING): **(15+15 points)**

In this exercise, we consider the problem of BIN COVERING in an online scenario. Analogous to the situation for online bin packing, we are given a sequence of items of unknown weights $a_1, \ldots, a_n \in [0,1]$ and want to assign these items to bins in an online fashion; however, the bins do not have limited capacity. In the BIN COVERING problem, we want to *maximize* the number of *covered* bins, i.e., the number of bins that receive items of total weight at least 1.

a) Find an online algorithm for BIN COVERING with an absolute competitive ratio of 2 and prove its competitive ratio. Prove that no deterministic online algorithm can have an absolute competitive ratio $c < 2$.

b) Prove that no deterministic online algorithm for BIN COVERING can have an asymptotic competitive ratio $c < 3/2$.

Exercise 3 (Online Graph Coloring): **(20 points)**

In this exercise, we consider the problem of Graph Coloring in an online scenario. Our input sequence consists of the vertices $v_1, \dots, v_n$ of an undirected graph

$$G = (\{v_1, \dots, v_n\}, E).$$

Together with each vertex v_i, we are given the list E_i of all edges connecting v_i to previously given vertices $v_1, \dots, v_{i-1}$. Edges from v_i to vertices v_j with $j > i$ are only revealed once vertex v_j is given. The number of edges and vertices in the graph is not known in advance.

When we are given v_i, we have to choose a color $c(v_i) \in \mathbb{N}$ for v_i in such a way that no vertex adjacent to v_i has color $c(v_i)$. As usual, this choice is final; we cannot change the color later on. We want to minimize the number of colors used. For an example, see Figure 1.

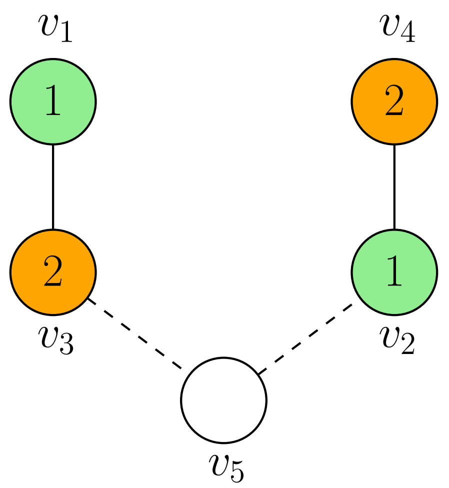

Figure 1: A snapshot from the execution of an online graph coloring algorithm. The algorithm has colored the vertices $v_1, \dots, v_4$ using colors 1 and 2, and is now given a new vertex v_5 connected to v_2 and v_3. It cannot use color 1 or 2 for v_5 and therefore has to introduce a new color for v_5.

We want to show that there is no deterministic c-competitive algorithm for this problem for any constant c. For any constant number $2 \leq k \in \mathbb{N}$ of colors and any deterministic online algorithm A, devise a strategy for an adversary that satisfies the following requirements.

- The strategy always produces a forest $T_{A,k}$.
- The online algorithm A uses at least k colors on $T_{\mathcal{A},k}$.

Offline, every forest can be colored with 2 colors; therefore, this implies that A's competitive ratio is at least $k/2$.