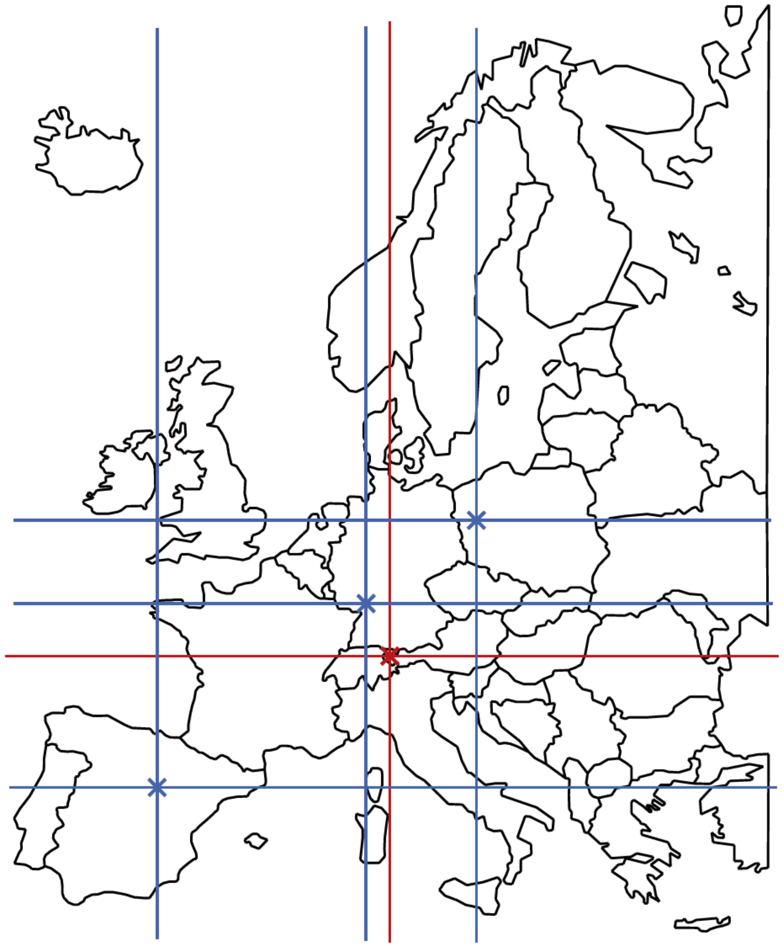
Technische Universität Braunschweig

Computational Geometry – Exercise Meeting #4

February 2nd, 2022

Planar Point Location – Motivation

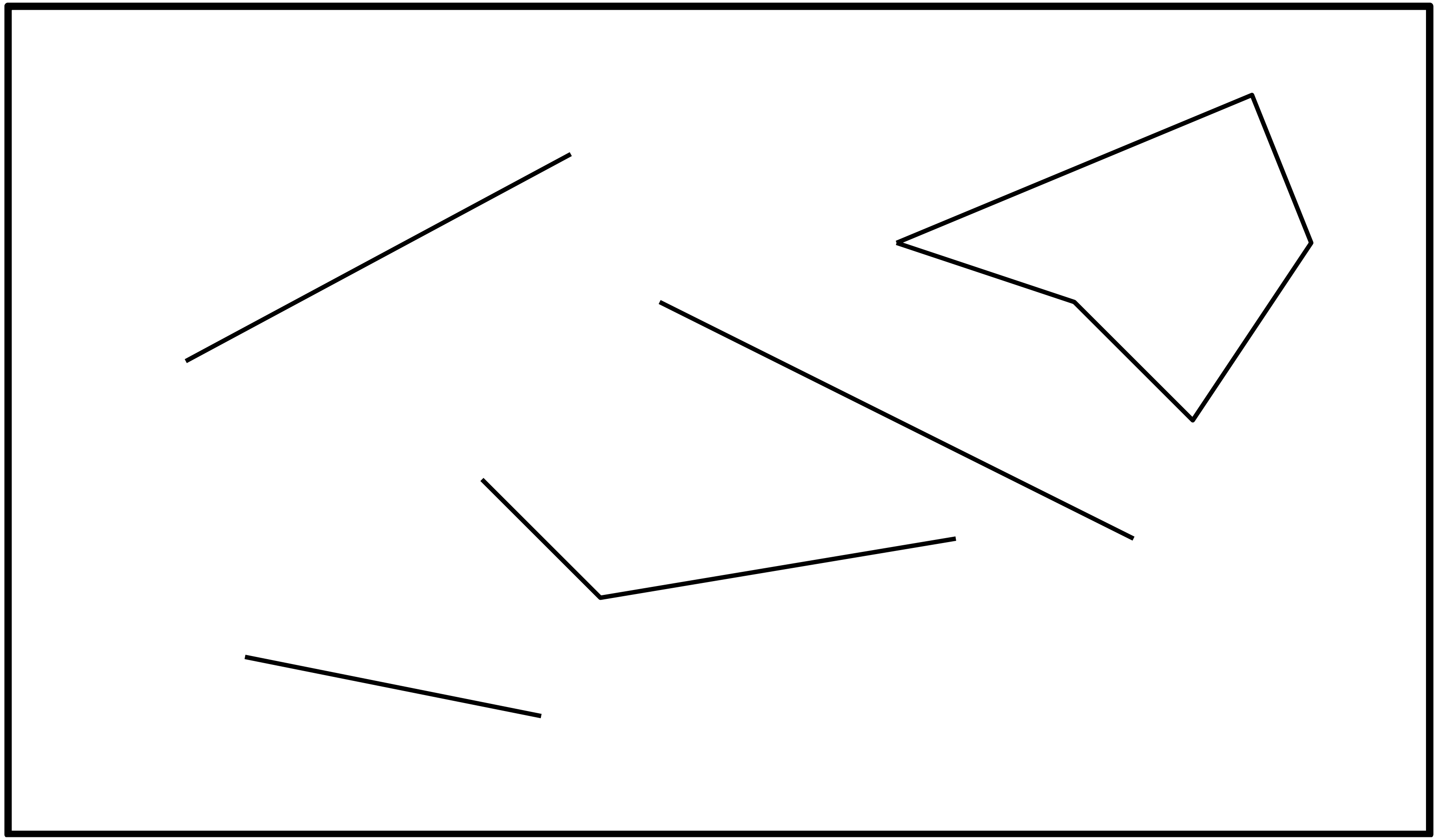


Given a position $p = (p_x, p_y)$ in a map, determine in which country p lies.

more precisely:
Find a data structure for efficiently answering such point location queries.

The map is modeled as a subdivision of the plane into disjoint polygons.

Trapezoidal Maps – Intuition



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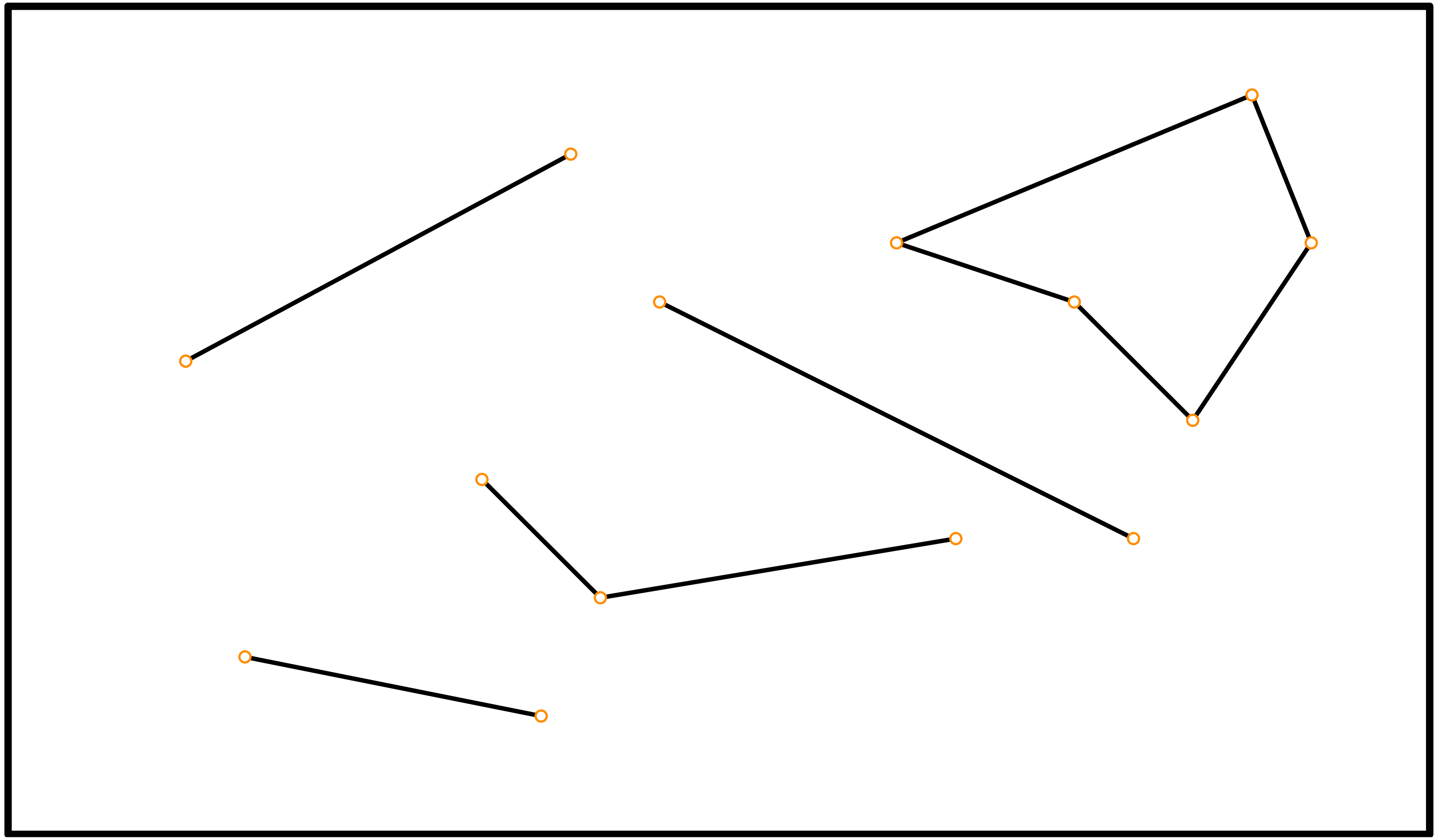

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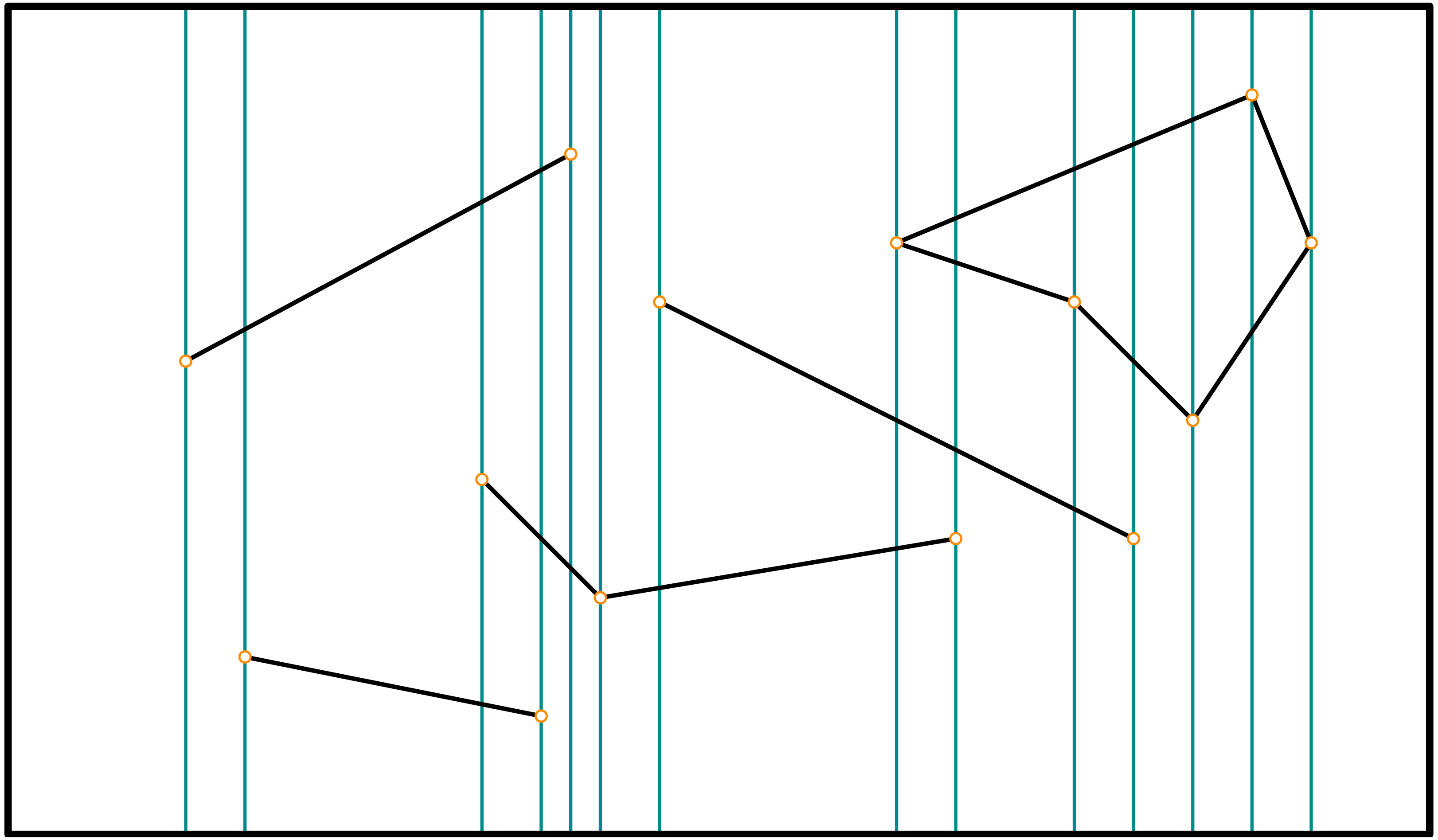

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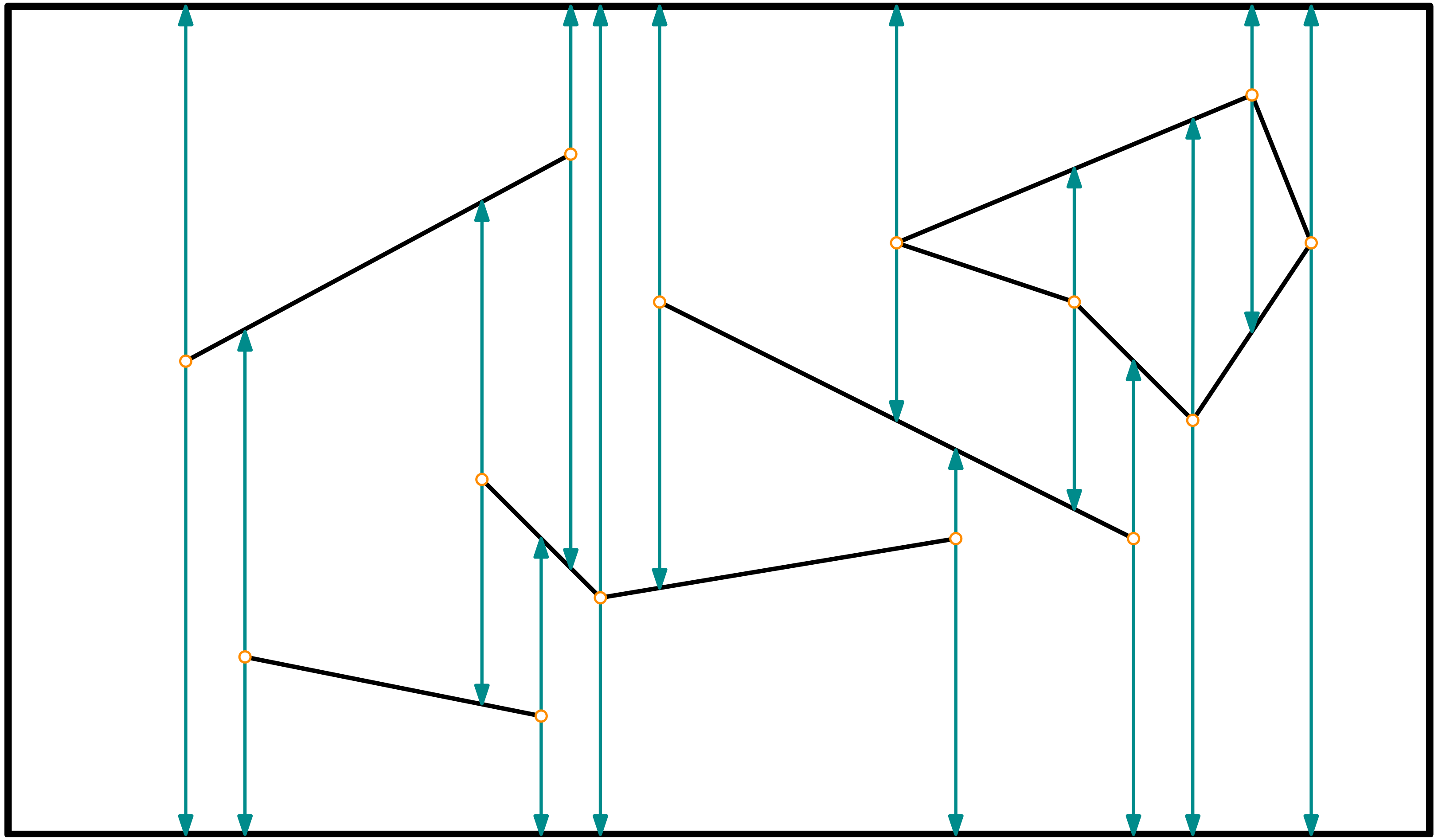

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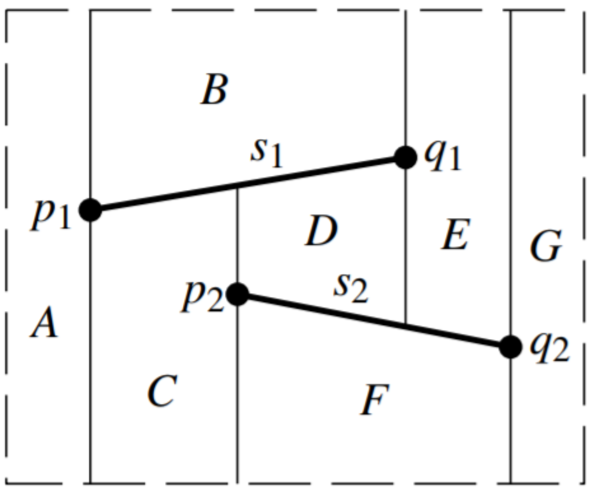
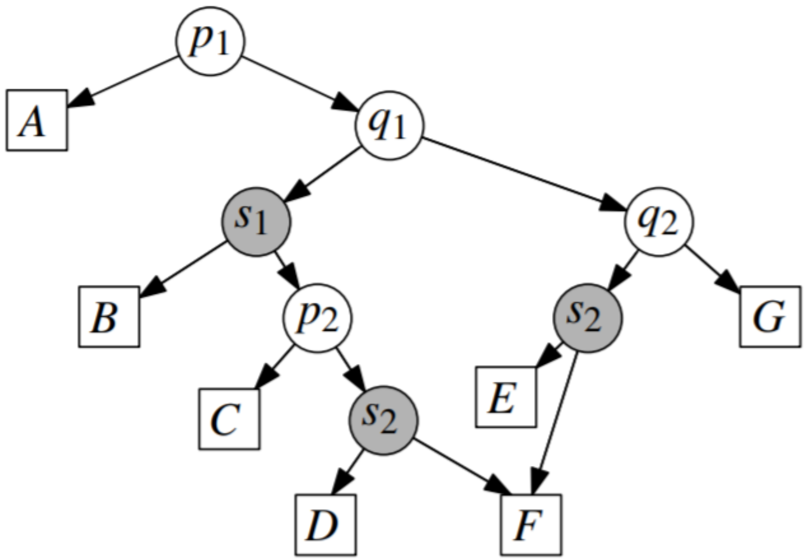
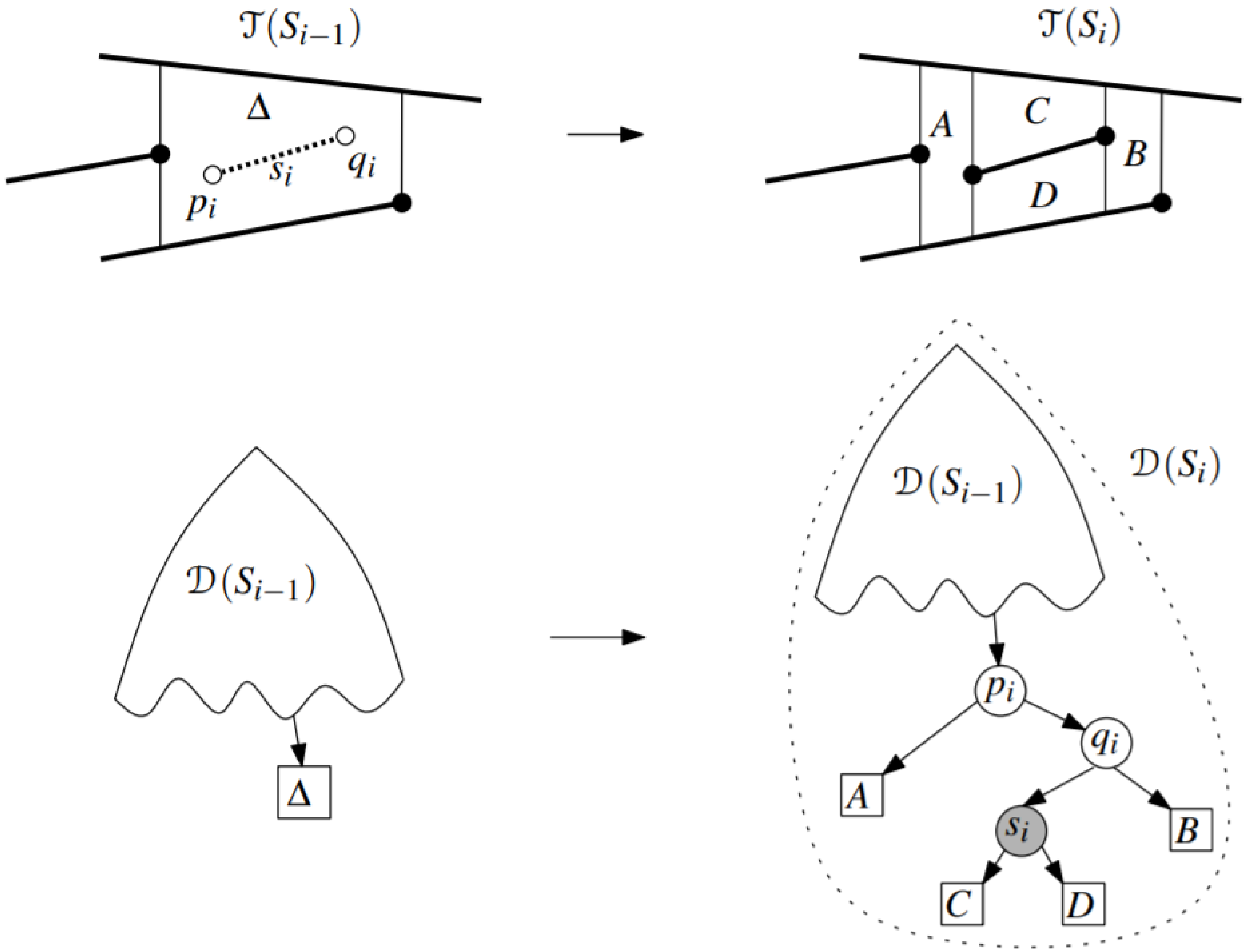


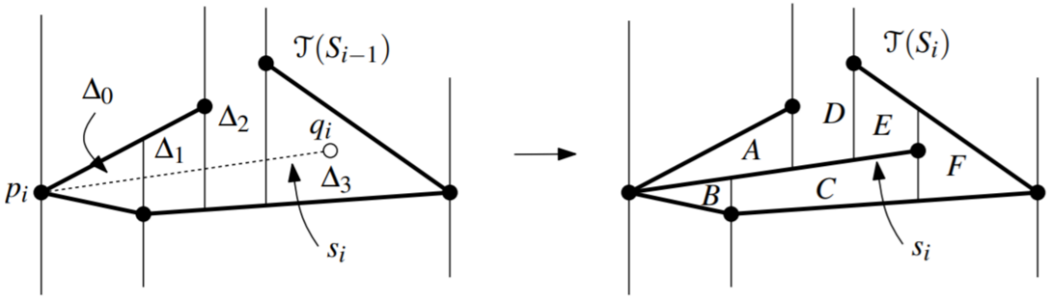


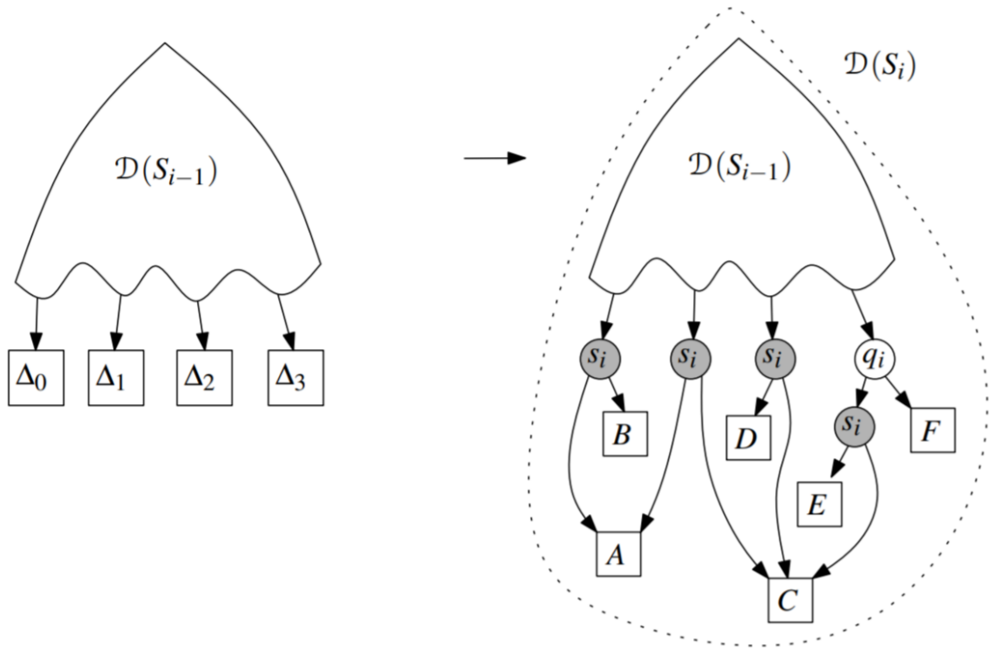
Lemma 6.2 *The trapezoidal map $\mathcal{T}(S)$ of a set S of n line segments in general position contains at most $6n+4$ vertices and at most $3n+1$ trapezoids.*

Trapezoidal Maps – Update complexity

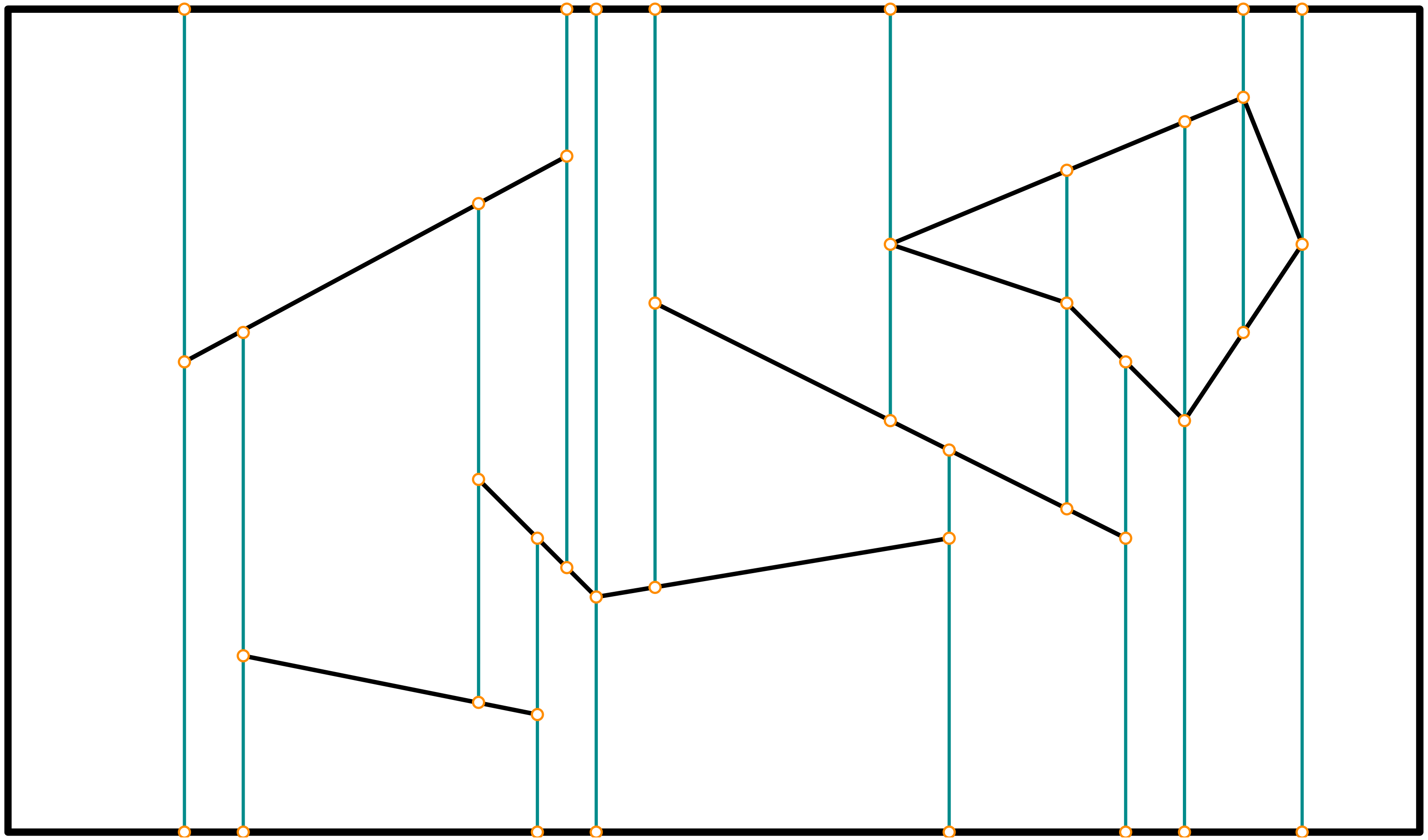


Trapezoidal Maps – Update complexity

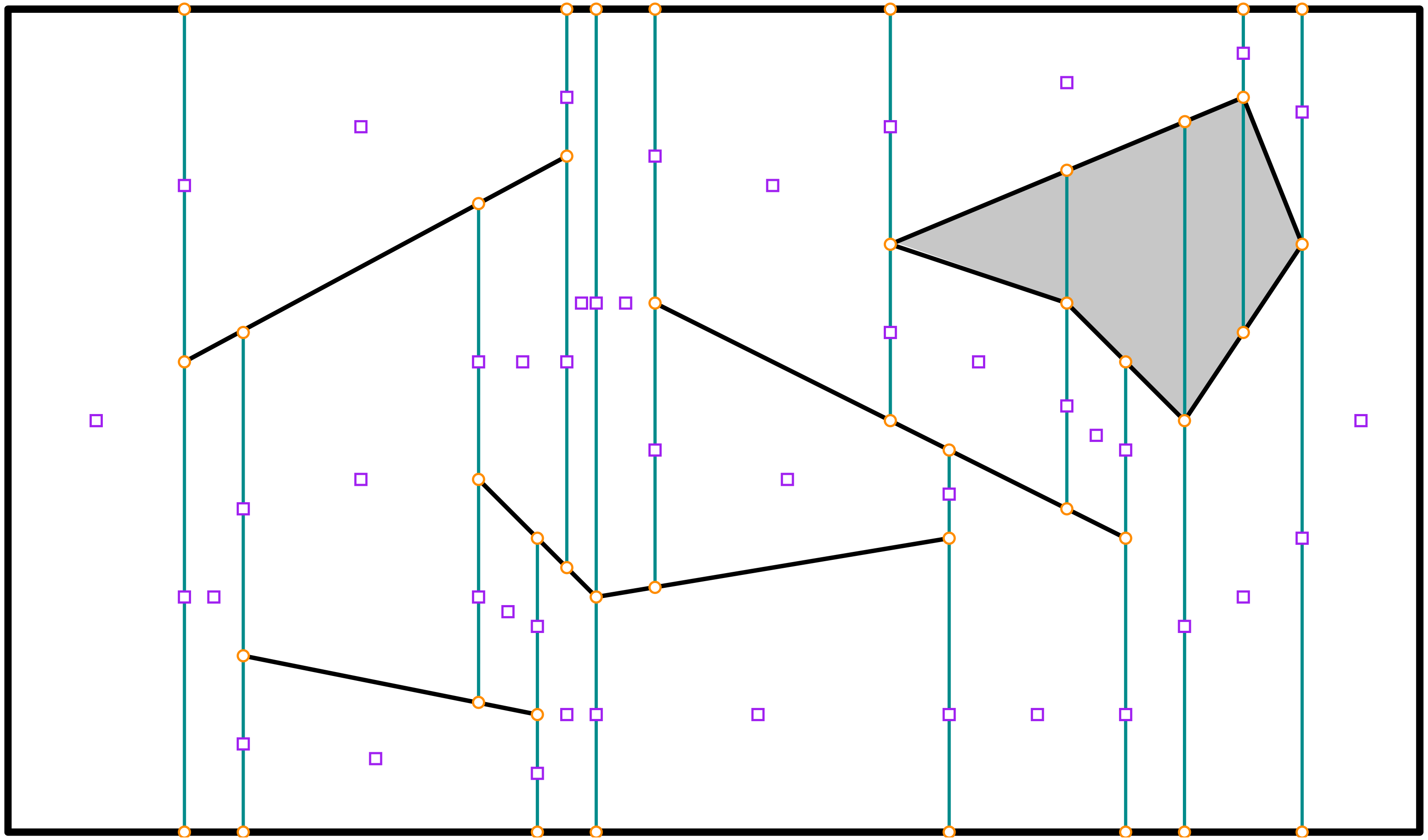




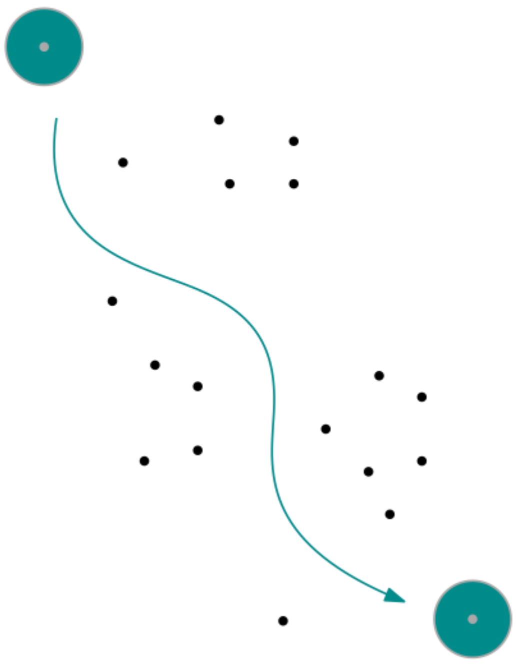
Trapezoidal Maps – Intuition

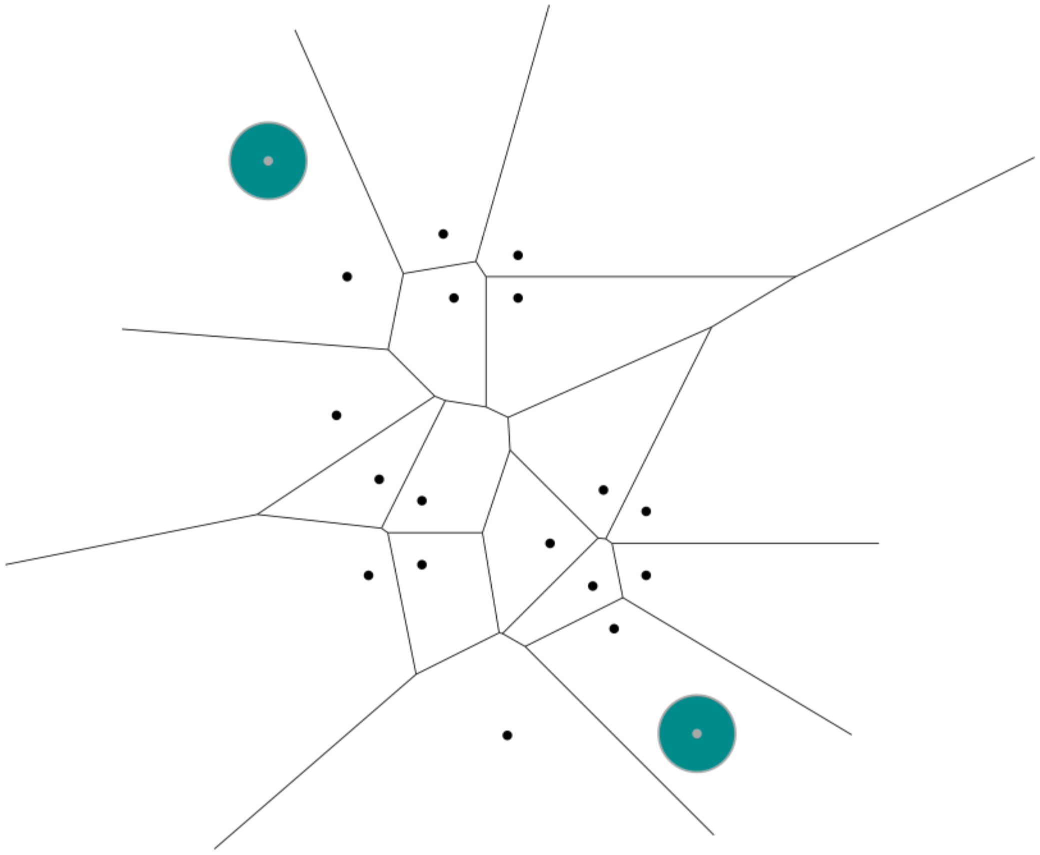


Trapezoidal Maps – Motion Planning



Motion Planning – Circle among points





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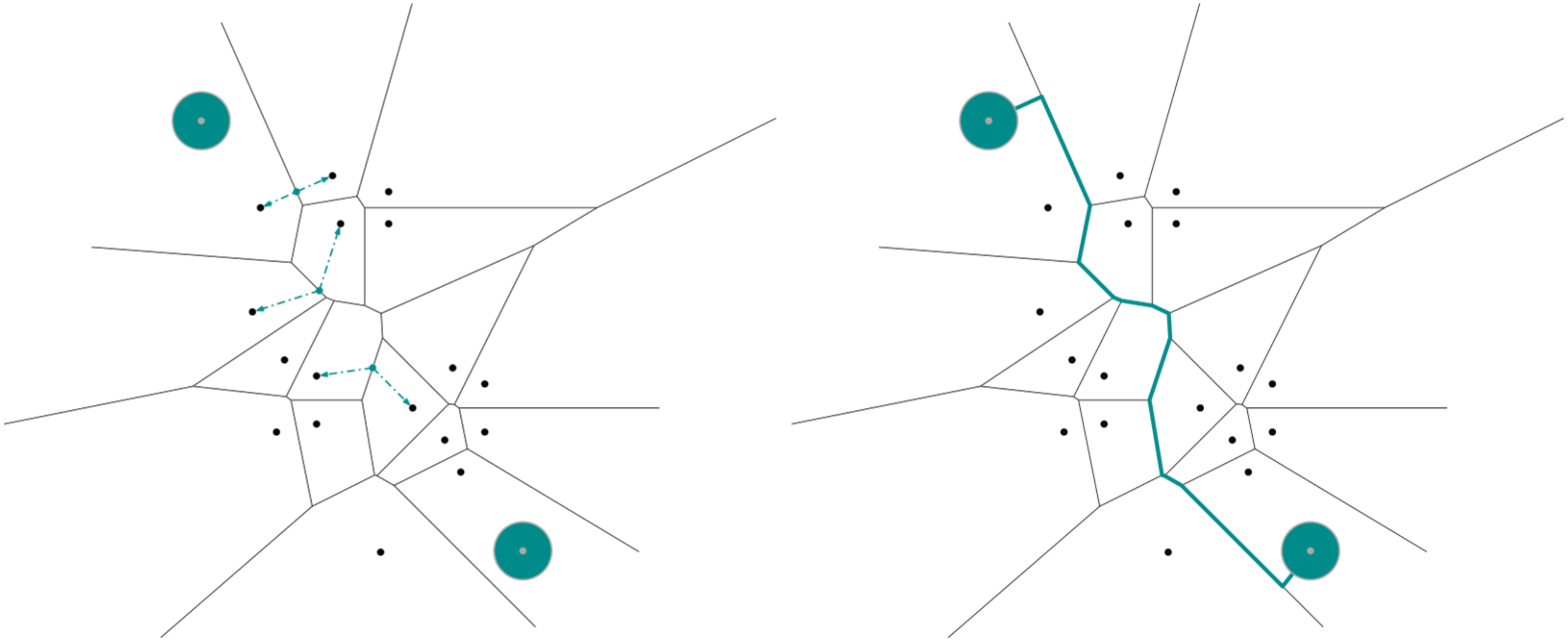
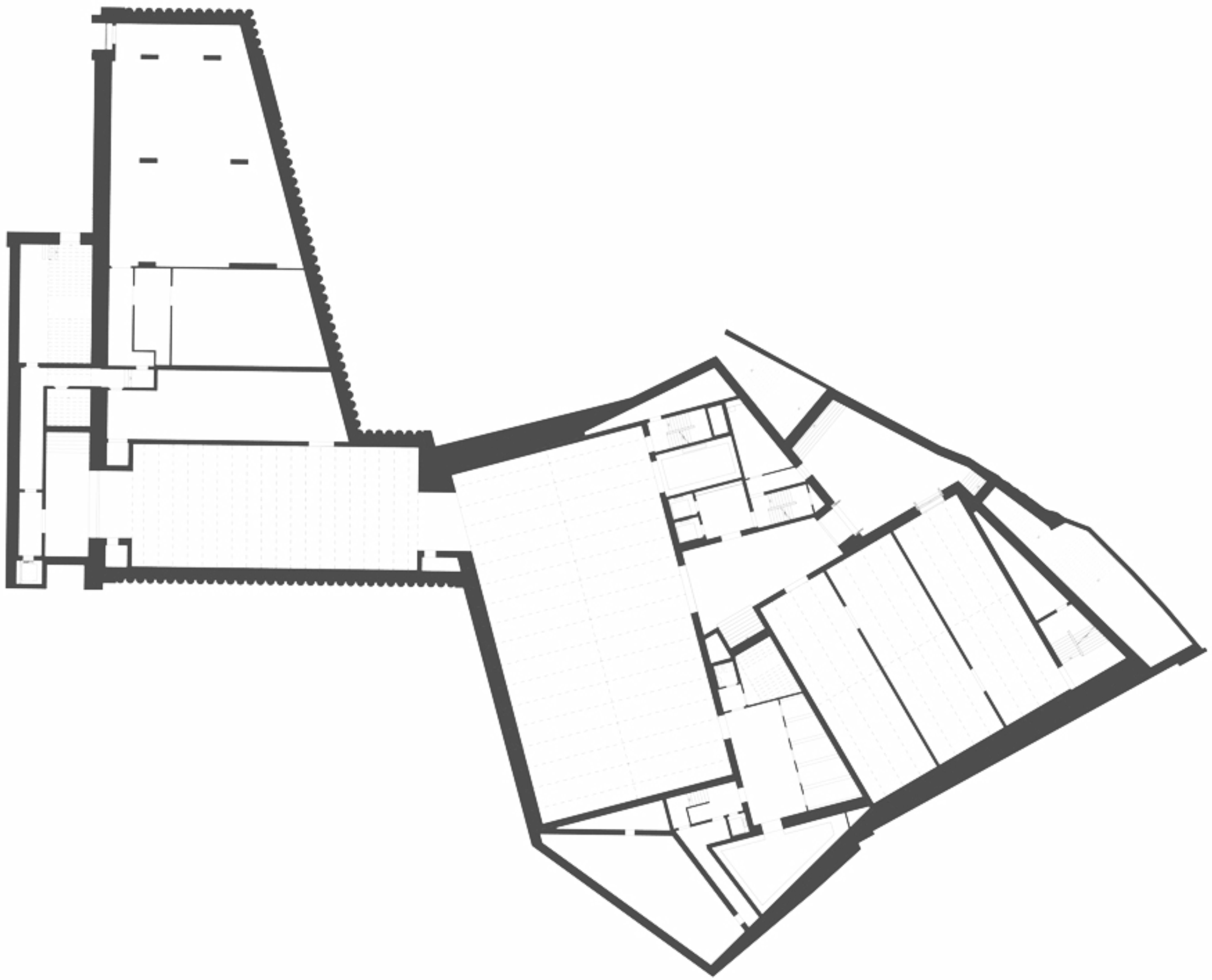
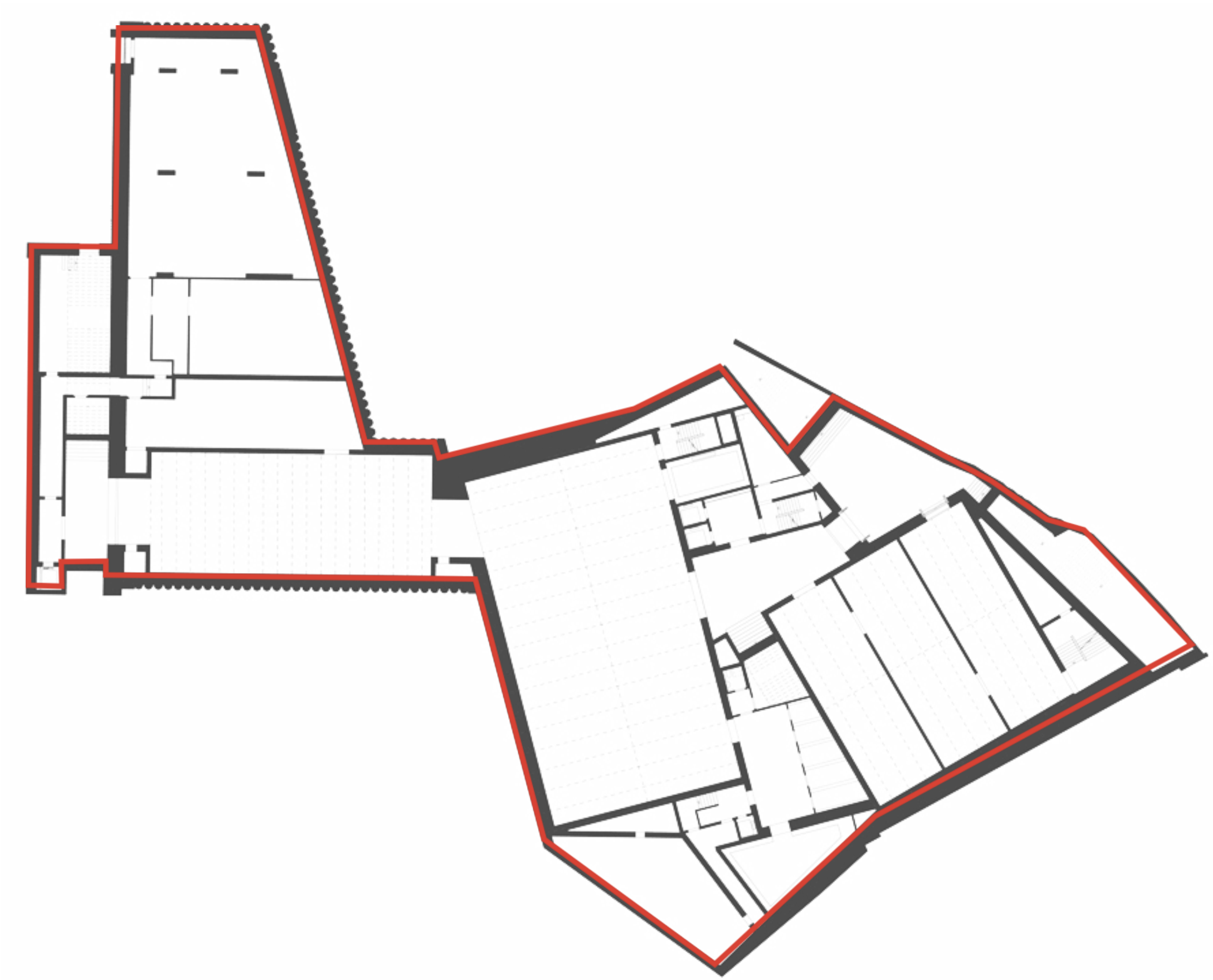
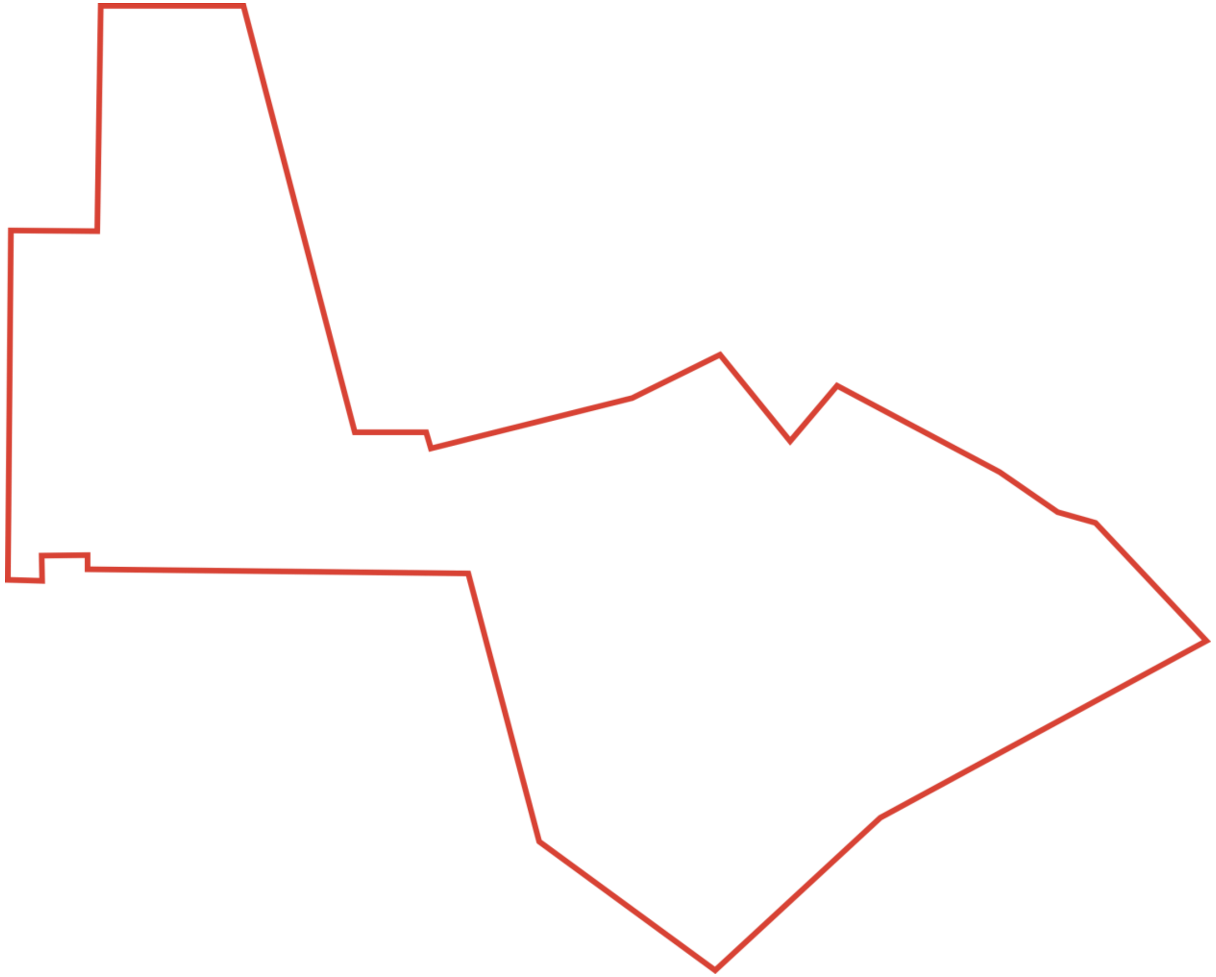
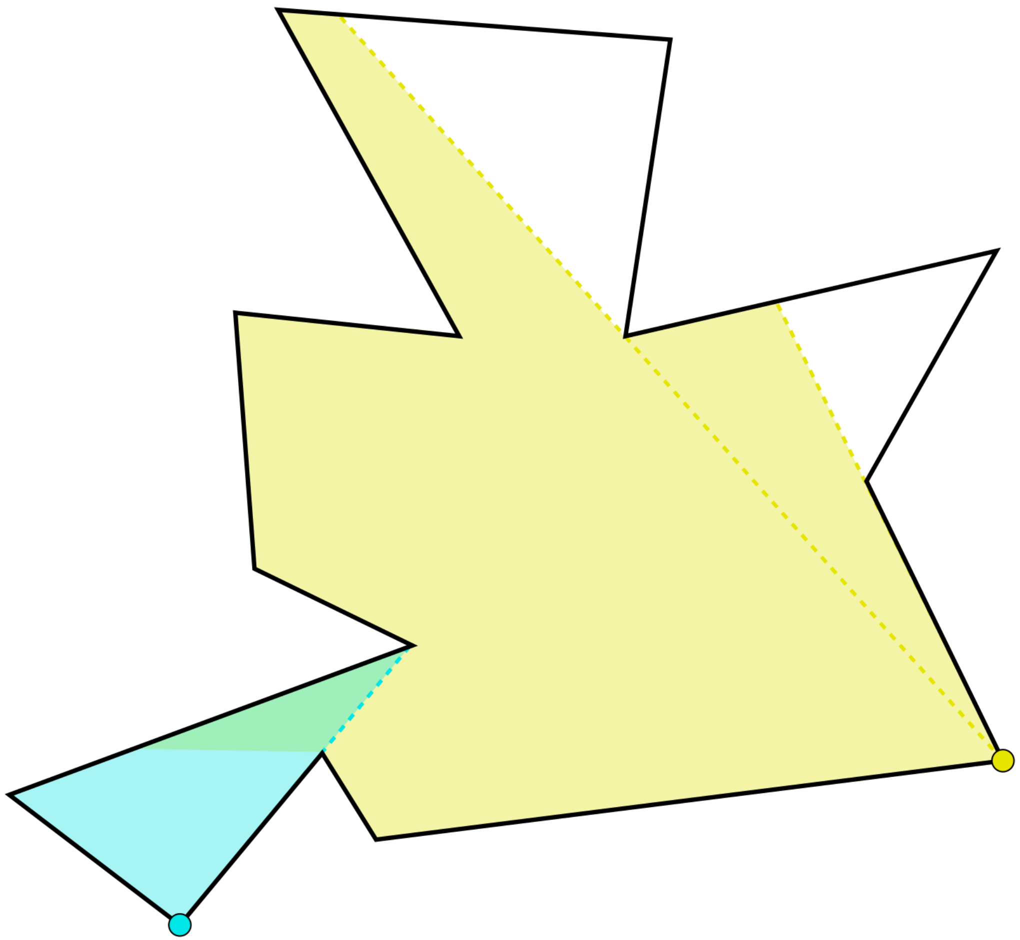








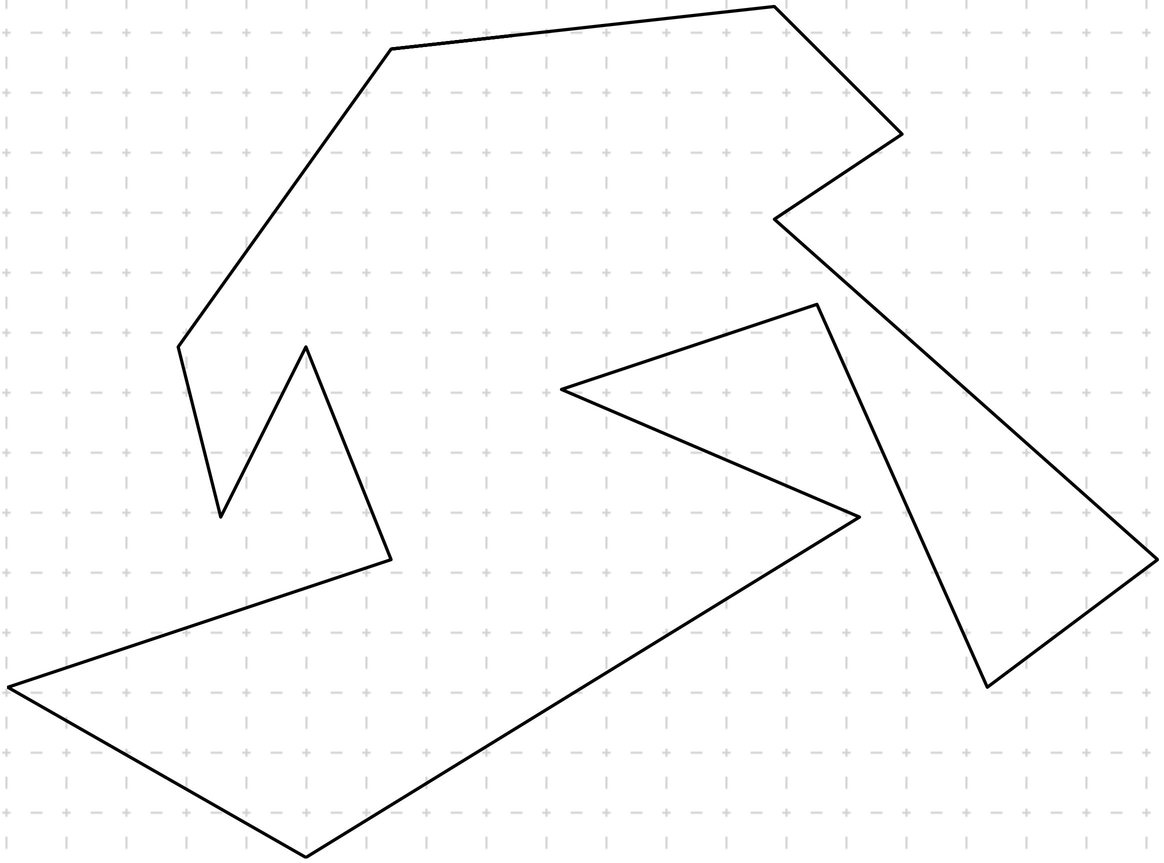
Art Gallery Problem – Simple polygons



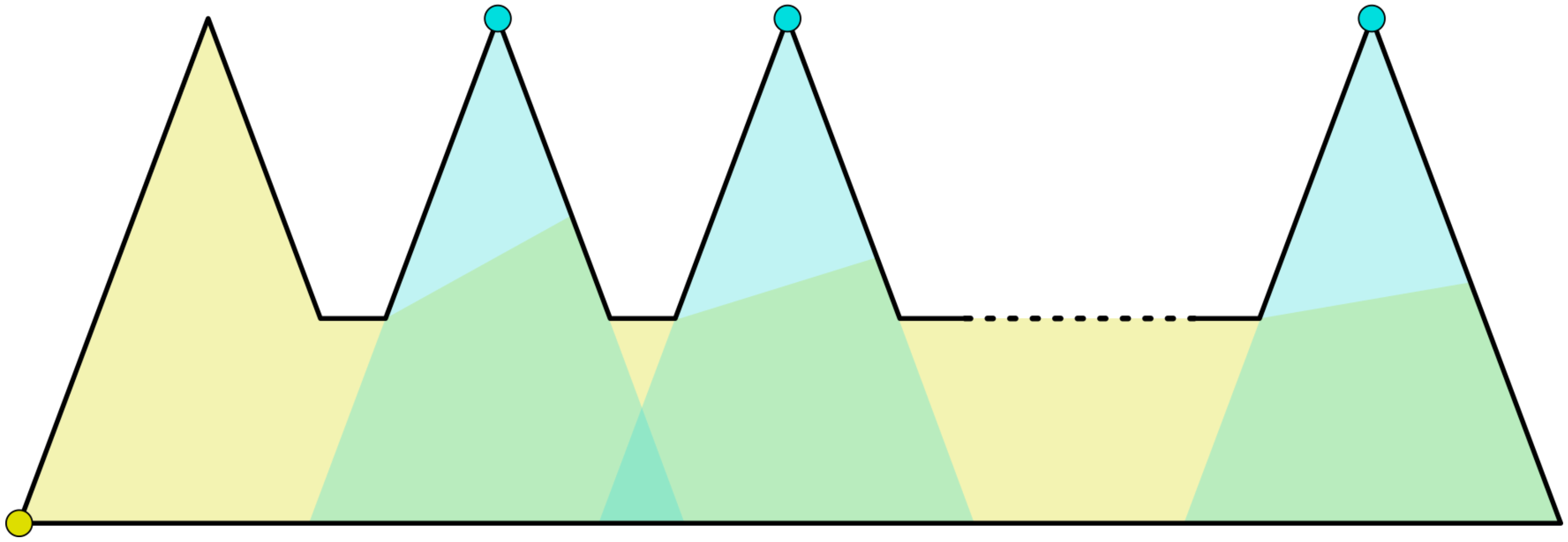
Simple polygon:
- No intersection of edges
- No holes

Guard and guard cover:
- Represented by points
- Placed on vertices of the polygon
- Cover contains all points that are visible from the guard

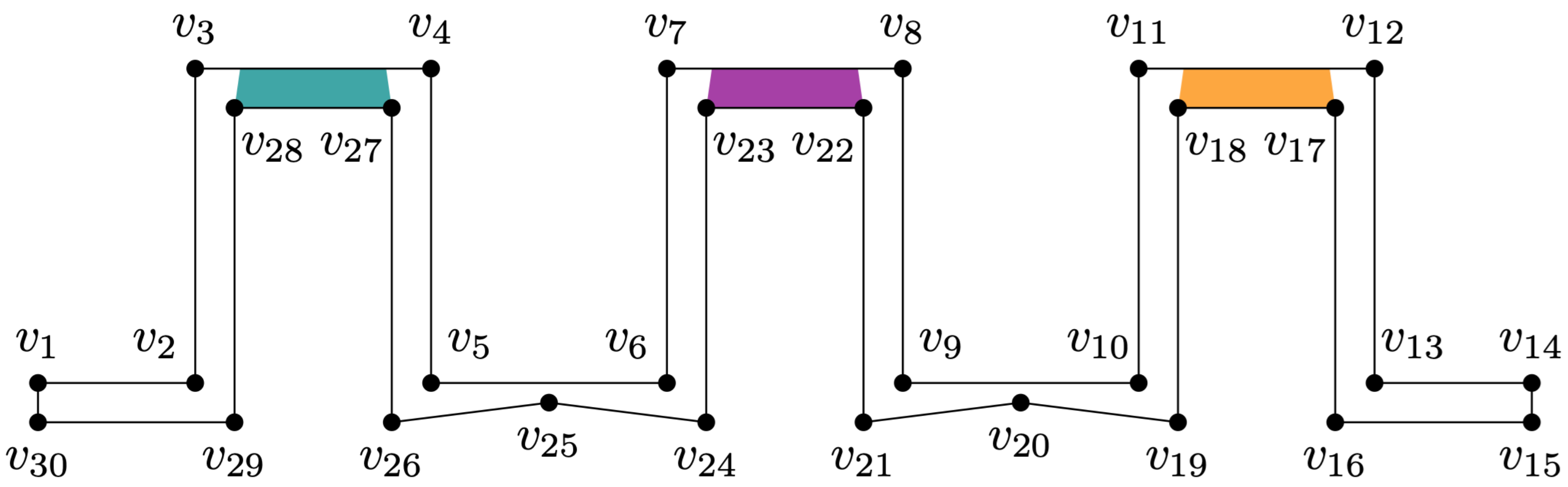
Art Gallery Problem – How to approach simple polygons?



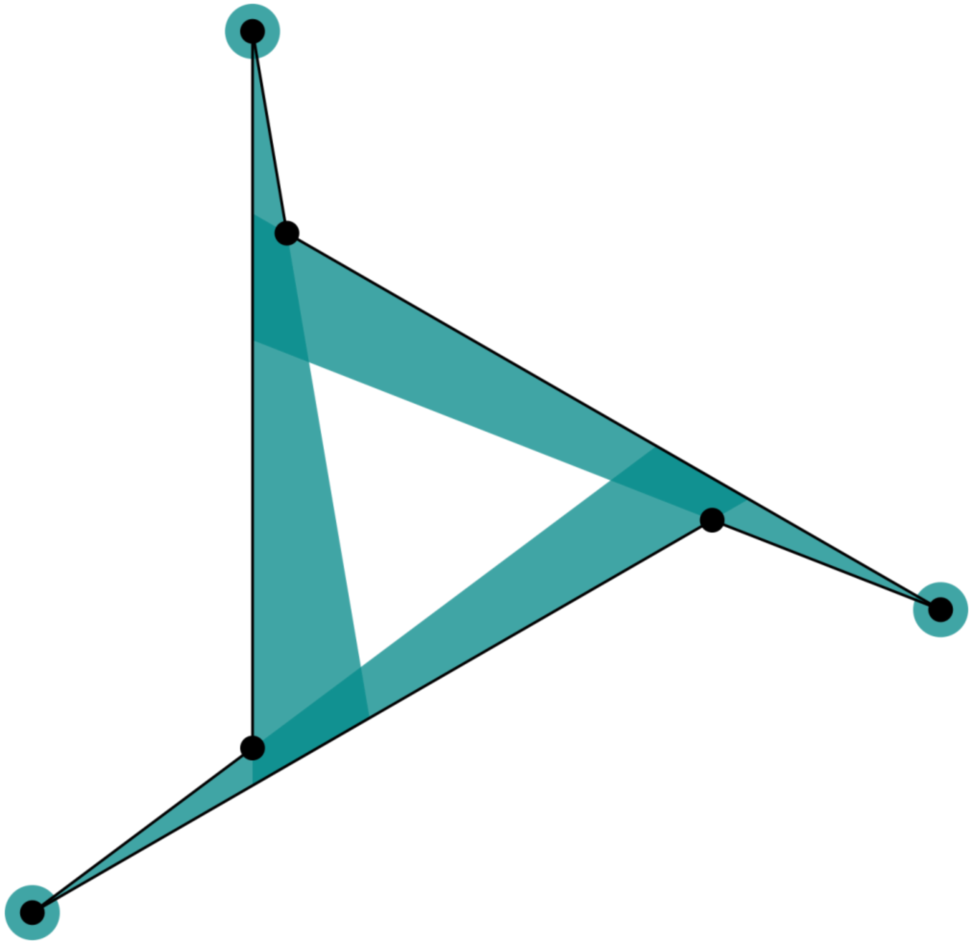
Art Gallery Problem – Lower bound (necessity)



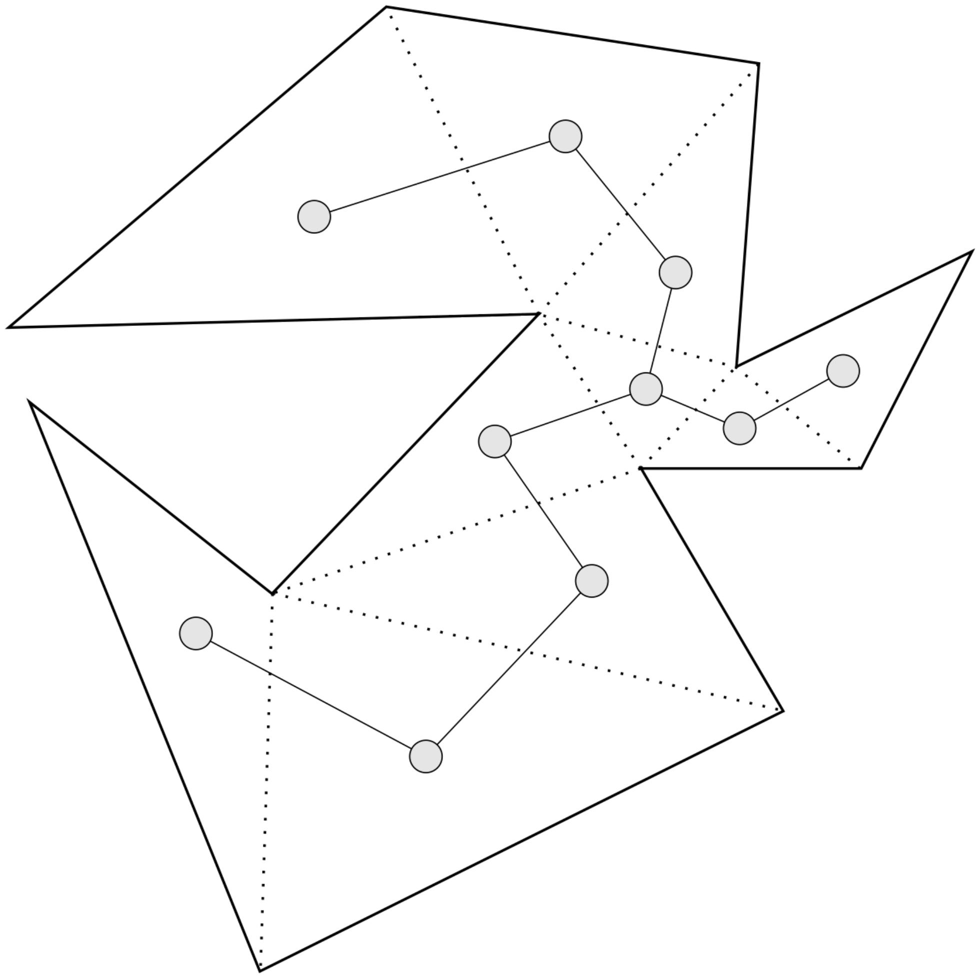
Art Gallery Problem – “Every 3rd”-approach does not work



Art Gallery Problem – Edge cover

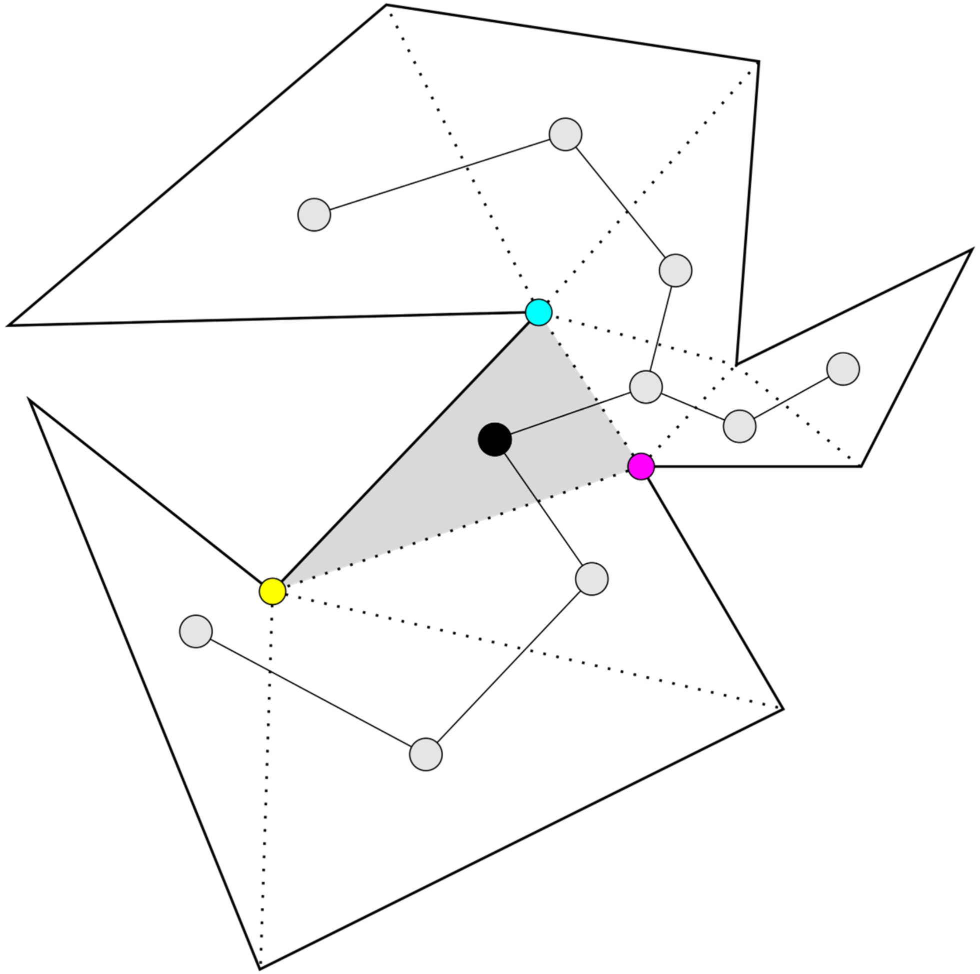


Art Gallery Problem – Upper bound (sufficiency)



- Consider the dual graph of the triangulation
- Select any vertex of the dual graph and color the triangle vertices in three different colors
- Perform BFS on dual graph, color uncolored vertices in remaining color (graph is a tree)

Art Gallery Problem – Upper bound (sufficiency)



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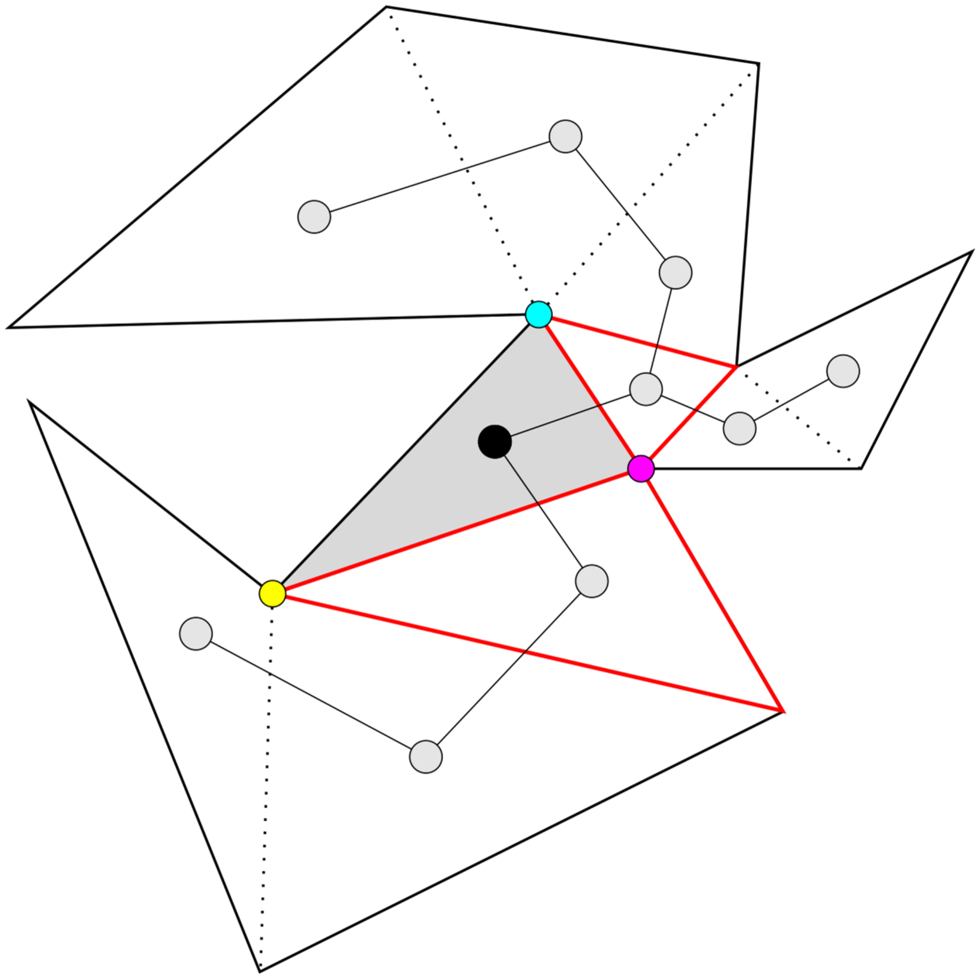

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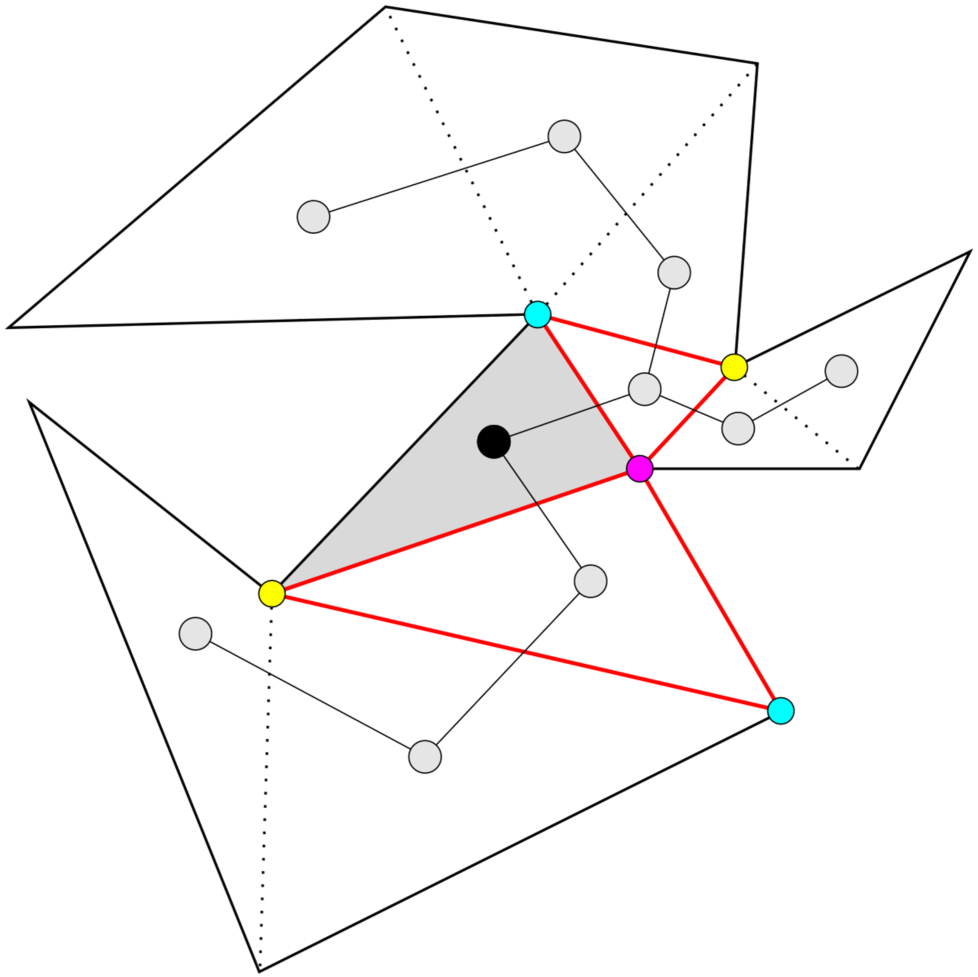

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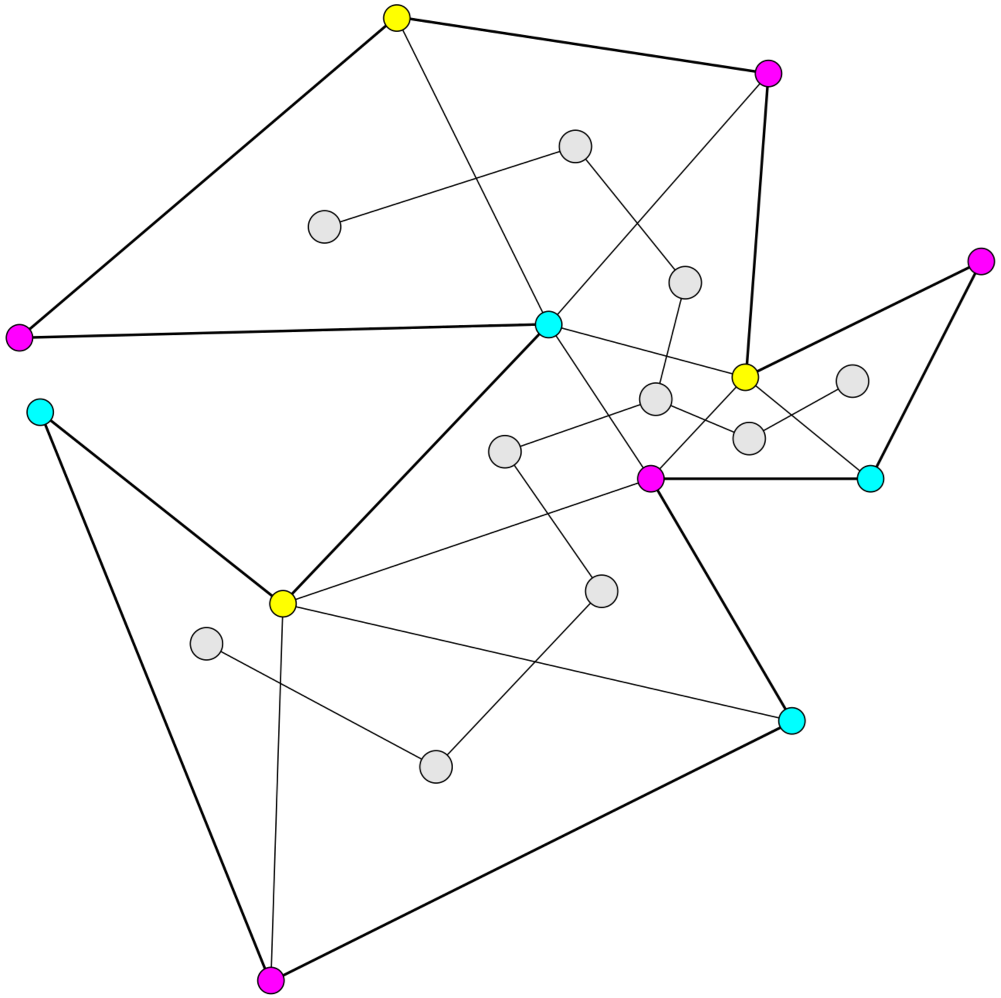

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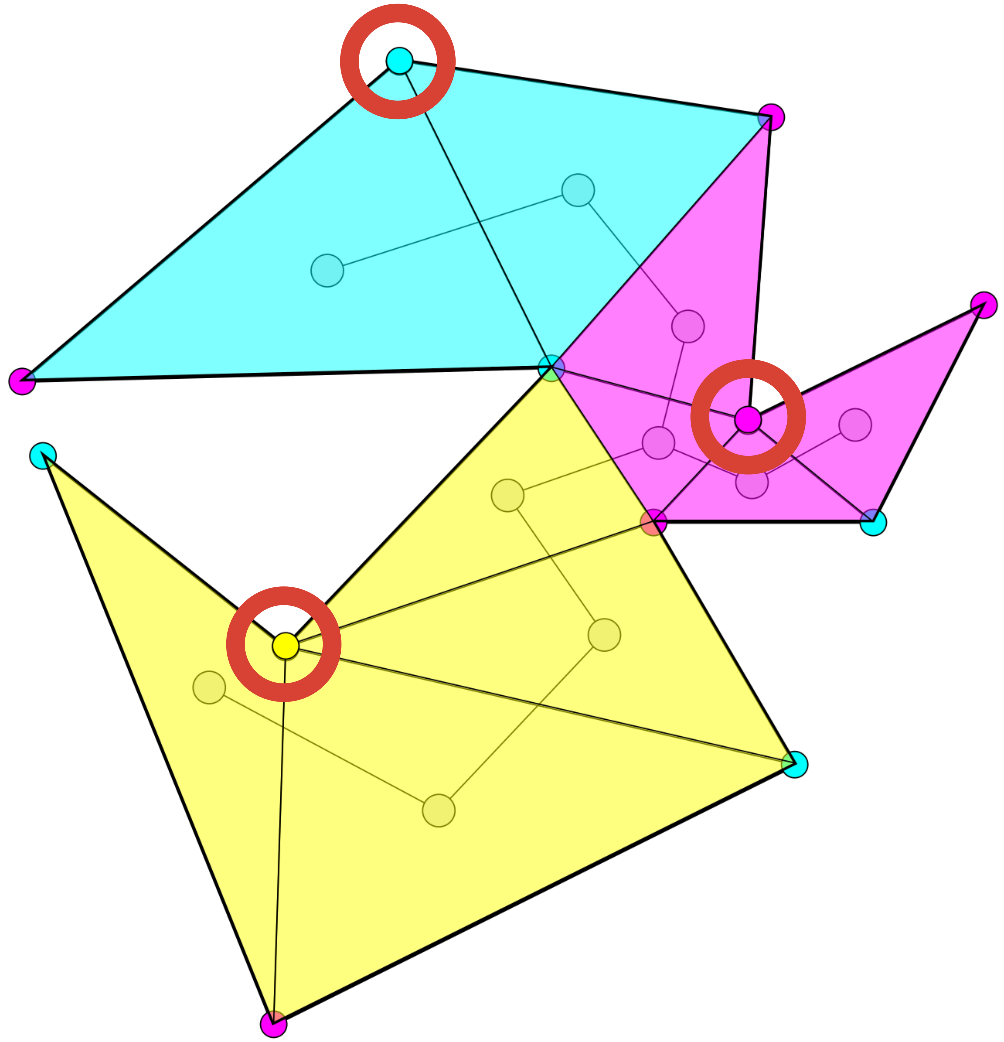

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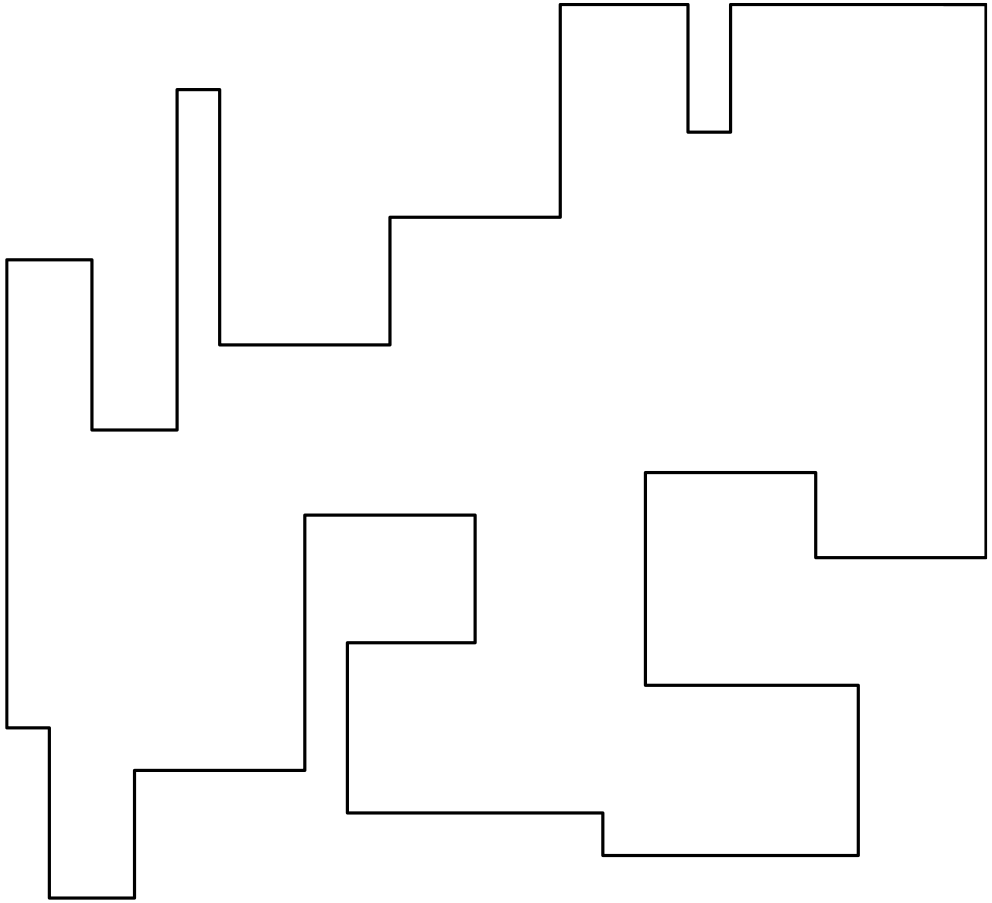
Art Gallery Problem – Orthogonal polygons



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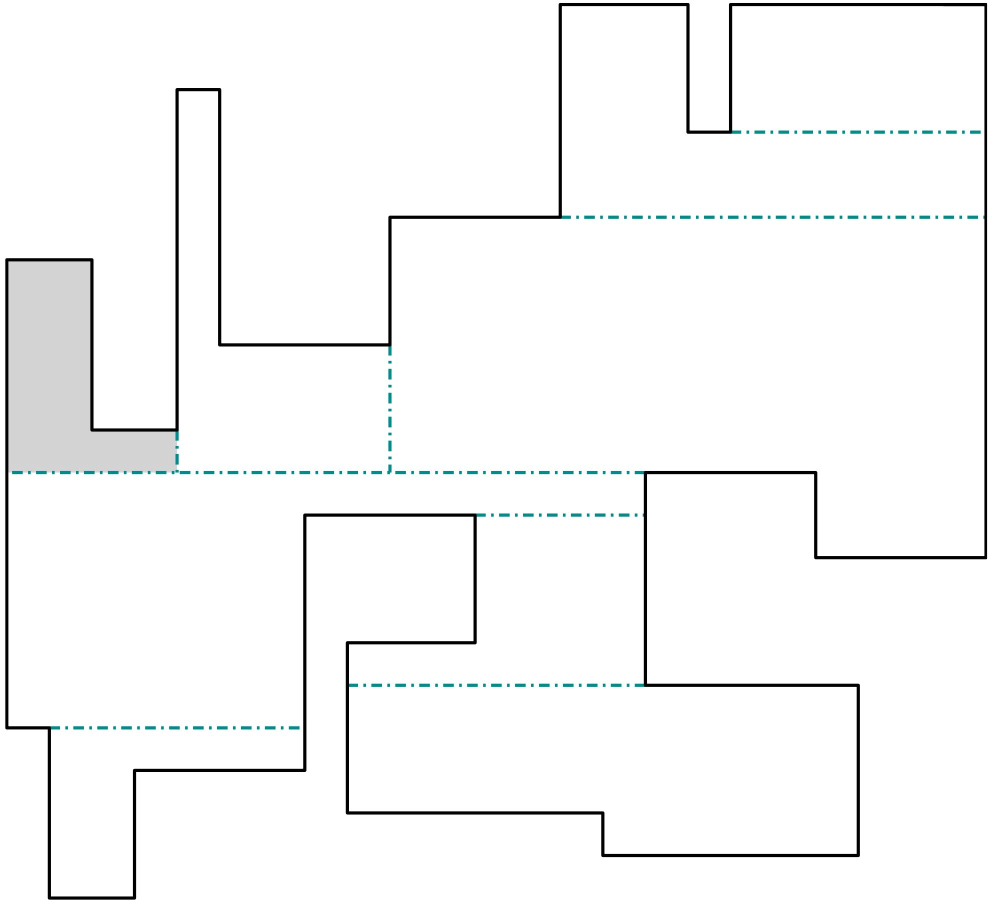

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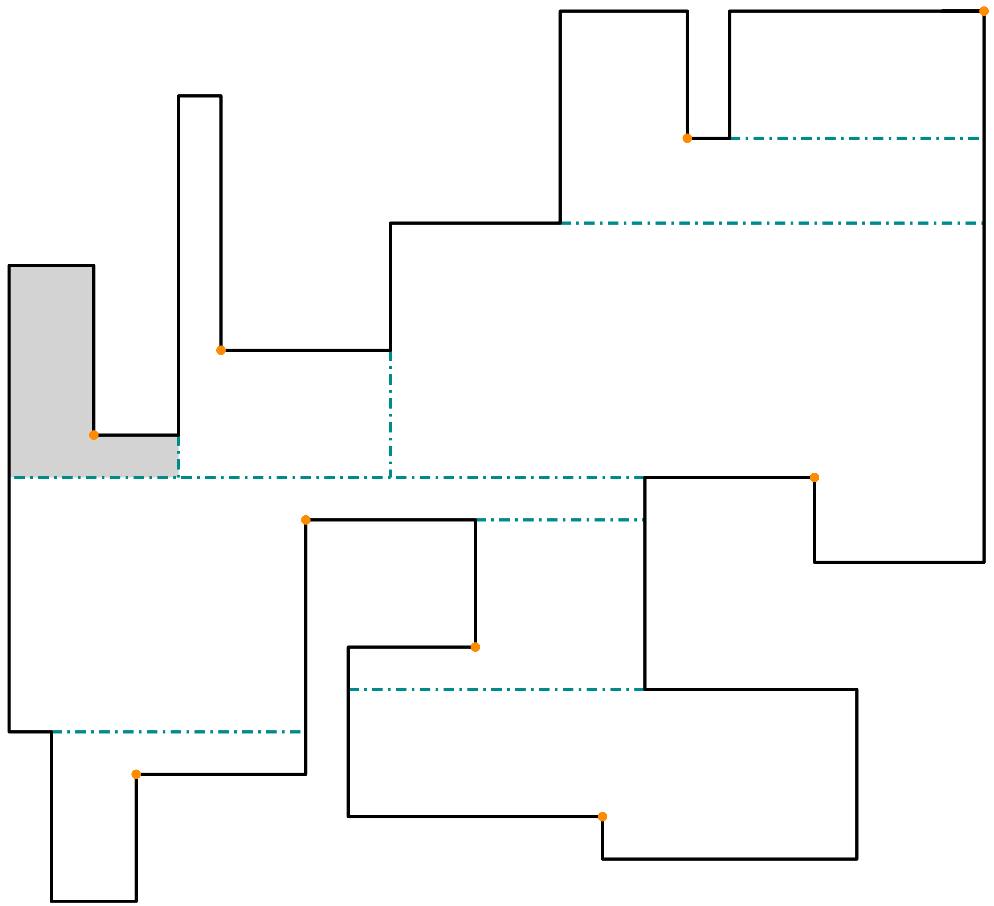
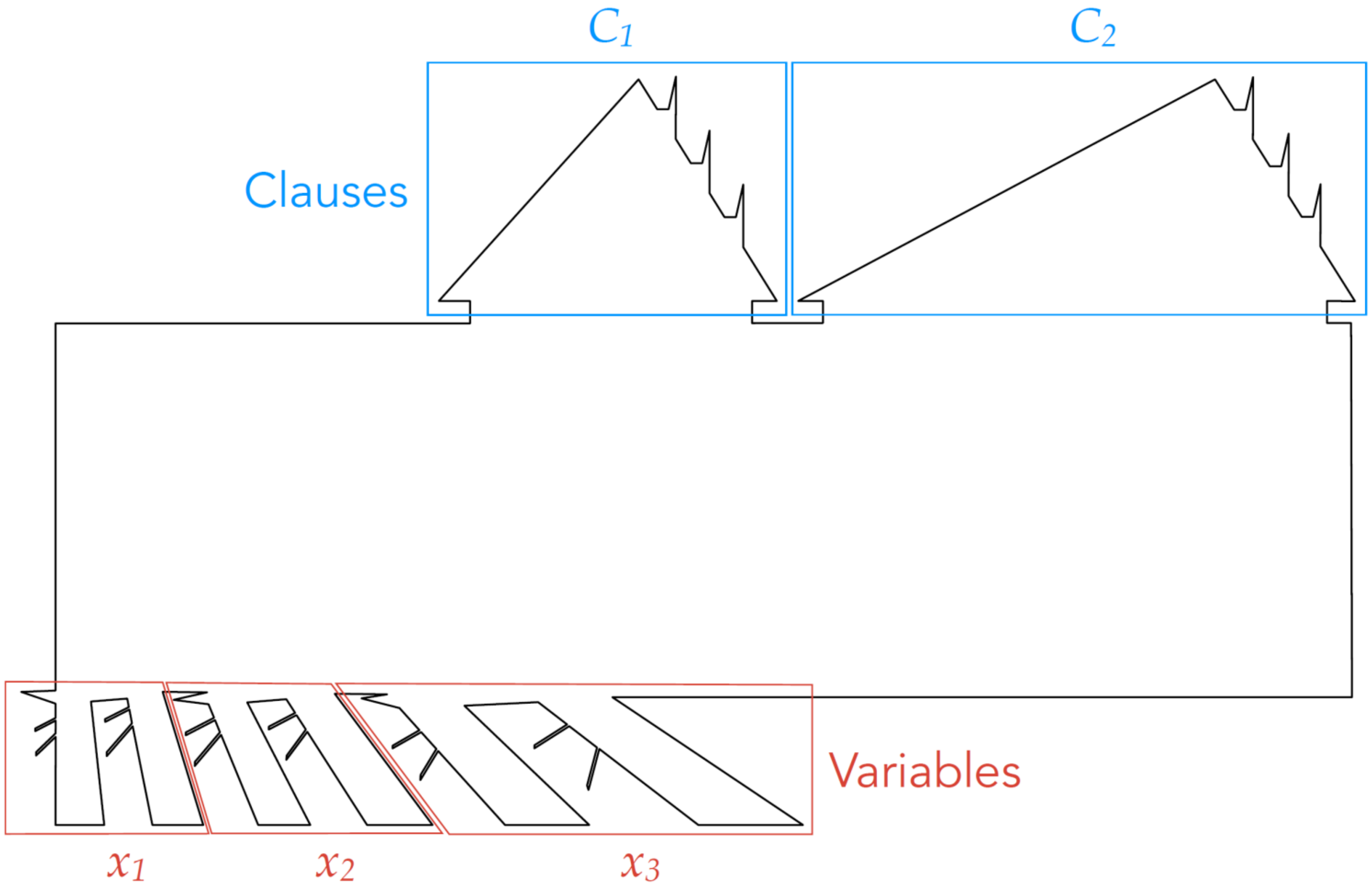


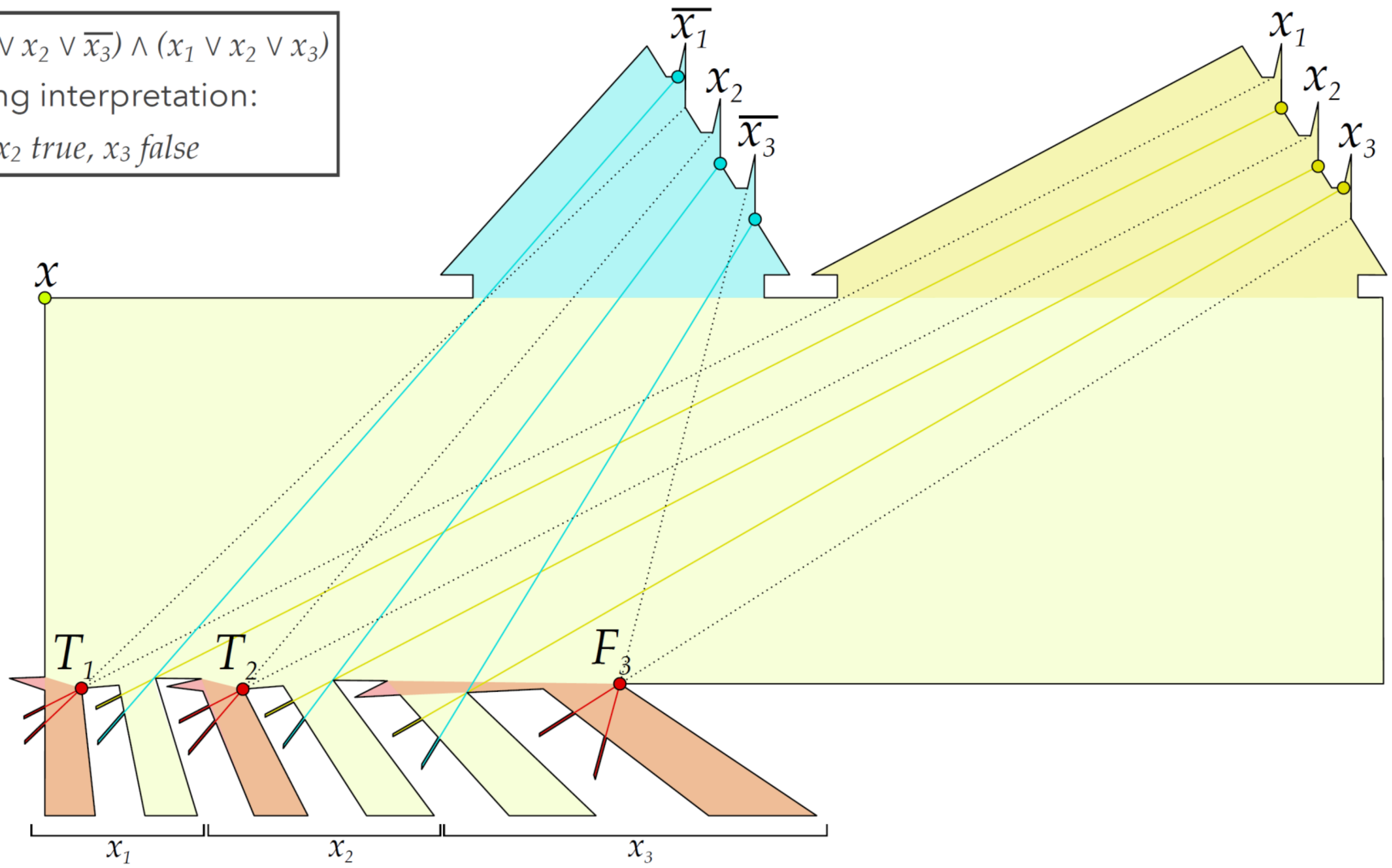
Art Gallery Problem – NP-hardness

Example. $F = (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$



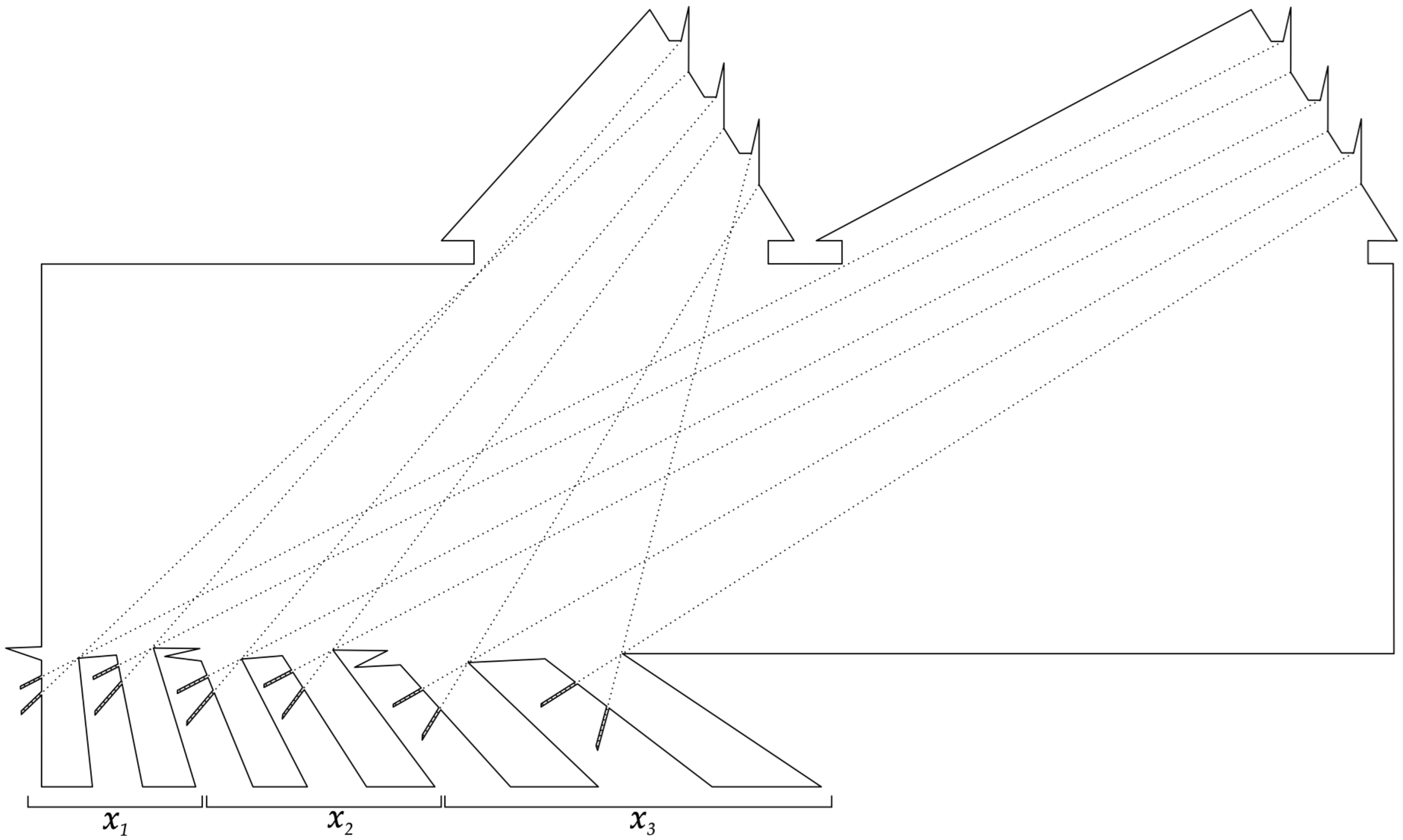
Art Gallery Problem – NP-hardness

$F = (\overline{x_1} \vee x_2 \vee \overline{x_3}) \wedge (x_1 \vee x_2 \vee x_3)$
Satisfying interpretation:
x_1 *true*, x_2 *true*, x_3 *false*



Proof „⇐“: Consider a satisfying assignment of F

Art Gallery Problem – NP-hardness



Proof „⇒“: The polygon is covered with at most $3m + n + 1$ guards

Art Gallery Problem – Irrational guards

Indeed, Sándor Fekete posed at MIT in 2010 and at Dagstuhl in 2011 an open problem, asking whether there are polygons requiring irrational coordinates in an optimal guard set [1, 17]. The question has been raised again by Günter Rote at EuroCG 2011 [26]. It has also been mentioned by Rezende *et al.* [13]: "it remains an open question whether there are polygons given by rational coordinates that require optimal guard positions with irrational coordinates". A similar question has been raised by Friedrichs *et al.* [19]: "[...] it is a long-standing open problem for the more general Art Gallery Problem (AGP): For the AGP it is not known whether the coordinates of an optimal guard cover can be represented with a polynomial number of bits".

Our results. We answer the open question of Sándor Fekete, by proving the following main result of our paper. Recall that a polygon $\mathcal{P}$ is called *monotone* if there exists a line l such that every line orthogonal to l intersects $\mathcal{P}$ at most twice.

Theorem 1. *There is a simple monotone polygon $\mathcal{P}$ with integer coordinates of the vertices such that*

(i) $\mathcal{P}$ can be guarded by 3 guards placed at points with irrational coordinates, and

(ii) an optimal guard set of $\mathcal{P}$ with guards at points with rational coordinates has size 4.

Irrational Guards are Sometimes Needed*

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Abstract

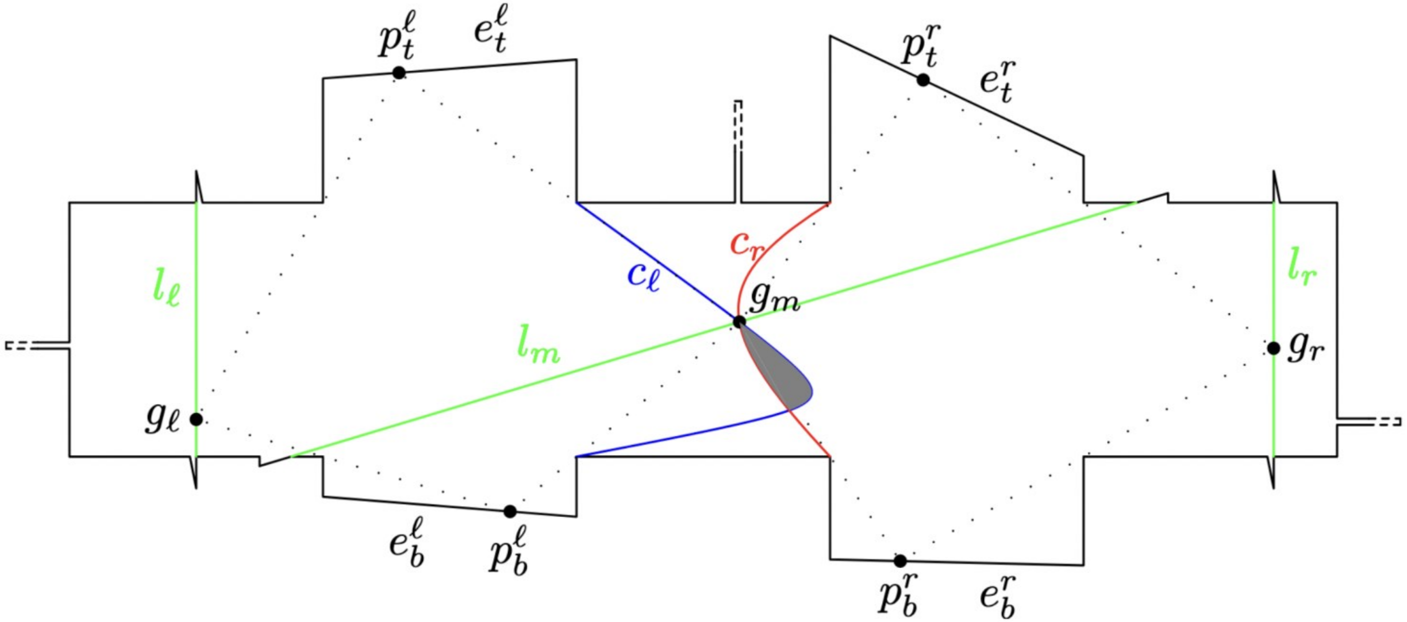
In this paper we study the *art gallery problem*, which is one of the fundamental problems in computational geometry. The objective is to place a minimum number of guards inside a simple polygon so that the guards together can see the whole polygon. We say that a guard at position x sees a point y if the line segment xy is contained in the polygon.

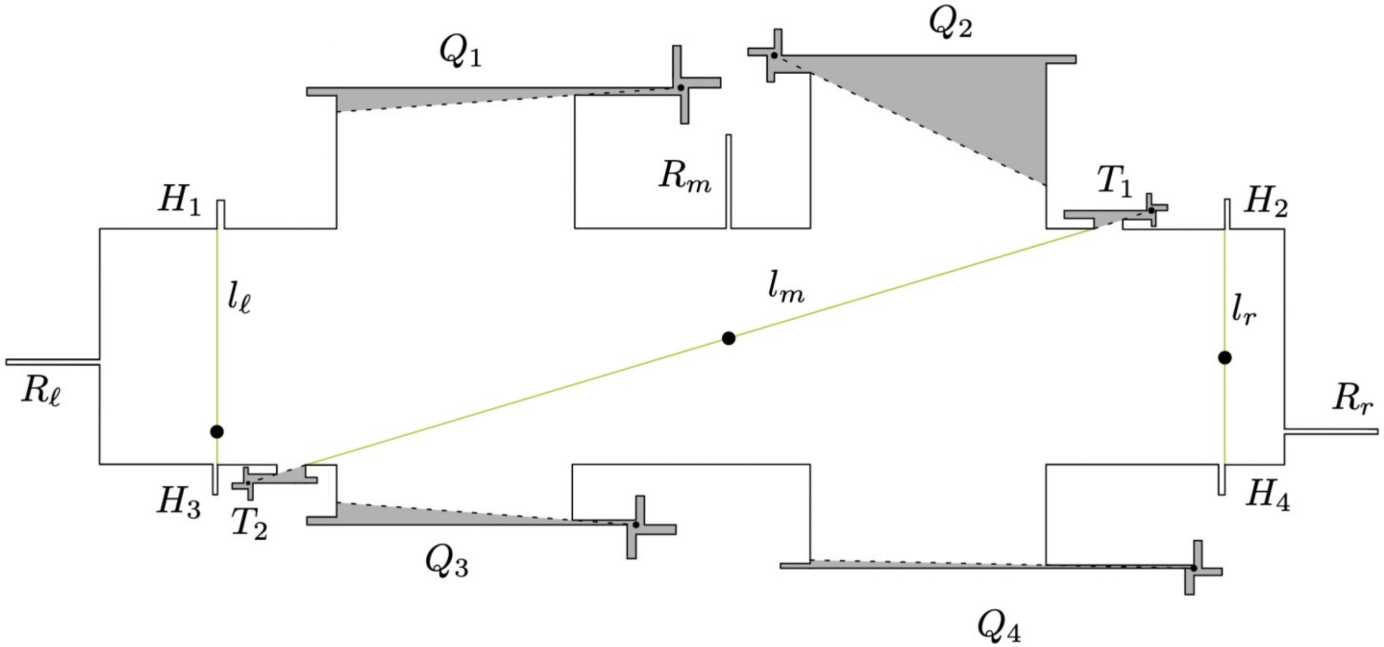
Despite an extensive study of the art gallery problem, it remained an open question whether there are polygons given by integer coordinates that require guard positions with irrational coordinates in any optimal solution. We give a positive answer to this question by constructing a *monotone* polygon with integer coordinates that can be guarded by three guards only when we allow to place the guards at points with irrational coordinates. Otherwise, four guards are needed. By extending this example, we show that for every n, there is a polygon which can be guarded by $3n$ guards with irrational coordinates but needs $4n$ guards if the coordinates have to be rational. Subsequently, we show that there are rectilinear polygons given by integer coordinates that require guards with irrational coordinates in any optimal solution.

1998 ACM Subject Classification F.2.2 Nonnumerical Algorithms and Problems

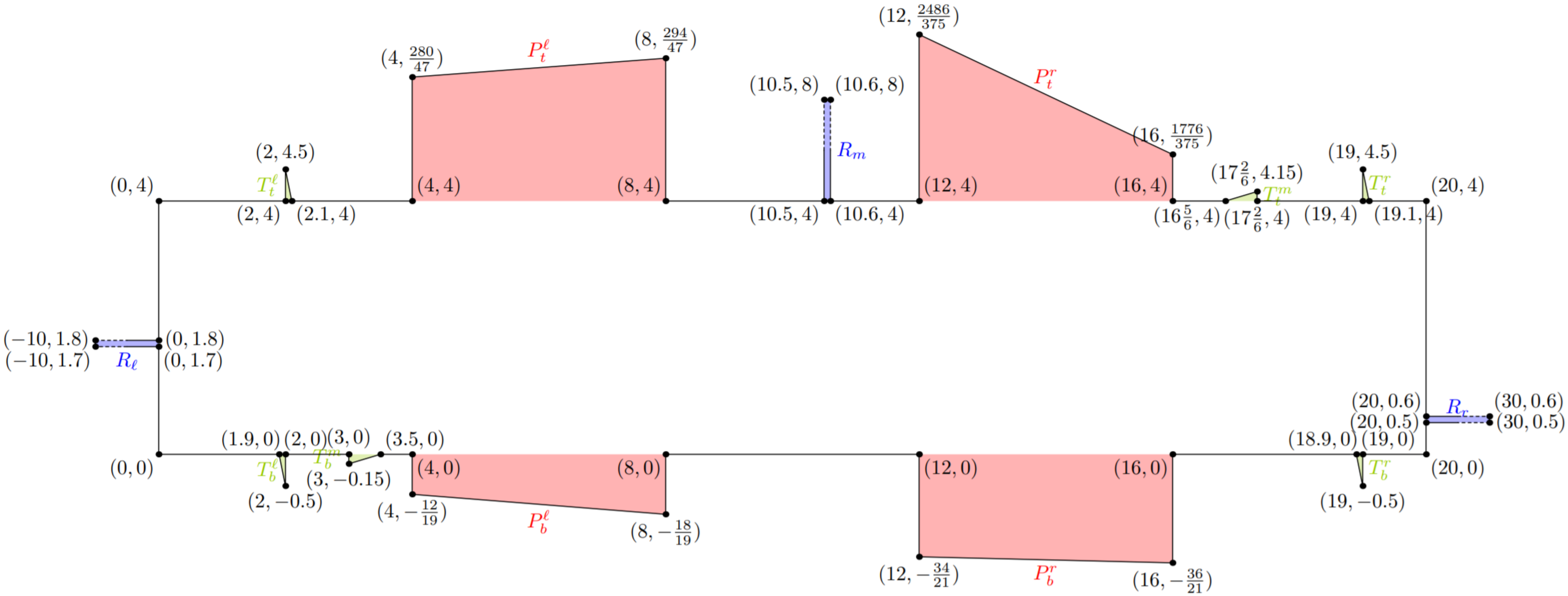
Keywords and phrases art gallery problem, computational geometry, irrational numbers

Digital Object Identifier 10.4230/LIPIcs.SoCG.2017.3





https://doi.org/10.4230/LIPIcs.SoCG.2017.3



$(12, \frac{2486}{375})$
$(8, \frac{294}{47})$
$(4, \frac{280}{47})$
P_t^ℓ
$(10.5, 8)$ $(10.6, 8)$
P_t^r
R_m
$(16, \frac{1776}{375})$
$(2, 4.5)$
$(17\frac{2}{6}, 4.15)$
$(19, 4.5)$
$(0, 4)$ T_t^ℓ $(4, 4)$ $(8, 4)$ $(12, 4)$ $(16, 4)$ T_t^m T_t^r $(20, 4)$
$(2, 4)$ $(2.1, 4)$ $(10.5, 4)$ $(10.6, 4)$ $(16\frac{5}{6}, 4)$ $(17\frac{2}{6}, 4)$ $(19, 4)$ $(19.1, 4)$
$(-10, 1.8)$ $(0, 1.8)$
$(-10, 1.7)$ R_ℓ $(0, 1.7)$
$(20, 0.6)$ R_r $(30, 0.6)$
$(20, 0.5)$ $(30, 0.5)$
$(1.9, 0)$ $(2, 0)$ $(3, 0)$ $(3.5, 0)$ $(18.9, 0)$ $(19, 0)$
$(0, 0)$ T_b^ℓ T_b^m $(4, 0)$ $(8, 0)$ $(12, 0)$ $(16, 0)$ T_b^r $(20, 0)$
$(3, -0.15)$
$(2, -0.5)$ $(19, -0.5)$
$(4, -\frac{12}{19})$
P_b^ℓ
$(8, -\frac{18}{19})$
$(12, -\frac{34}{21})$ P_b^r $(16, -\frac{36}{21})$

Art Gallery Problem – ∃ℝ-completeness

The Art Gallery Problem is ∃ℝ-complete

MIKKEL ABRAHAMSEN and ANNA ADAMASZEK, University of Copenhagen
TILLMANN MILTZOW, Utrecht University

The *Art Gallery Problem* (AGP) is a classic problem in computational geometry, introduced in 1973 by Victor Klee. Given a simple polygon $\mathcal{P}$ and an integer k, the goal is to decide if there exists a set G of k *guards* within $\mathcal{P}$ such that every point $p \in \mathcal{P}$ is seen by at least one guard $g \in G$. Each guard corresponds to a point in the polygon $\mathcal{P}$, and we say that a guard g *sees* a point p if the line segment pg is contained in $\mathcal{P}$.

We prove that the AGP is ∃ℝ-complete, implying that (1) any system of polynomial equations over the real numbers can be encoded as an instance of the AGP, and (2) the AGP is not in the complexity class NP unless NP = ∃ℝ. As a corollary of our construction, we prove that for any real algebraic number α, there is an instance of the AGP where one of the coordinates of the guards equals α in any guard set of minimum cardinality. That rules out many natural geometric approaches to the problem, as it shows that any approach based on constructing a finite set of candidate points for placing guards has to include points with coordinates being roots of polynomials with arbitrary degree. As an illustration of our techniques, we show that for every compact semi-algebraic set $S \subseteq [0,1]^2$, there exists a polygon with corners at rational coordinates such that for every $p \in [0,1]^2$, there is a set of guards of minimum cardinality containing p if and only if $p \in S$.

In the ∃ℝ-hardness proof for the AGP, we introduce a new ∃ℝ-complete problem ETR-INV. We believe that this problem is of independent interest, as it has already been used to obtain ∃ℝ-hardness proofs for other problems.

4

CCS Concepts: • **Theory of computation → Computational geometry**; **Problems, reductions and completeness**; **Complexity classes;**

Additional Key Words and Phrases: Art gallery problem, existential theory of the reals

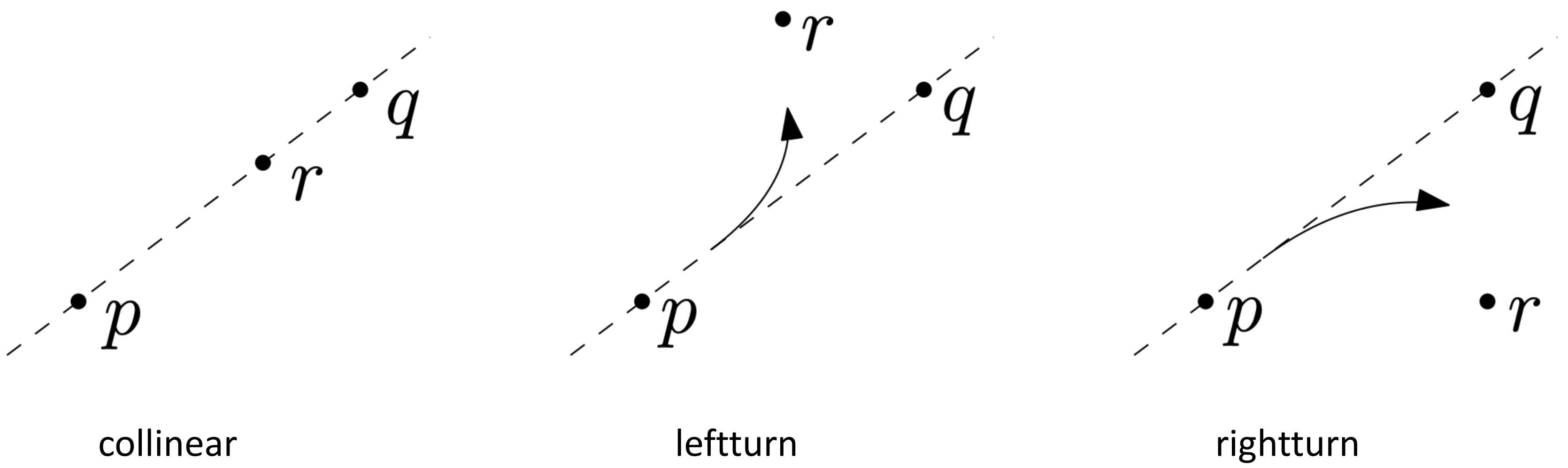
ACM Reference format:
Mikkel Abrahamsen, Anna Adamaszek, and Tillmann Miltzow. 2021. The Art Gallery Problem is ∃ℝ-complete. *J. ACM* 69, 1, Article 4 (December 2021), 70 pages.
https://doi.org/10.1145/3486220

Other complete problems for the existential theory of the reals include:

- the art gallery problem of finding the smallest number of points from which all points of a given polygon are visible.[22]
- the packing problem of deciding whether a given set of polygons can fit in a given square container.[23]
- recognition of unit distance graphs, and testing whether the dimension or Euclidean dimension of a graph is at most a given value.[9]
- stretchability of pseudolines (that is, given a family of curves in the plane, determining whether they are homeomorphic to a line arrangement);[4][24][25]
- both weak and strong satisfiability of geometric quantum logic in any fixed dimension >2;[26]
- Model checking interval Markov chains with respect to unambiguous automata[27]
- the algorithmic Steinitz problem (given a lattice, determine whether it is the face lattice of a convex polytope), even when restricted to 4-dimensional polytopes;[28][29]
- realization spaces of arrangements of certain convex bodies[30]
- various properties of Nash equilibria of multi-player games[31][32][33]
- embedding a given abstract complex of triangles and quadrilaterals into three-dimensional Euclidean space;[17]
- embedding multiple graphs on a shared vertex set into the plane so that all the graphs are drawn without crossings;[17]
- recognizing the visibility graphs of planar point sets;[17]
- (projective or non-trivial affine) satisfiability of an equation between two terms over the cross product;[34]
- determining the minimum slope number of a non-crossing drawing of a planar graph;[35]
- recognizing graphs that can be drawn with all crossings at right angles;[36]
- the partial evaluation problem for the MATLANG+eigen matrix query language.[37]
- the low-rank matrix completion problem.[38]

https://doi.org/10.1145/3486220

Order-types and Order-type realizability

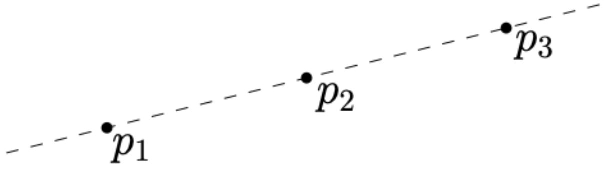


- Order-type of points $p_1, \dots, p_n$: Mapping of each triple of points to its orientation
- Order-type Realizability: Given an order-type for a set of abstract points, are there coordinates fulfilling the given order-type(s)?

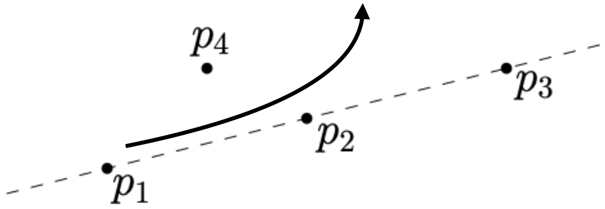
https://arxiv.org/abs/1406.2636

Order-types and Order-type realizability

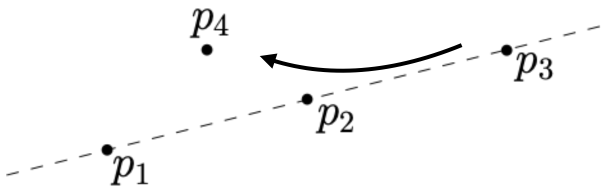
- Example: Given abstract points p_1, p_2, p_3 and p_4
  - Given order-type
  - (p_1, p_2, p_3) collinear



  - (p_1, p_2, p_4) leftturn



  - (p_1, p_3, p_4) leftturn



  - (p_2, p_3, p_4) rightturn