Technische Universität Braunschweig

# Computational Geometry – Exercise Meeting #3

January 19th, 2022

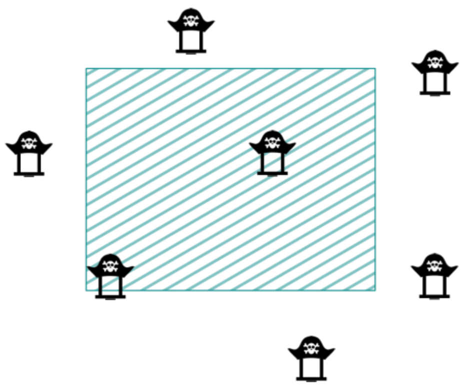
**What did you think of the question sheet?**
(Wording, type of questions, etc.)

**Would you prefer more ordinary homework?**

**Do you have any ideas for next time?**

# Maximizing distance to sites in a bounded area

Provided a set of **sites** $\boldsymbol{P}$ in the plane and an axis-aligned **rectangle** $\boldsymbol{R}$, find a point inside of $R$ with **maximal distance to the nearest member of** $\boldsymbol{P}$. Assume that no three points in $P$ are collinear.



# Maximizing distance to sites in a bounded area

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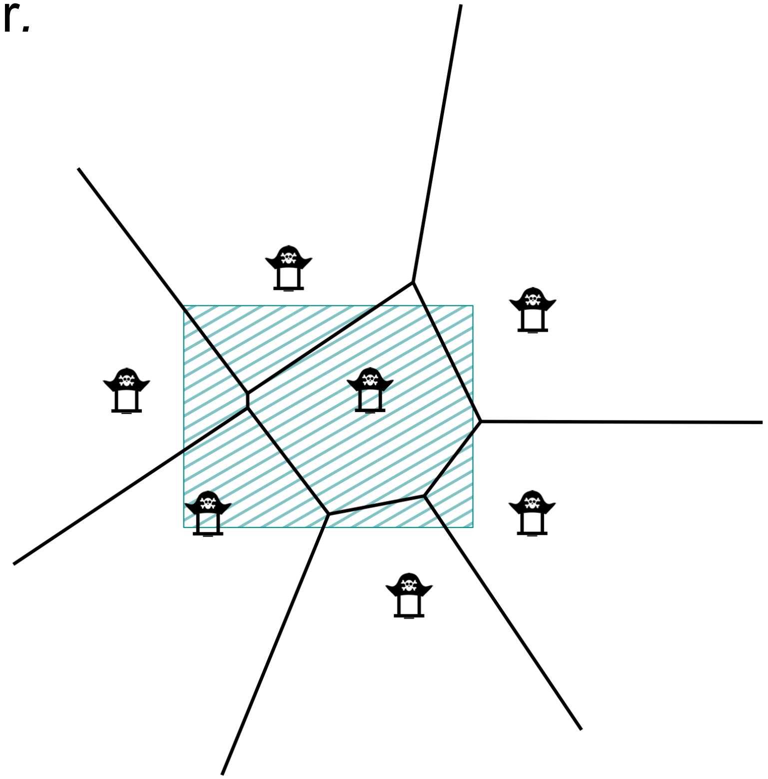
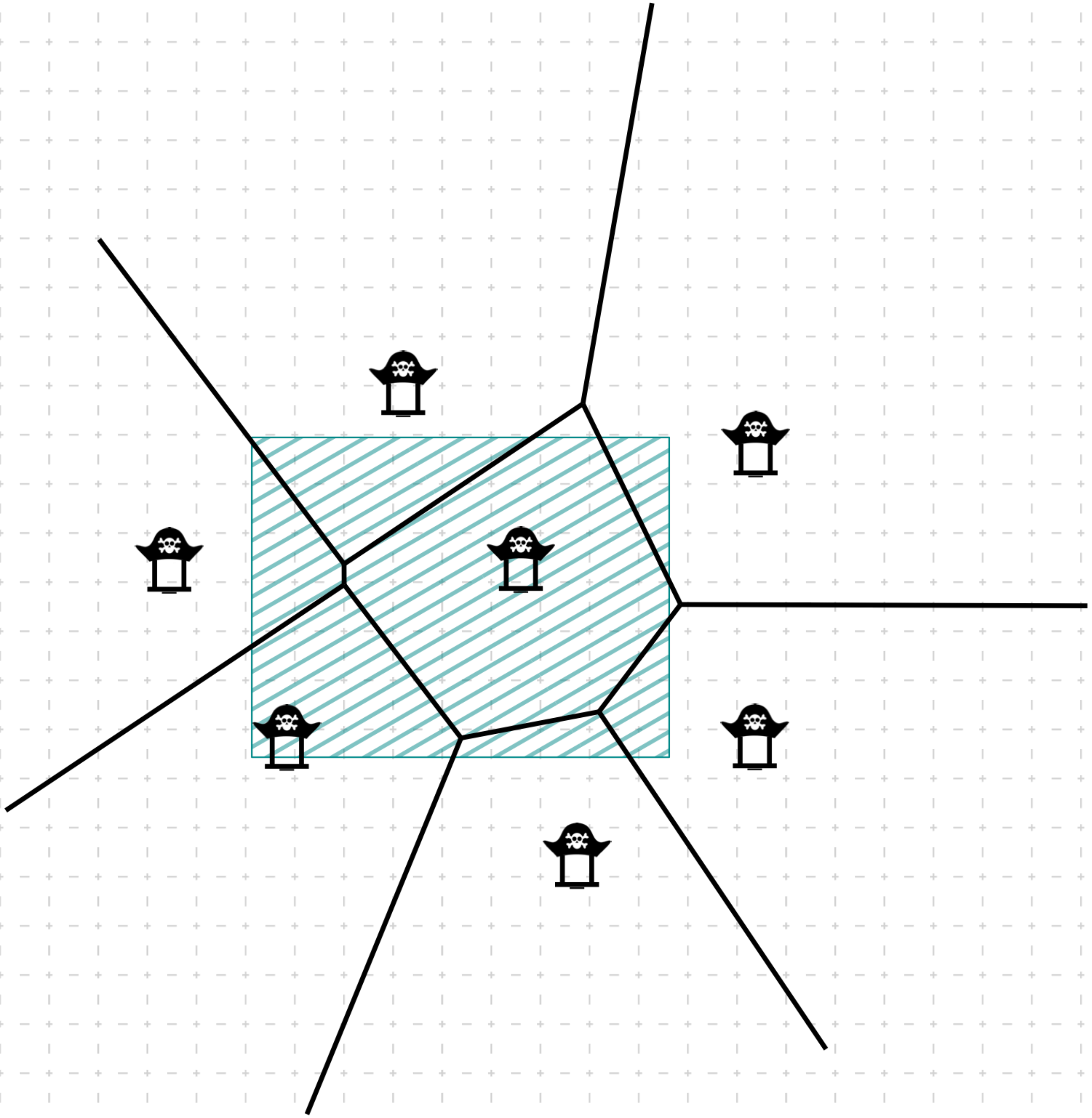
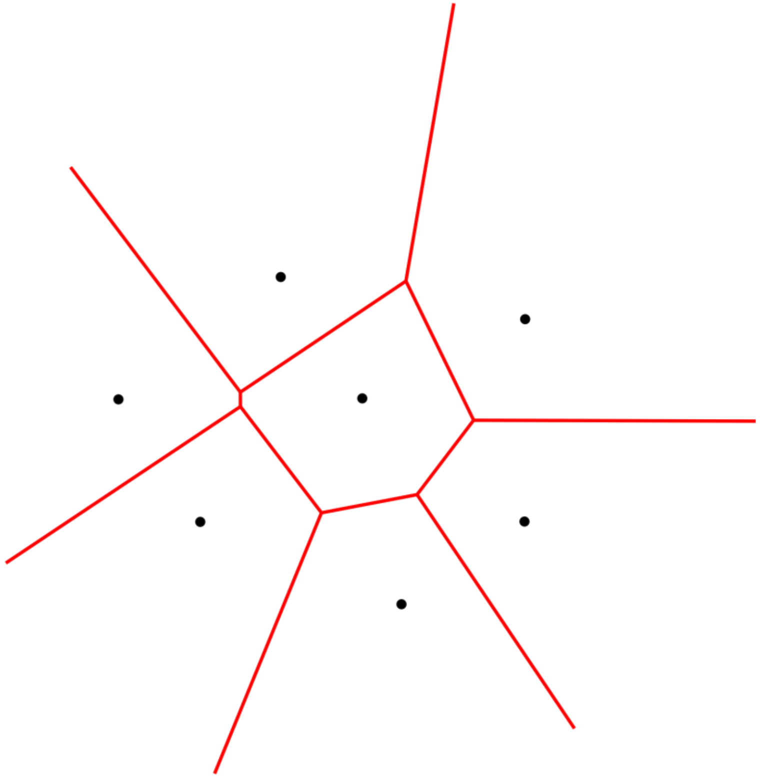




# Refresh – Higher order Voronoi diagrams



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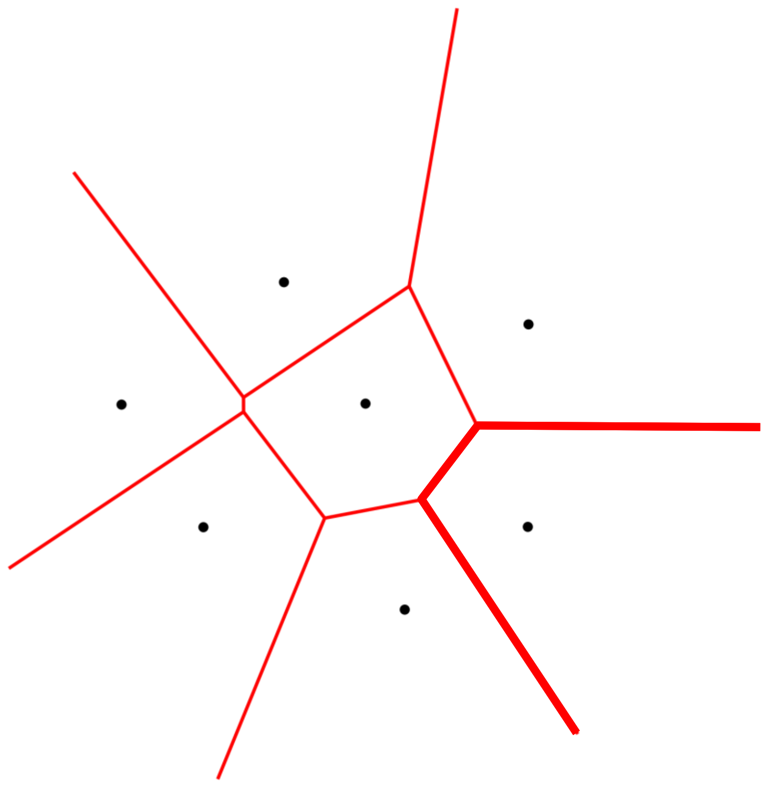
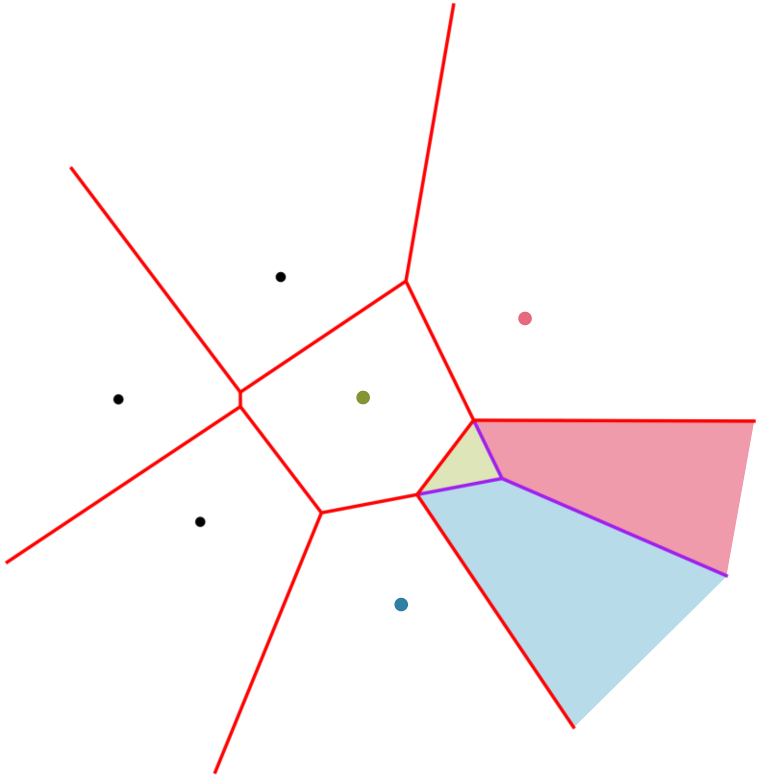
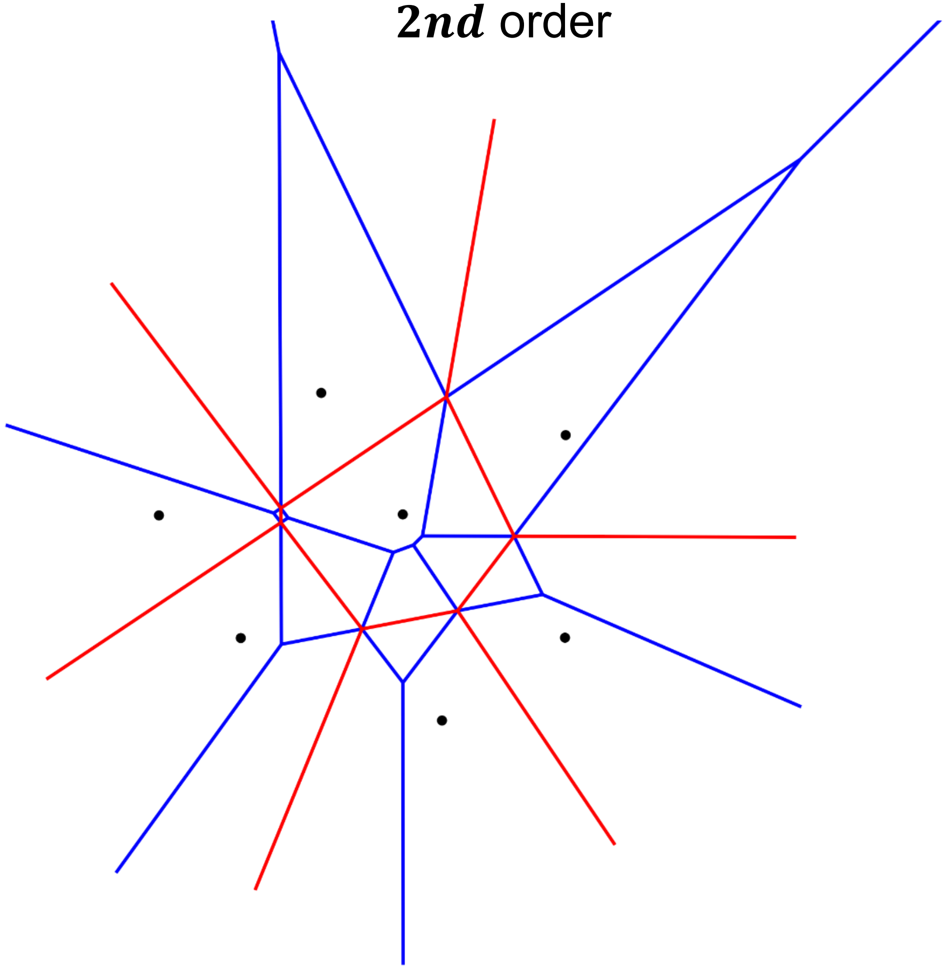


# Refresh – Higher order Voronoi diagrams



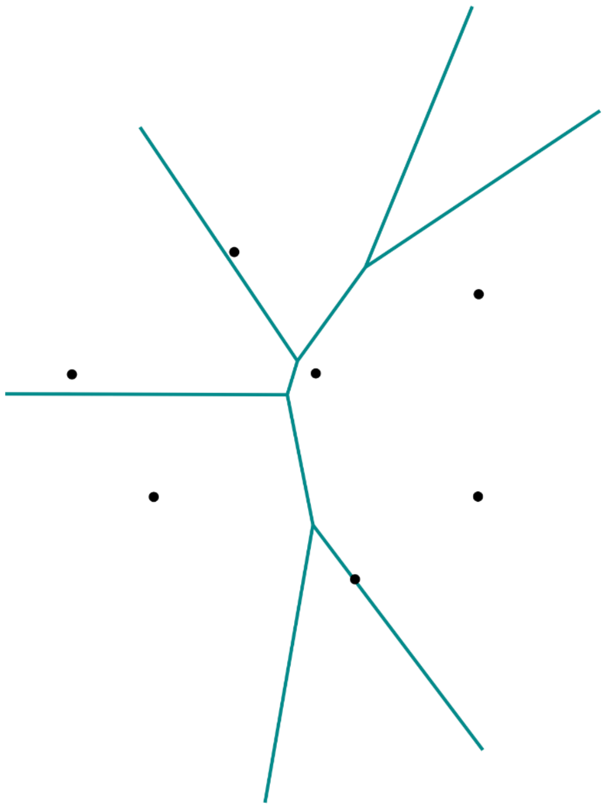
# Refresh – Higher order Voronoi diagrams

$2nd$ order



# Refresh – Farthest Point Voronoi diagrams

$(\boldsymbol{n-1})\boldsymbol{th}$ order

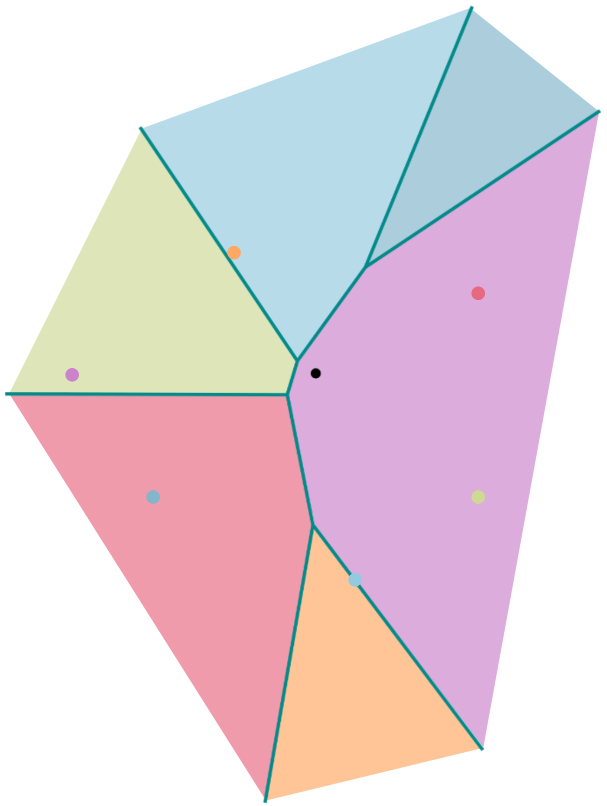


Can you think of any relation to convex hulls?

Can be computed incrementally, or directly in $O(n \log n)$

# Farthest point Voronoi diagrams – Properties

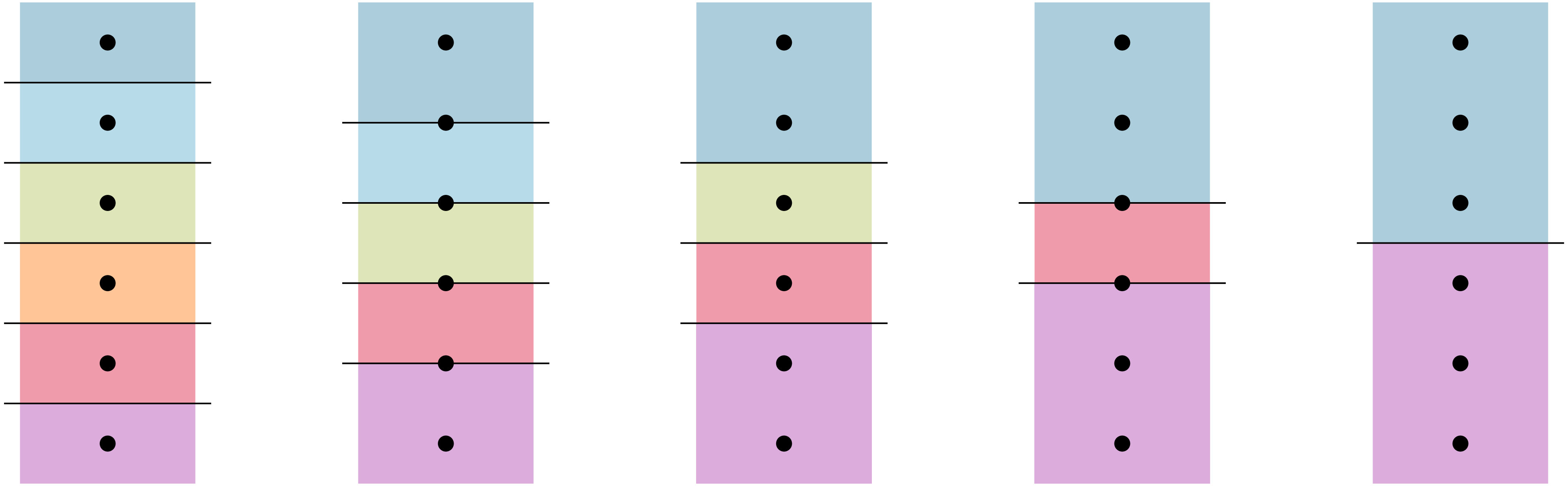
$(n-1)th$ order



The $(n-1)th$ order Voronoi region of a point is non-empty exactly if the point is part of the set's convex hull.

All cells are unbounded, the planar graph is a tree.

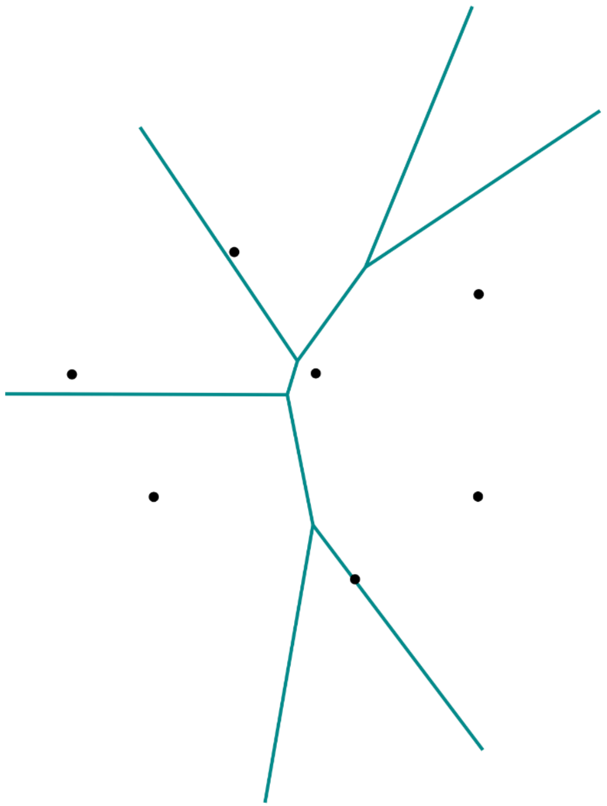
# Notes from last week – What about collinear points?



If the **entire set** is collinear, the $i$th order Voronoi diagram is a set of $(n-i)$ lines, i.e. one single line for $i = n - 1$.
Otherwise, all diagrams are connected planar graphs, the farthest point diagram being a tree.

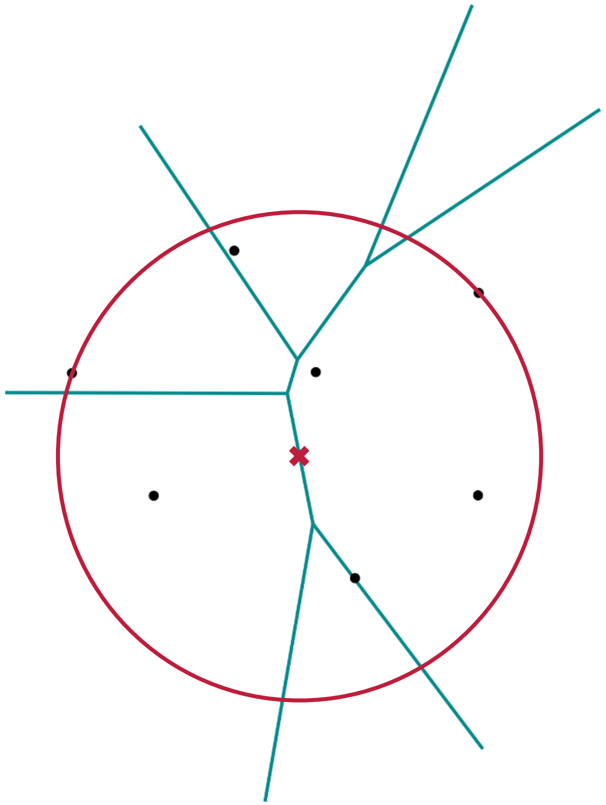
# Farthest point Voronoi diagrams – Properties

$(n-1)th$ order

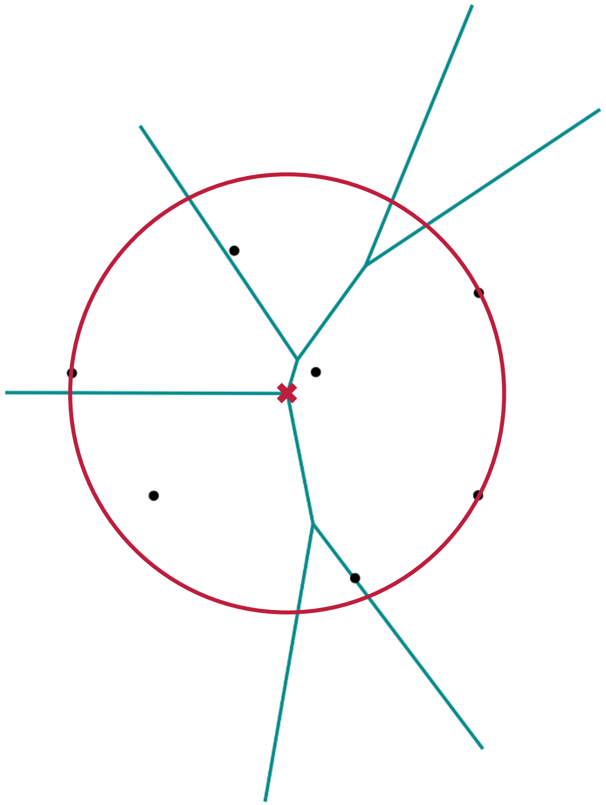


What can we say about the edges and vertices?

# Farthest point Voronoi diagrams – Properties



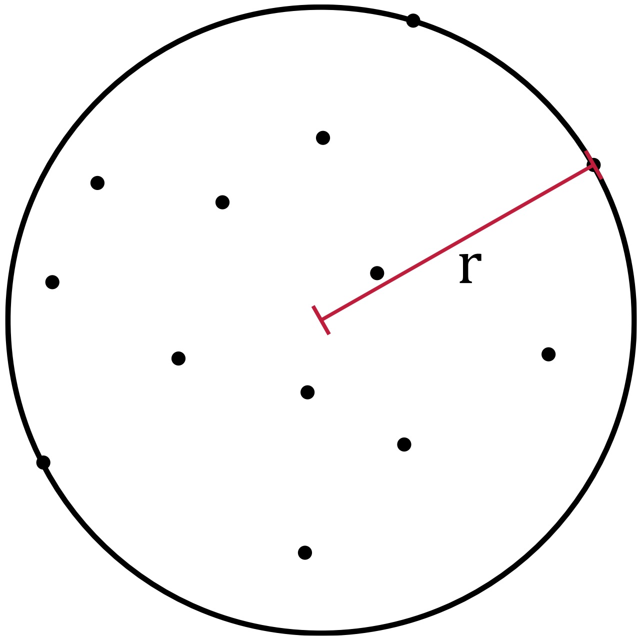
Edges are equidistant to two sites, closer to all others.



Vertices are equidistant to at least three sites, closer to all others.

# Smallest enclosing disk ($1$-center problem)

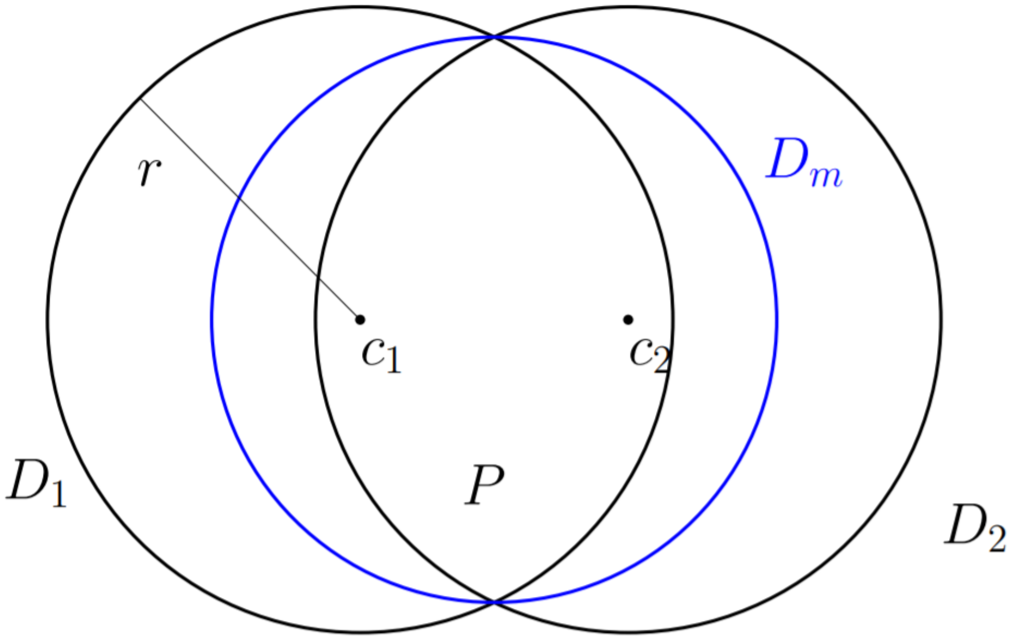
Provided a set of **points $\boldsymbol{P}$** in the plane, **find a disk $\boldsymbol{md(D)}$** with **minimal radius $\boldsymbol{r}$** that contains all members of $P$. Assume that no three points in $P$ are collinear.



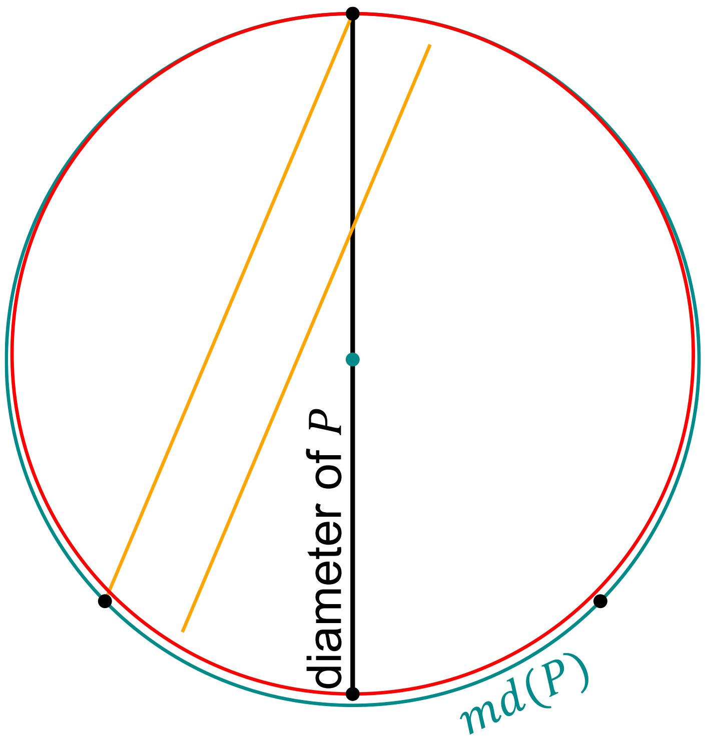

What can we say about an optimal disk $D$?

# Smallest enclosing disk – Uniqueness

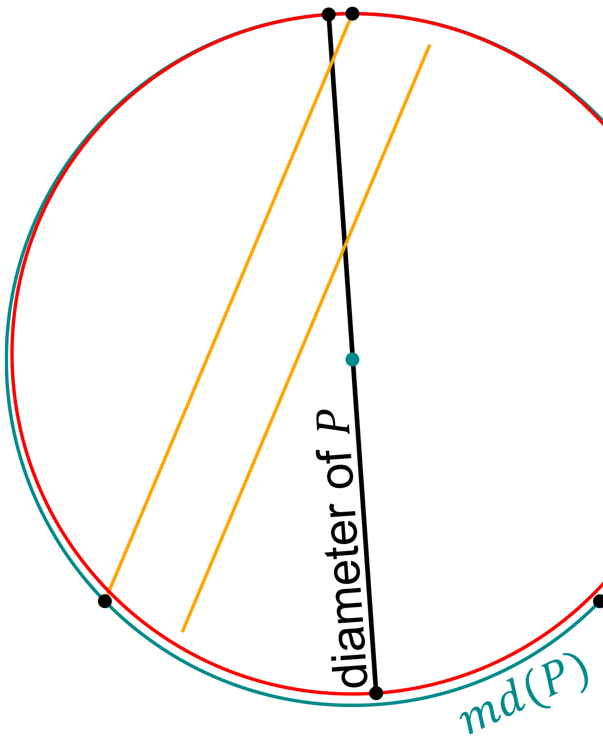
*For any point set $P$, the smallest enclosing disk $md(P)$ is unique.*



# Smallest enclosing disk – Relation to diameter

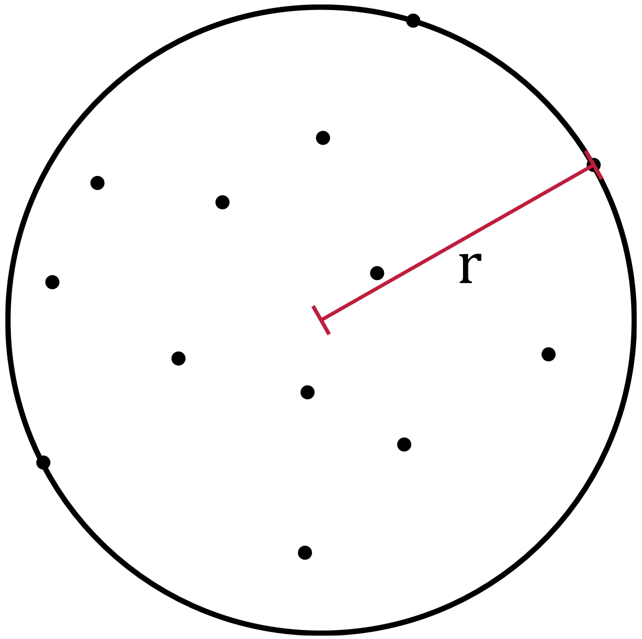


# Smallest enclosing disk – Relation to diameter



# Finding the smallest enclosing disk

Provided a set of **points** $\boldsymbol{P}$ in the plane, **find a disk** $\boldsymbol{md(D)}$ with **minimal radius** $\boldsymbol{r}$ that contains all members of $P$. Assume that no three points in $P$ are collinear.

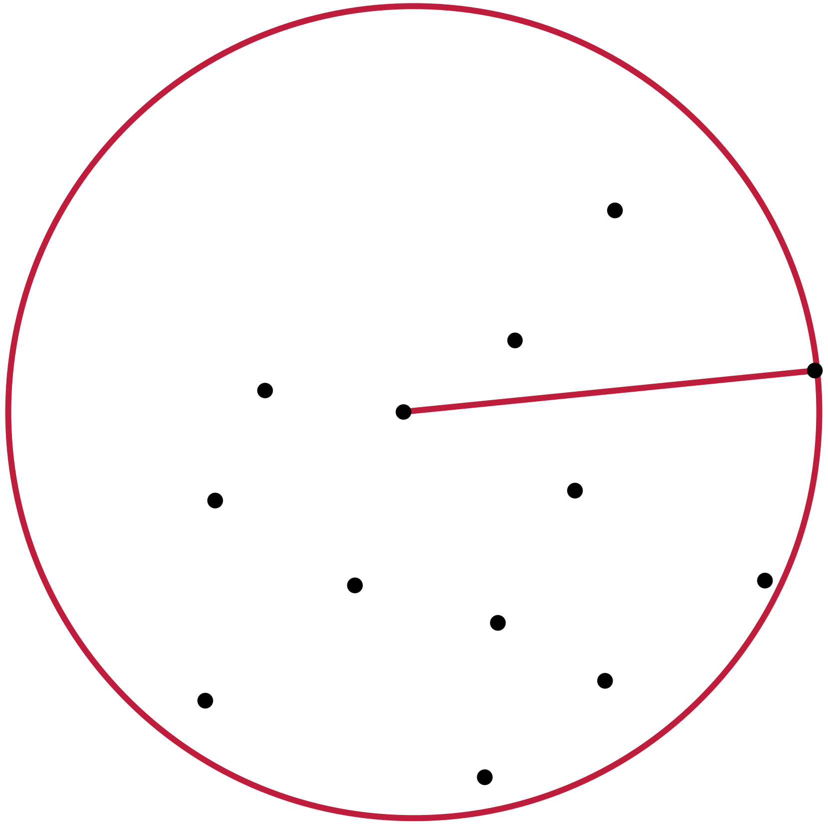



How long would a naive approach take, at most?

Any ideas how we can find an approximate min disk?

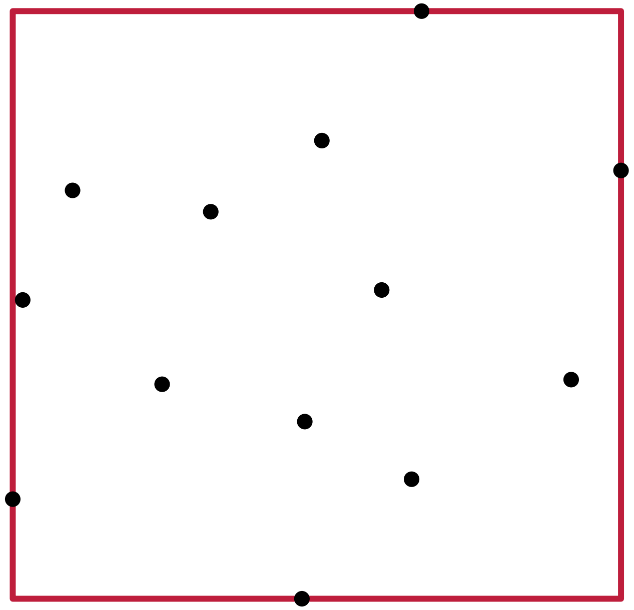
# Smallest enclosing disk – $2$-Approximation

1. Pick any point $p \in P$
2. Find the farthest point $p'$
3. Draw a circle.



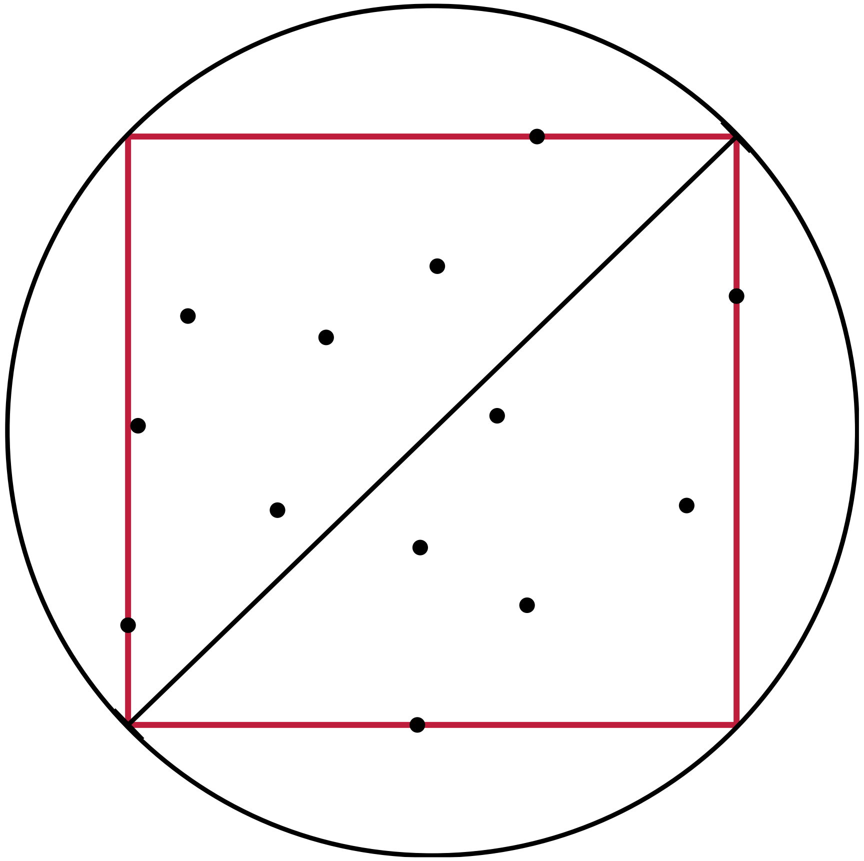
# Smallest enclosing disk – $\sqrt{2}$-Approximation by bounding box

1. Compute an axis-aligned bounding box



# Smallest enclosing disk – $\sqrt{2}$-Approximation by bounding box

1. Compute an axis-aligned bounding box
2. Place a circle on the corners.



# Smallest enclosing disk – Optimal solution in $O(n \log n)$

CLOSEST-POINT PROBLEMS

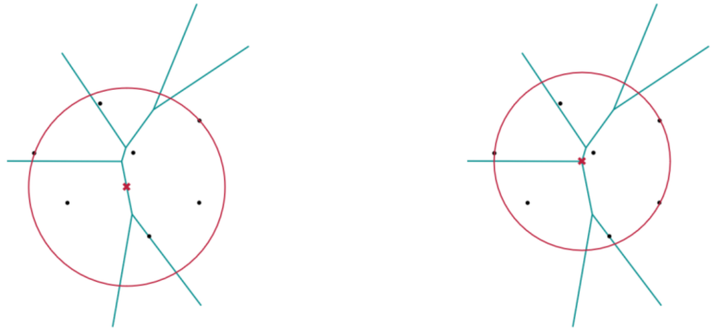
Michael Ian Shamos† and Dan Hoey

Department of Computer Science, Yale University
New Haven, Connecticut 06520

Abstract

A number of seemingly unrelated problems involving the proximity of N points in the plane are studied, s as finding a Euclidean minimum spanning tree, the smallest circle enclosing the set, $k$ nearest and farthest neighbors, the two closest points, and a proper straight-line triangulation. For most of the problems consi a lower bound of O(N log N) is shown. For all of them the best currently-known upper bound is $O(N^2)$ or wors The purpose of this paper is to introduce a single geometric structure, called the Voronoi diagram, which ca constructed rapidly and contains all of the relevant proximity information in only linear space. The Vorono diagram is used to obtain O(N log N) algorithms for all of the problems.

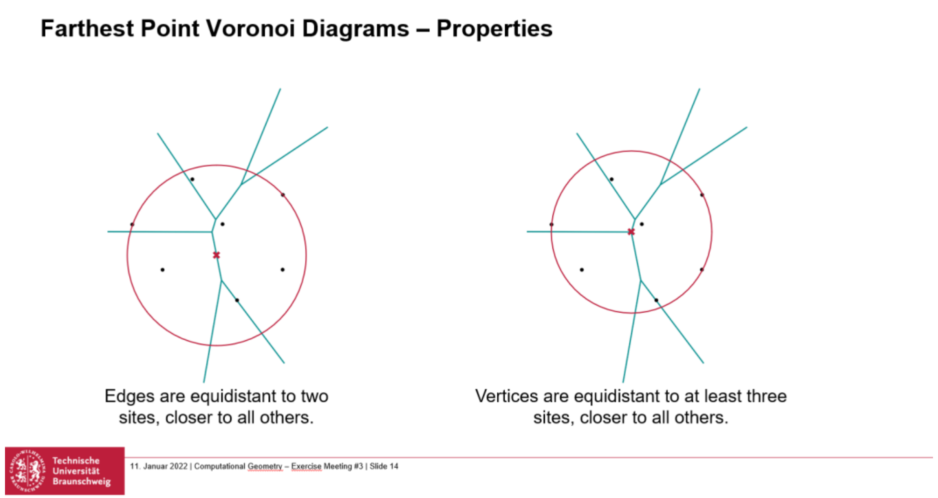
## Farthest Point Voronoi Diagrams – Properties



Edges are equidistant to two sites, closer to all others.

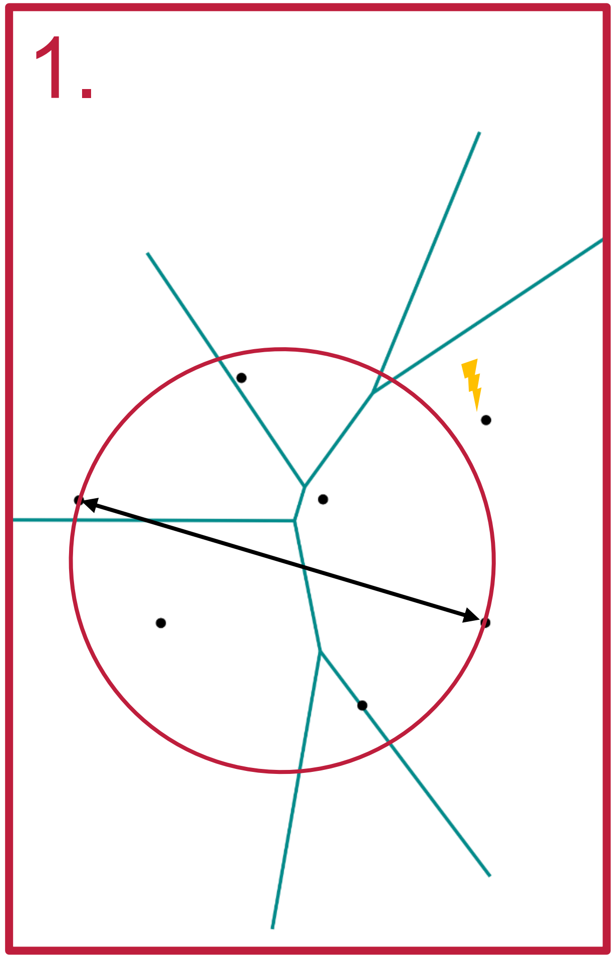
Vertices are equidistant to at least three sites, closer to all others.

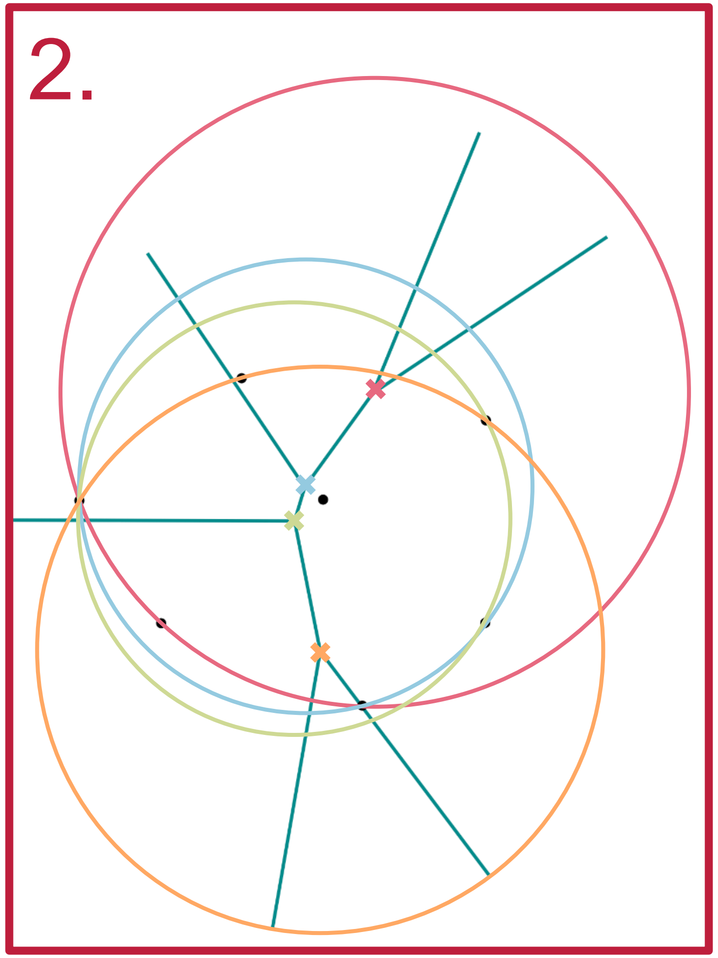
# Farthest point Voronoi diagrams – Properties




**$md(P)$ is defined by the farthest pair or by three sites, so:**

1. Check if the smallest disk on the farthest pair fits.
2. Otherwise, check all circles induced by highest-order Voronoi vertices.



# Smallest enclosing disk – Optimal solution in expected linear time

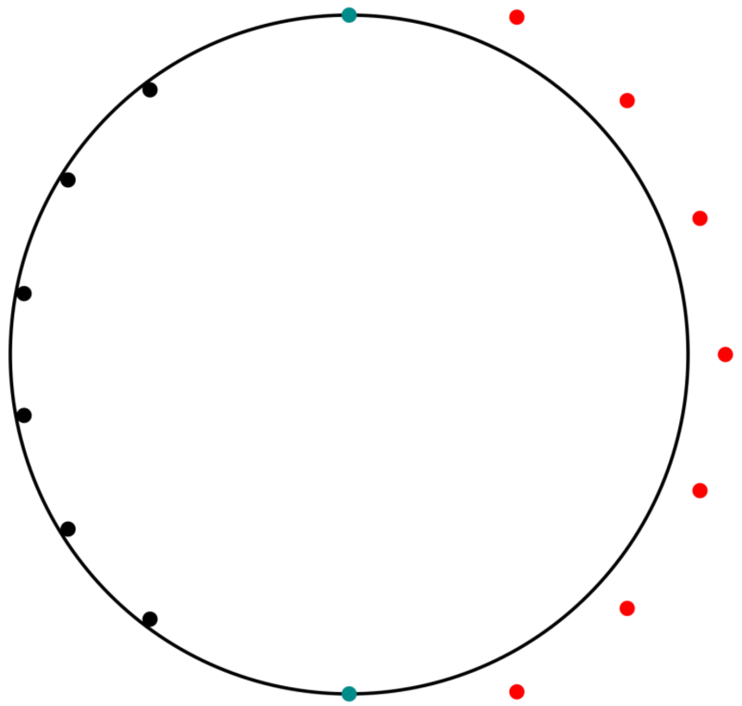
APPEARED IN "New Results and New Trends in Computer Science", (H. Maurer, Ed.), *Lecture Notes in Computer Science* **555** (1991) 359-370.

## Smallest enclosing disks (balls and ellipsoids)

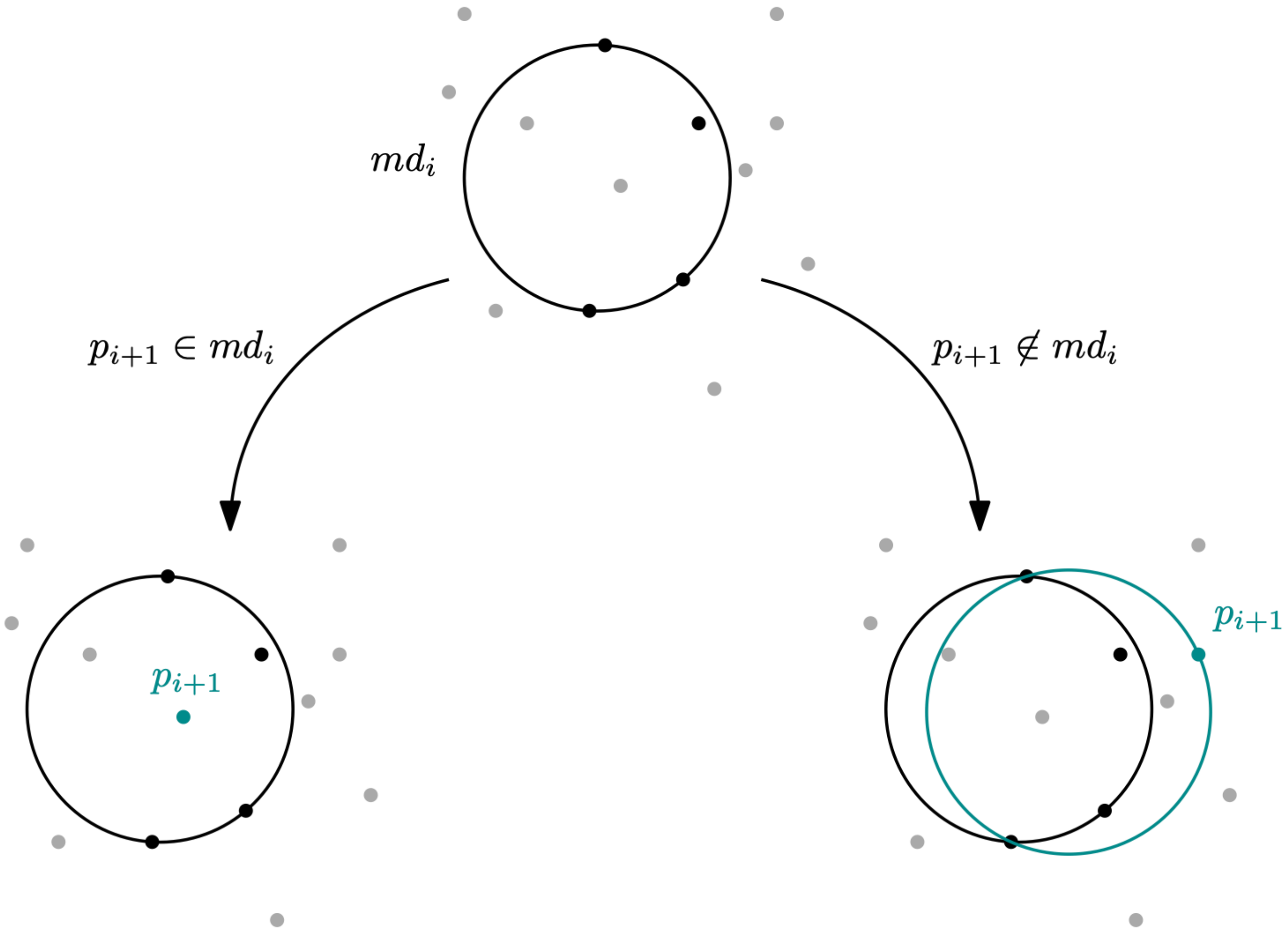
EMO WELZL*
Institut für Informatik, Freie Universität Berlin
Arnimallee 2-6, W 1000 Berlin 33, Germany
e-mail: emo@tcs.fu-berlin.de

**Abstract**

A simple randomized algorithm is developed which computes the smallest enclosing disk of a finite set of points in the plane in expected linear time. The algorithm is based on Seidel's recent Linear Programming algorithm, and it can be generalized to computing smallest enclosing balls or ellipsoids of point sets in higher dimensions in a straightforward way. Experimental results of an implementation are presented.

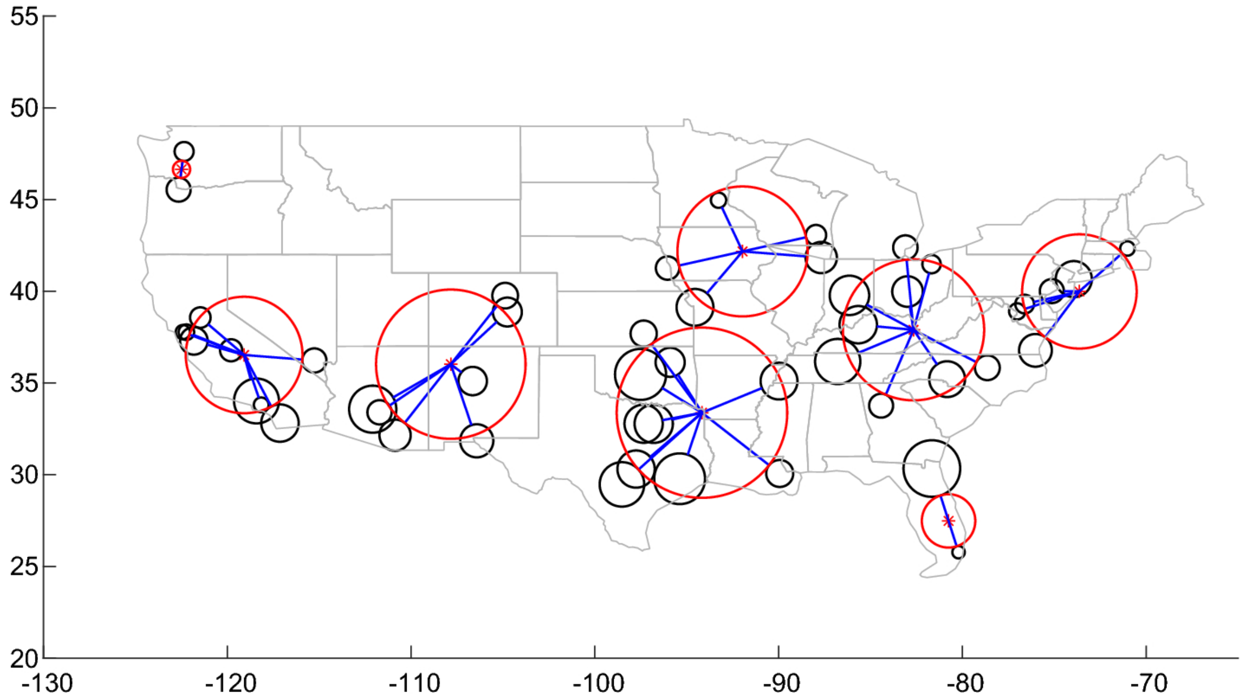


# Smallest enclosing disk – Optimal solution in expected linear time



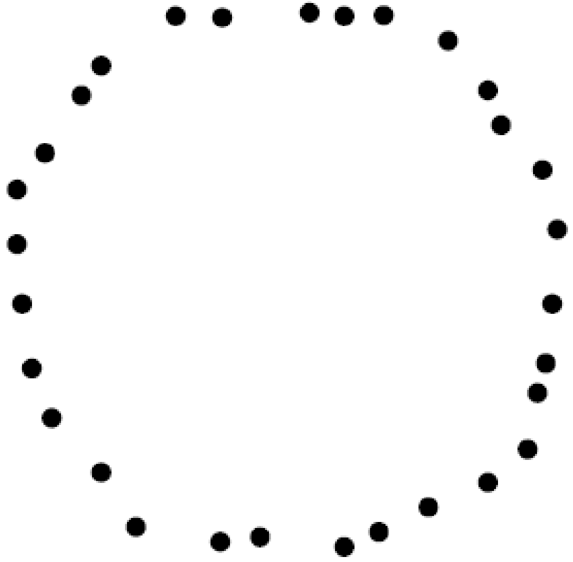
# $k$-center problem (NP-hard)

Provided a set of **points $P$** in the plane, **find $k$ disks $D_i$** with **minimal radii $r_i$** that, when combined, contain all members of $P$.

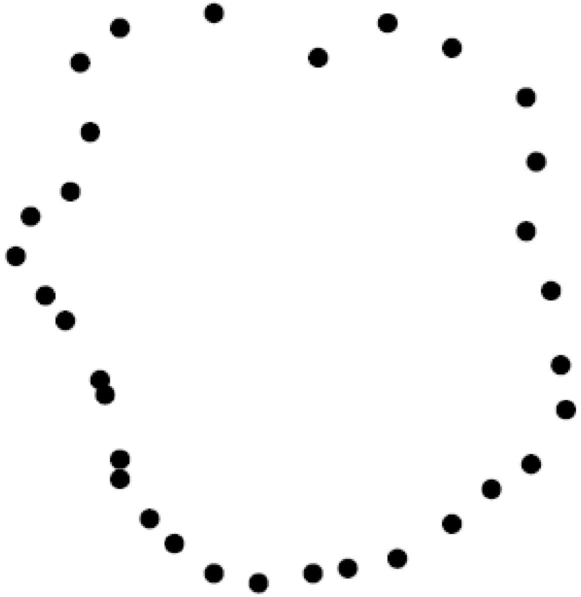



https://doi.org/10.1007/s10898-019-00834-6

# Roundness

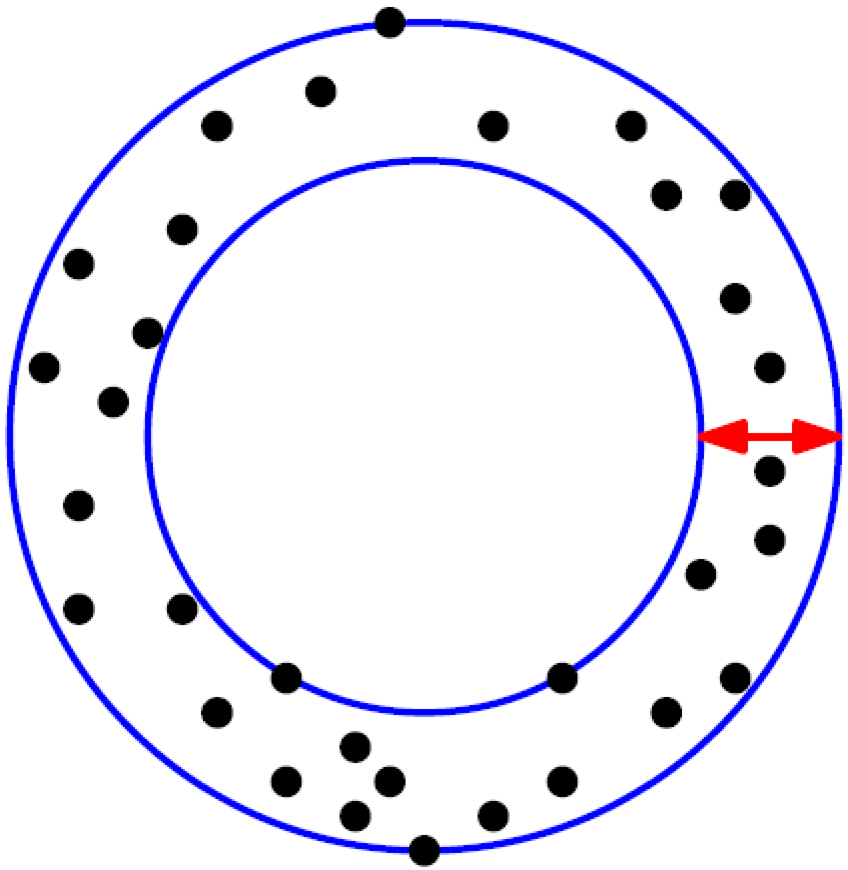


round



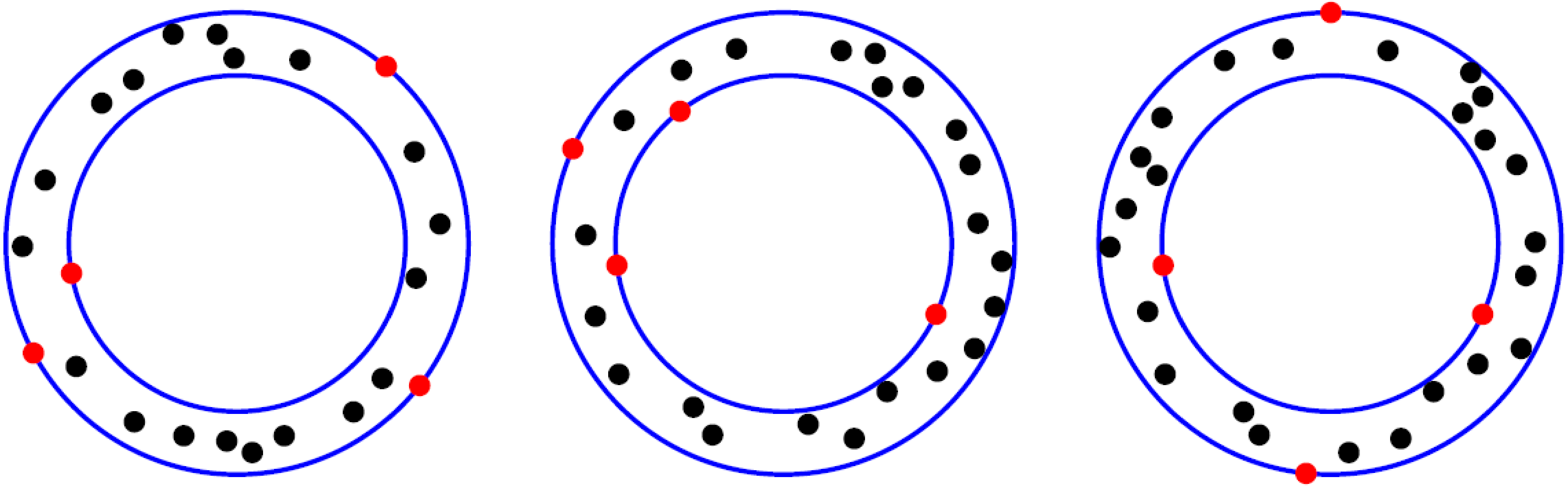
not so round

# Roundness – Metric: Smallest annulus

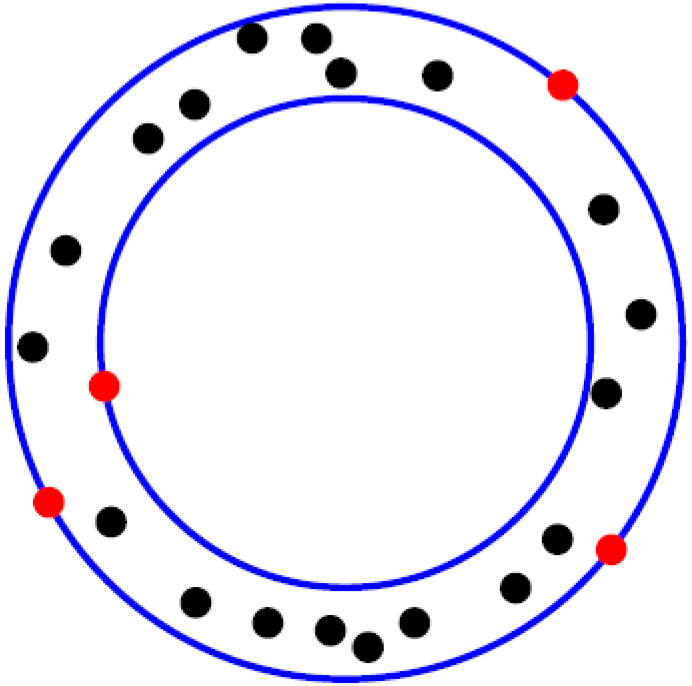


What can we say about the inner- and outer circles of a smallest-width annulus?

# Roundness – Three kinds of annuli



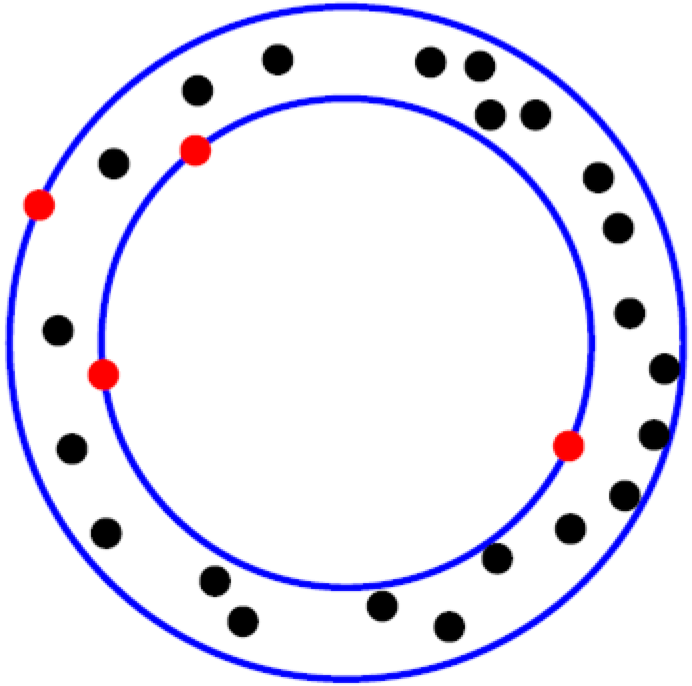
# Roundness – Three kinds of annuli



Outer circle via three points:

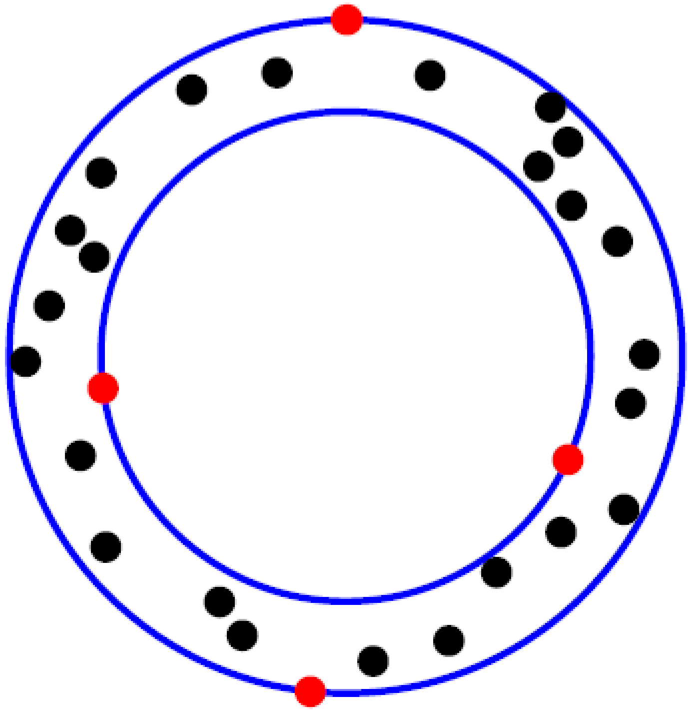
- Center lies on a vertex of the farthest point Voronoi diagram
- Inner circle is defined by closest point to this vertex

# Roundness – Three kinds of annuli



Inner circle via three points:
- Center lies on a vertex of the first-order Voronoi diagram
- Outer circle is defined by farthest point from this vertex

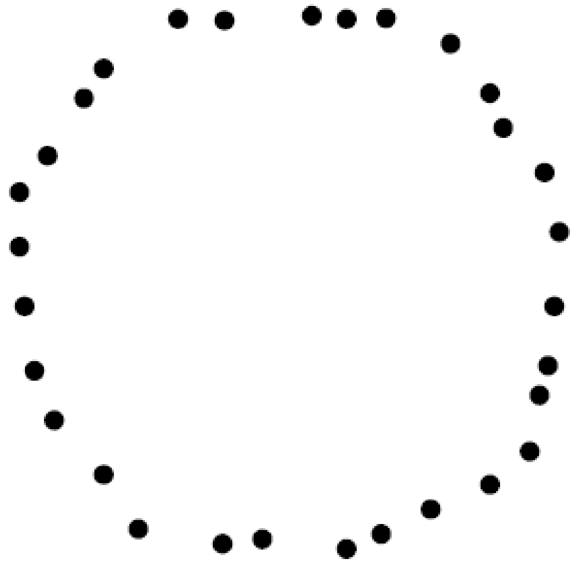
# Roundness – Three kinds of annuli



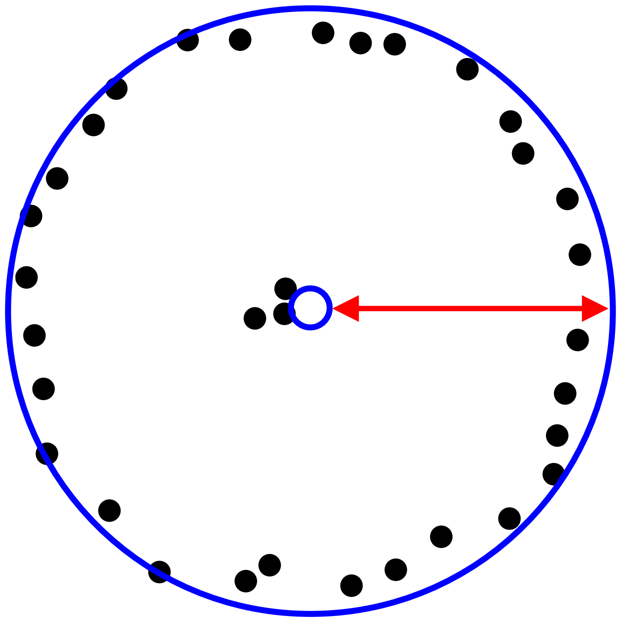
Each circle via two points:

- Center lies on the intersection of an edge of the first-order Voronoi diagram and the farthest point diagram

# Notes from last week – What about outliers?



round



not round
at all...?

# Notes from last week – What about outliers?

Minimum-Width Annulus with Outliers:
Circular, Square, and Rectangular Cases*

Hee-Kap Ahn† Taehoon Ahn† Sang Won Bae‡ Jongmin Choi†
Mincheol Kim† Eunjin Oh§ Chan-Su Shin¶ Sang Duk Yoon‖

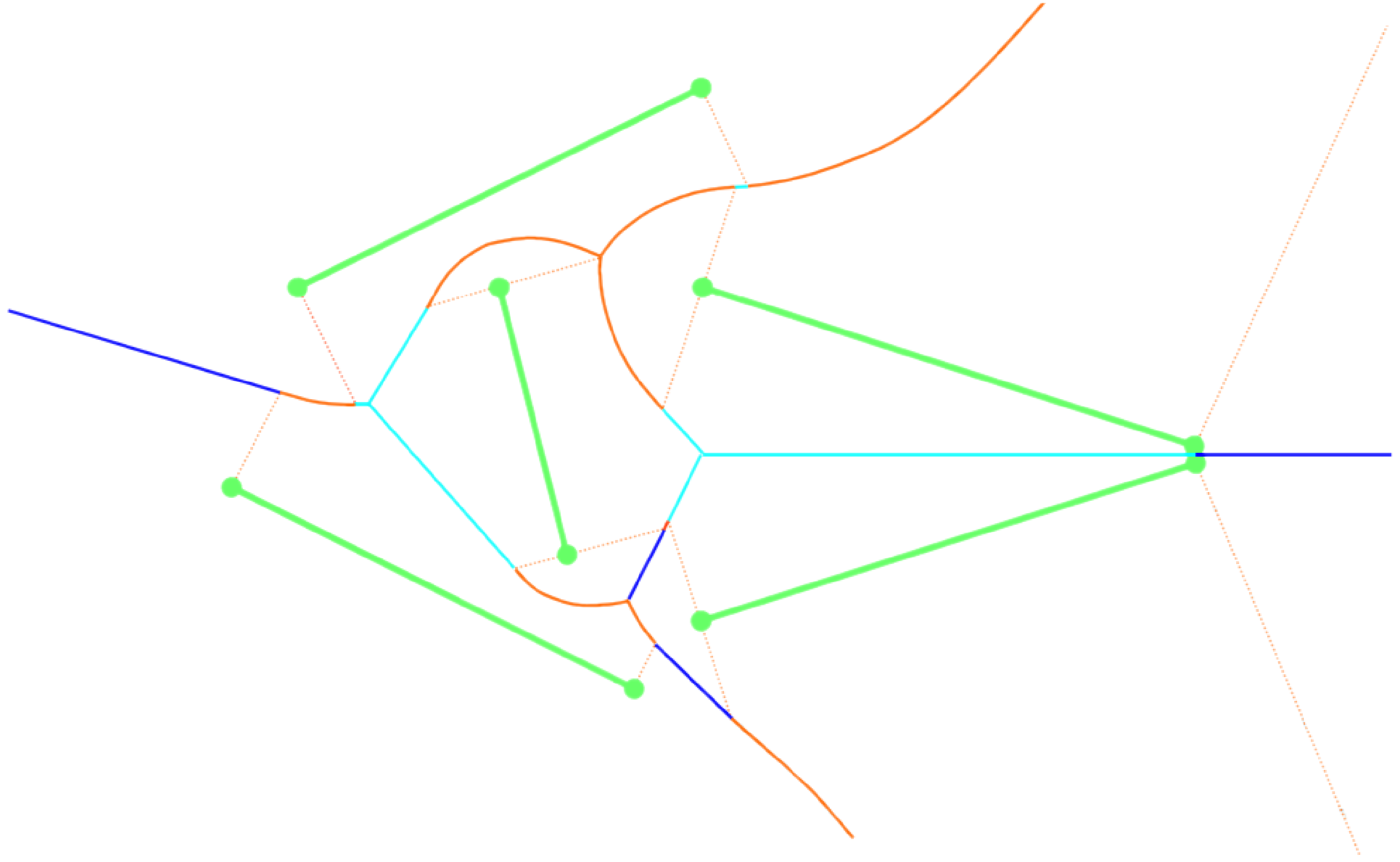
**Abstract**

We study the problem of computing a minimum-width annulus with outliers. Specifically, given a set of $n$ points in the plane and an integer $k$ with $1 \leq k \leq n$, the problem asks to find a minimum-width annulus that contains at least $n - k$ input points. The $k$ excluded points are considered as outliers of the input points. In this paper, we are interested in particular in annuli of three different shapes: circular, square, and rectangular annuli. For the three cases, we present first and improved algorithms to the problem.

*“[...] $n$ points in the plane and an integer $k$ with $1 \leq k \leq n$, the problem asks to find a minimum-width annulus that contains at least $n - k$ input points.”*

http://algo.postech.ac.kr/~heekap/Papers/annulus_outlier.pdf

# More on Voronoi diagrams – Line segments



Source: https://cw.fel.cvut.cz/b181/_media/courses/cg/lectures/07-voronoi-ii.pdf

# Extra – Proof of Uniqueness for Min Disk

*For any point set $P$, the smallest enclosing disk $md(P)$ is unique.*

**Proof.**

Suppose there are two different smallest enclosing disks $D_1 = (c_1, r)$ and $D_2 = (c_2, r)$, with $P \subset D_1$ and $P \subset D_2$.

The disk $D_m$ with center $(c_1 + c_2)/2$ and radius $sqrt(r^2 - a^2)$, where $a$ is half the distance of $c_1$ and $c_2$, also contains $P$.

Contradiction, since the radius of $D_m$ is smaller. □

Technische Universität Braunschweig