Finding Longest Geometric Tours

Sándor P. Fekete

Abstract We discuss the problem of finding a longest tour for a set of points in a geometric space. In particular, We show that a longest tour for a set of n points in the plane can be computed in time $O(n)$ if distances are determined by the Manhattan metric, while the same problem is NP-hard for points on a sphere under Euclidean distances.

1 Introduction: Short and Long Roundtrips

The Traveling Salesman Problem (TSP) is one of the classic problems of combinatorial optimization. Given a complete graph $G = (V,E)$ with edge weights $c(e)$ for all edges $e \in E$, find a shortest roundtrip through all vertices, i.e., a cyclic permutation π from the symmetric group S_n of all n vertices $v_1,\ldots,v_n$, such that the total tour length $\sum_{i=1}^{n} c(\{v_i, v_{\pi(i)}\})$ is minimized.

The difficulties of finding a good roundtrip are well known. The classical Odyssey is illustrated in Figure 1: according to legend, it took Ulysses many years to complete his voyage. One justification is the computational complexity of the TSP: it is one of the most famous NP-hard problems, so it does indeed take many years of CPU time to find provably optimal solutions for non-trivial instances.

However, there is an even more convincing justification for Ulysses' failure to be home in a more timely fashion: it was not him who chose his route. Instead, malevolent gods caused a deliberately long voyage—so the real objective was to *maximize* the traveled distance. This motivates the MaxTSP: Find a roundtrip that visits all vertices in a weighted graph, such that the total tour length is maximized.

In this chapter, we study longest tours in a geometric setting, in which the vertices are points in two- or three-dimensional space, and the edge weights are induced

Sándor P. Fekete
Department of Computer Science, TU Braunschweig, 38106 Braunschweig, Germany, e-mail: s.fekete@tu-bs.de

by the distance between them. As it turns out, the difficulty of the corresponding MaxTSP depends greatly on the involved geometry.

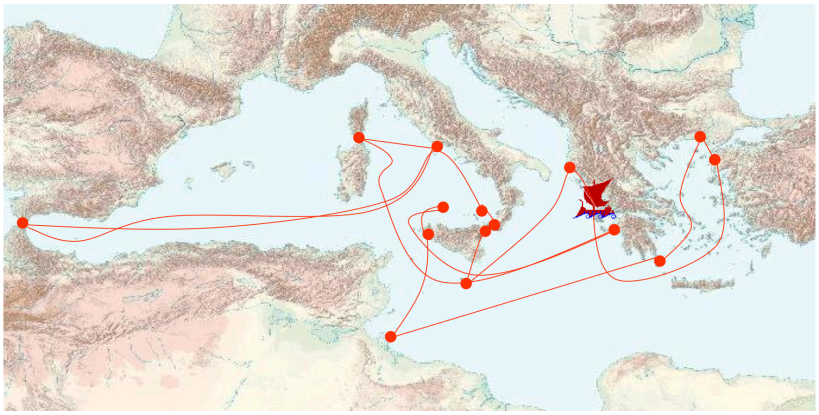

Fig. 1 The Odyssey: a tour through 16 locations in the Mediterrenean.

2 Traveling in Manhattan

The cost of traveling in a geometric space is measured by the geometric distance between points. A particularly simple way is to measure the axis-parallel distances separately, as one does when traveling along streets and avenues in Manhattan, giving rise to L_1 or *Manhattan distances*. A natural alternative is to use L_2 or *Euclidean distances*, which correspond to the length of straightline connections. When trying to find a shortest tour, this distinction does not make a difference in terms of the resulting problem complexity.

Theorem 1. *It is NP-complete to decide whether a set of n distinct points in the integer planar grid allows a roundtrip of length n.*

This amounts to deciding whether a *grid graph* has a Hamiltionian cycle; see Figure 2. The corresponding distance is the same in Manhattan or Euclidean distances.

In the following we sketch why it is considerably easier to find a *longest* tour for a planar point set with Manhattan distances.

Theorem 2. *When distances are measured according to the Manhattan metric, finding a longest roundtrip for a set P of n points in the plane can be achieved in time $O(n)$.*

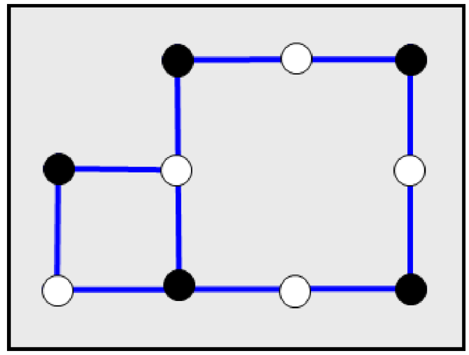

Fig. 2 A set of integer grid points induces a *grid graph*, in which two vertices are adjacent if and only if they have distance 1. Grid graphs are bipartite, as we can split the set of vertices into those with odd coordinate sum (black) and even coordinate sum (white). It is NP-complete to decide whether a given grid graph with n vertices has a tour of length n, i.e., a Hamiltonian cycle.

One key idea is to consider a *Manhattan median* $c = (x_c, y_c)$ for P, i.e., a point for which x_c is a median of all x-coordinates, and y_c is a median of all y-coordinates. Because c minimizes both the sum of all x- and y-distances to points in P, it induces a *Minimum Steiner Star*, as follows; we write $L_1(p,q)$ for the Manhattan distance between p and q.

Lemma 1. *A Manhattan median c minimizes the total distance $\sum_{i=1}^{n} L_1(c, p_i)$ to all points in P.*

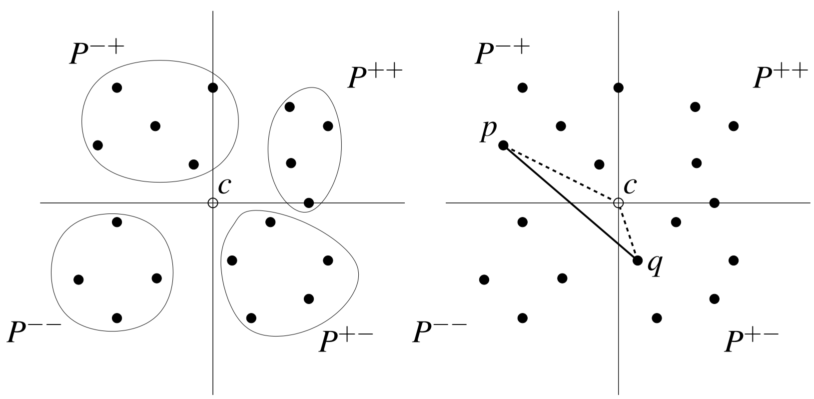

Fig. 3 (Left) A two-dimensional median c for a planar point set splits it into four quadrants, with equal numbers of points in opposite quadrants. (Right) In Manhattan distances, connecting two points in opposite quadrants incurs the same cost as connect the median to both of them.

Finding c is possible in linear time. Because a median splits P into two equal subsets according to either x- or y-distances, it follows that it induces a split into four quadrants, such that there is an equal number of points in opposite quadrants; see Figure 3 (Left). Furthermore, the weight of the corresponding Steiner Star, i.e., the sum of all distances from the median, induces an upper bound on the length of a longest tour.

Lemma 2. *For a set P of n points in the plane, we have the dual relationship* $\max_{\pi \in S_n^{cyclic}} \sum_{i=1}^{n} L_1(p_i, p_{\pi(i)}) \leq 2 \min_{c \in \mathbb{R}^2} \sum_{i=1}^{n} L_1(c, p_i)$.

This is a simple consequence of triangle inequality, as shown in Figure 3 (Right): each edge between two points p and q can be mapped to a path via a third point c, so $L_1(p,q) \leq L_1(p,c) + L_1(c,q)$. Note that for Manhattan distances, this inequality actually holds with *equality*, provided that p and q lie in opposite quadrants respective to c. This observation allows us to find an optimal permutation that consists of *two* cycles instead of one.

Lemma 3. *For a set P of n points in the plane, there is a permutation* $\overline{\pi}$ *consisting of two cycles for which* $\sum_{i=1}^{n} L_1(p_i, p_{\overline{\pi}(i)}) = 2 \min_{c \in \mathbb{R}^2} \sum_{i=1}^{n} L_1(c, p_i)$.

As shown in Figure 4 (Left), such a solution consists of two subtours: cycles that go back and forth between opposite quadrants, which is possible because of the equal number of points in those quadrants.

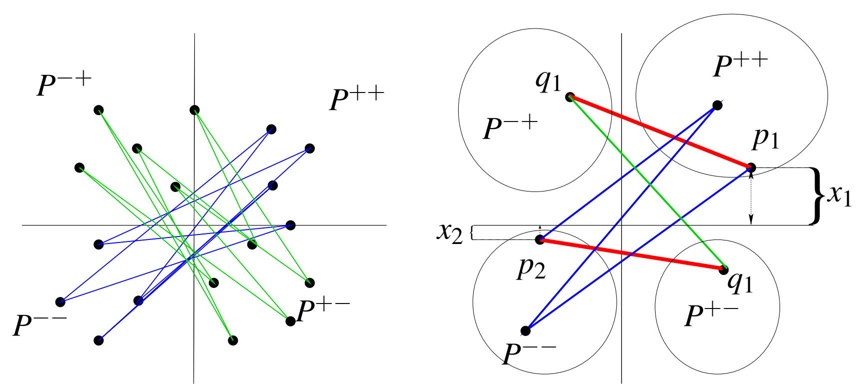

Fig. 4 (Left) Because opposite quadrants contain equal numbers of points, traveling back and forth between quadrants induces an optimal pair of subtours, shown in blue and green. (Right) In a tour, there must be connections between adjacent quadrants (shown in red), inducing an adjustment of the upper bound.

There is one final step left for obtaining an optimal tour. Observe that in order to form one connected cycle, any tour must contain a pair of edges (p_1, q_1) and (p_2, q_2) that connect *adjacent* quadrants, as shown in Figure 4 (Right). This causes a gap in the triangle inequality, depending on the distance to a coordinate axis; in the example, this works out to $L_1(p_1, q_1) + 2|x_1| = L_1(p_1, c) + L_1(c, q_1)$ and $L_1(p_2, q_2) + 2|x_2| = L_1(p_2, c) + L_1(c, q_2)$. As a consequence, we must adjust the upper bound of $\sum_{i=1}^{n} L_1(c, p_i)$ by subtracting twice the smallest possible coordinate distances for one point from each of two opposite coordinate halfplanes, i.e.,

$2|x_1|+2|x_2|$ in the example. Finding such a pair is easily possible in linear time. Now the corresponding pair of edges (p_1,q_1) and (p_2,q_2) can be used for merging both subtours into one tour that meets this upper bound, meaning that it is optimal.

Theorem 2 has a very powerful generalization for *polyhedral norms*, which are induced by using a symmetric convex polyhedron as the unit ball.

Theorem 3. *When distances are measured according to a polyhedral norm with a fixed number f of facets, finding a longest roundtrip for a set P of n points in a d-dimensional space for some fixed d can be achieved in time $O(n^{f-2}\log n)$.*

3 Traveling around the Globe

In a graph setting, the TSP is NP-complete, and it is not difficult to see that this is also the case for the MaxTSP: If M is a sufficiently large number, replacing each edge weight $c(e)$ of a TSP instance by $M-c(e)$ yields a MaxTSP instance with the same optimal tours. But how can we use this simple idea of inverting short and long edges in a geometric setting? In fact, Theorem 2 indicates that this may not even be possible.

The key idea lies in switching from from the plane to a sphere, and consider Euclidean distances. Consider Figure 5, which displays a sign posted at the airport of Auckland in New Zealand: It shows that furthest distances are to London and Frankfurt, which are both almost antipodal to the current location, i.e., at close to the theoretical maximum distance of 20.000km. Moreover, both cities are far from Auckland, but close to each other.

Utilizing this observation, we can conclude the following.

Theorem 4. *When distances are measured according to the Euclidean metric, finding a longest roundtrip for a set P of n points on a sphere is an NP-hard problem.*

We sketch a reduction from the Hamiltonicity of Grid Graphs, i.e., a consequence of Theorem 1. Given a set P of planar distinct grid points, like the one shown in Figure 2. We embed two "small" copies of P at antipodal locations of a sphere, as indicated in Figure 6 (Left). This leaves corresponding pairs at maximum possible distance, as shown in Figure 6 (Middle). Now observe that grid graphs are bipartite: each grid point has either even or odd sum of coordinates, and moving to an adjacent grid point changes parity. Omitting all *even* points of P from one of the two locations, but all *odd* points from the other (as depicted in Figure 6 (Right)) leaves maximum possible distances between points that are almost antipodal (like Auckland and Frankfurt)—meaning that they are adjacent in the original grid graph. Therefore, there is a tour that uses only edges of this maximum possible length, if and only if the original grid graph has a Hamiltonian cycle. (The complete proof requires more involved trigonometry for a full argument, but that is a mere matter of math.)

Fig. 5 Large distances around the globe: a picture taken at Auckland airport, which is close to being 20.000km away from, i.e., almost antipodal to, London and Frankfurt.

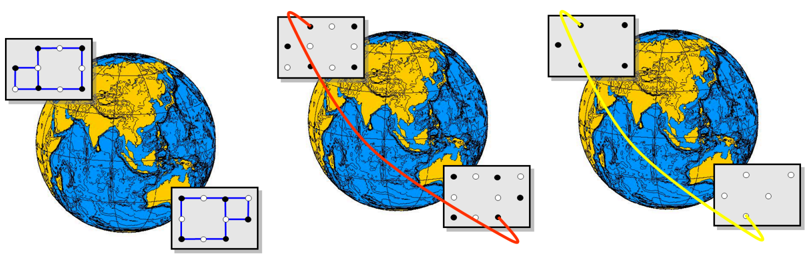

Fig. 6 Showing NP-hardness of the MaxTSP for points in 3D and Euclidean distances: (Left) Embedding two copies of a given grid graph G, such that corresponding vertices become antipodal points. (Middle) Longest distances within the resulting point set connect antipodal points. (Right) Exploiting bipartiteness of grid graphs for mapping edges in the original grid graph to longest edges in the remaining point set.

How hard is it to find a longest tour for Euclidean distances in the plane? This has been unresolved for more than a decade: it is Problem #49 of THE OPEN PROBLEMS PROJECT, and has been a challenge since 2003.

Problem 1. What is the complexity of finding a longest roundtrip for a set P of n points in the plane, when distances are measured according to the Euclidean metric?

4 Further Reading

There are different books describing various aspects of the TSP. A classic overview is provided by Lawler et al. [18]. A detailed exposition of the computational aspects involved in solving instances to optimality was presented by Applegate et al. [1], while Cook [7] gives an entertaining survey with various historical and anecdotal notes. The Odyssey instance was first presented by Grötschel and Padberg [14, 15, 16] and is contained in the benchmark library TSPLIB [21].

Papdimitriou [20] was the first to prove NP-hardness of the TSP for Euclidean distances in the plane; the NP-hardness result for Hamiltonicity in grid graphs is due to Itai et al. [17].

A couple of years before Arora [3] and Mitchell [19] independently showed that the geometric TSP can be approximated arbitrarily well (i.e., for any $\varepsilon > 0$, there is a polynomial-time algorithm that computes a tour within a factor of $(1+\varepsilon)$ of the optimum), Barvinok [5] already did the same for the MaxTSP. Independently, Serdyuokov [22, 23, 24] gave several results for the MaxTSP. The bottleneck version of the MaxTSP (find a tour with a shortest edge that is as long as possible) was considered by Arkin et al. [2].

Barvinok et al. [4] were the first to provide a polynomial-time algorithm for the MaxTSP with polyhedral norms in fixed-dimensional space, as stated in Theorem 3. The strong geometric duality between Minimum Steiner Star and Maximum Matching was first observed by Tamir and Mitchell [25]. Fekete and Meijer [10, 11] proved a tight bound for the corresponding ratio between Minimum Steiner Star and Maximum Matching for Euclidean distances, and demonstrated in [12, 13] that this can be exploited for finding good maximum-weight matchings.

The results of this chapter (in particular, Theorem 2 and Theorem 4) were first presented in [9]. A more detailed journal version can be found in [6], which also contains full details of the conference paper [4].

See [8] for The Open Problems Project.

References

1. D. L. Applegate, R. E. Bixby, V. Chvatal, and W. J. Cook. *The Traveling Salesman Problem: A Computational Study (Princeton Series in Applied Mathematics).* Princeton University Press, Princeton, NJ, USA, 2007.
2. E. M. Arkin, Y.-J. Chiang, J. Mitchell, S. Skiena, and T.-C. Yang. On the Maximum Scatter TSP. In *Proc. 8th ACM-SIAM Symp. Disc. Alg. (SODA 97)*, pages 211–220, 1997.
3. S. Arora. Polynomial-time approximation schemes for Euclidean TSP and other geometric problems. *J. ACM*, pages 753–782, 1998.
4. A. Barvinok, D. Johnson, G. Woeginger, and R. Woodroofe. The Maximum Traveling Salesman Problem under polyhedral norms. In *Proc. 6th Int. Integer Prog. Comb. Opt. Conf. (IPCO VI)*, volume 1412 of *Springer LNCS*, pages 195–201, 1998.
5. A. I. Barvinok. Two algorithmic results for the traveling salesman problem. *Mathematics of Operations Research*, 21(1):65–84, 1996.

6. A. I. Barvinok, S. P. Fekete, D. S. Johnson, A. Tamir, G. J. Woeginger, and R. Wodroofe. The geometric maximum Traveling Salesman problem. *J. ACM*, 50:641–664, 2003.
7. W. J. Cook. *In Pursuit of the Traveling Salesman: Mathematics at the Limits of Computation*. Princeton University Press, 2011.
8. E. D. Demaine, J. S. B. Mitchell, and J. O'Rourke. The open problems project, 2001. http://cs.smith.edu/ orourke/TOPP/.
9. S. P. Fekete. Simplicity and hardness of the maximum traveling salesman problem under geometric distances. In *Proc. 10th ACM-SIAM Sympos. Discrete Algorithms*, pages 337–345, 1999.
10. S. P. Fekete and H. Meijer. On minimum stars, minimum steiner stars, and maximum matchings. In *Proc. 15th Annu. ACM Sympos. Comput. Geom.*, pages 217–226. ACM, 1999.
11. S. P. Fekete and H. Meijer. On minimum stars and maximum matchings. *Discrete Comput. Geom.*, 23(3):389–407, 2000.
12. S. P. Fekete, H. Meijer, A. Rohe, and W. Tietze. Solving a "hard" problem to approximate an "easy" one: Heuristics for maximum matchings and maximum Traveling Salesman problems. In *Proc. 3rd Internat. Workshop Algorithm Eng. Exp.*, volume 2153 of *Lecture Notes Computer Science*, pages 1–16. Springer-Verlag, 2001.
13. S. P. Fekete, H. Meijer, A. Rohe, and W. Tietze. Solving a "hard" problem to approximate an "easy" one: Heuristics for maximum matchings and maximum Traveling Salesman problems. *J. Exp. Algorithms*, 7, 2002. 21 pages.
14. M. Grötschel and M. W. Padberg. Ulysses 2000: In search of optimal solutions to hard combinatorial problems. SC 93-34, Zuse Institute Berlin, Nov. 1993.
15. M. Grötschel and M. W. Padberg. Die optimierte Odyssee. *Spektrum der Wissenschaft, Digest*, 2:32–41, 1999.
16. M. Grötschel and M. W. Padberg. The optimized Odyssey. *AIROnews*, VI(2):1–7, 2001.
17. A. Itai, C. Papadimitriou, and J. Swarcfiter. Hamilton paths in grid graphs. *SIAM J. Comp.*, 11:676–686, 1982.
18. E. Lawler, J. Lenstra, A. Rinnooy Kan, and D. Shmoys. *The Traveling Salesman Problem*. Wiley, Chichester, 1985.
19. J. Mitchell. Guillotine subdivisions approximate polygonal subdivisions: Part ii – a simple PTAS for geometric k-MST, TSP, and related problems. *SIAM J. Comp.*, 28:1298–1309, 1999.
20. C. Papadimitriou. The Euclidean Traveling Salesman Problem is np-complete. *Theoretical Comp. Sci.*, 4:237–244, 1977.
21. G. Reinelt. TSPlib – A Traveling Salesman Problem library. *ORSA J. on Computing*, 3(4):376–384, 1991.
22. A. I. Serdyukov. An asymptotically exact algorithm for the Traveling Salesman Problem for a maximum in Euclidean space (russian). *Upravlyaemye Sistemy*, 27:79–87, 1987.
23. A. I. Serdyukov. Asymptotic properties of optimal solutions of extremal permutation problems in finite-dimensional normed spaces (russian). *Metody Diskret. Analiz.*, 51:105–111, 1991.
24. A. I. Serdyukov. The traveling salesman problem for a maximum in finite-dimensional real spaces (russian). *Diskret. Anal. Issled. Oper. 2*, 1:50–56, 1995.
25. A. Tamir and J. S. B. Mitchell. A maximum b-matching problem arising from median location models with applications to the roommates problem. *Math. Program.*, 80(2):171–194, 1998.