Visualizing Scissors Congruence

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Abstract

Consider two simple polygons with equal area. The Wallace–Bolyai–Gerwien theorem states that these polygons are scissors congruent, that is, they can be dissected into finitely many congruent polygonal pieces. We present an interactive application that visualizes this constructive proof.

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1 Introduction

At the dawn of the 19th century, William Wallace and John Lowry [1] posed the following:

> Is it possible in every case to divide each of two equal but dissimilar rectilinear figures, into the same number of triangles, such that those which constitute the one figure are respectively identical with those which constitute the other?

This sparked an active area of research, which culminated in the discovery of the following theorem, independently by Wallace-Lowry [1], Wolfgang Bolyai [2] and Paul Gerwien [3].

▶ **Theorem 1** (Wallace–Bolyai–Gerwien). *Any two simple polygons of equal area are scissors congruent, i.e. they can be dissected into a finite number of congruent polygonal pieces.*

David Hilbert himself recognized the importance of this theorem, including it as "Theorem 30" in his *The Foundations of Geometry* [4]. Furthermore, he posed a three-dimensional generalization of Wallace's question as number three of his famous 23 problems [5]: Given any two polyhedra of equal volume, can they be dissected into finitely many congruent tetrahedra? This problem was solved by Hilbert's own student Max Dehn, who provided (unlike the 2D case) a negative answer by constructing counterexamples [6].

The beauty of the original proof of WBG is that it is constructive: it describes an actual algorithm for constructing the polygonal pieces. To gain a deeper appreciation for this result, we built an interactive application that visualizes the algorithm in an intuitive and didactic manner. Instructors have taught the Wallace–Bolyai–Gerwien procedure using physical materials [7], and this application provides a digital analog.

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Figure 1 Visual representation of the algorithm: using scissor cuts, a triangle becomes a rectangle, which equidecomposes to another rectangle of the fixed height.

2 Algorithm

Indeed, the original proof demonstrates that any two simple polygons of equal area are "scissors congruent." We restate the theorem in a different manner that is more suited for visualization.

▶ **Corollary 2.** *Given two simple polygons of equal area S and T, there exists a finite sequence of cuts and rigid transformations that when applied to S result in T.*

This restatement motivates a constructive proof that can be formulated with an algorithm to rigidly transform S to T:

1. Compute some triangulation of S.
2. For each triangle in the trangulation, equidecompose that triangle into a rectangle.
3. For each rectangle generated above, equidecompose that rectangle into a rectangle of some fixed width w. If the starting rectangle is wider than $2w$, cut it in half and stack the two smaller rectangles on top of one another.
4. Stack all fixed width rectangles generated above into a single rectangle.
5. Perform the above steps in reverse order to equidecompose the rectangle into some triangulation of T.

Figure 1 shows the visual interpretation of each of these geometric procedures; see [8] for a more in-depth description and analysis.

3 Implementation

Our visualization application is implemented in HTML5 and JavaScript. It runs client-side in the web browser and can be accessed at `http://dmsm.github.io/scissors-congruence/`. The interface allows the user to input her own intial and terminal polygons. It then rescales the polygons so that they are of the same area, by calculating the optimal scaling factor for each polygon such that the following two constraints are satisfied: both polygons are of equal area, and the wider of the two is not too wide that is goes off the screen. Then, according to the above algorithm, it rigidly transforms her initial polygon into the terminal polygon (see Figure 2). The application takes advantage of the JavaScript libraries jQuery, Two.js, PolyK.js and Math.js in order to render the polygons in a fast and modular way.

References

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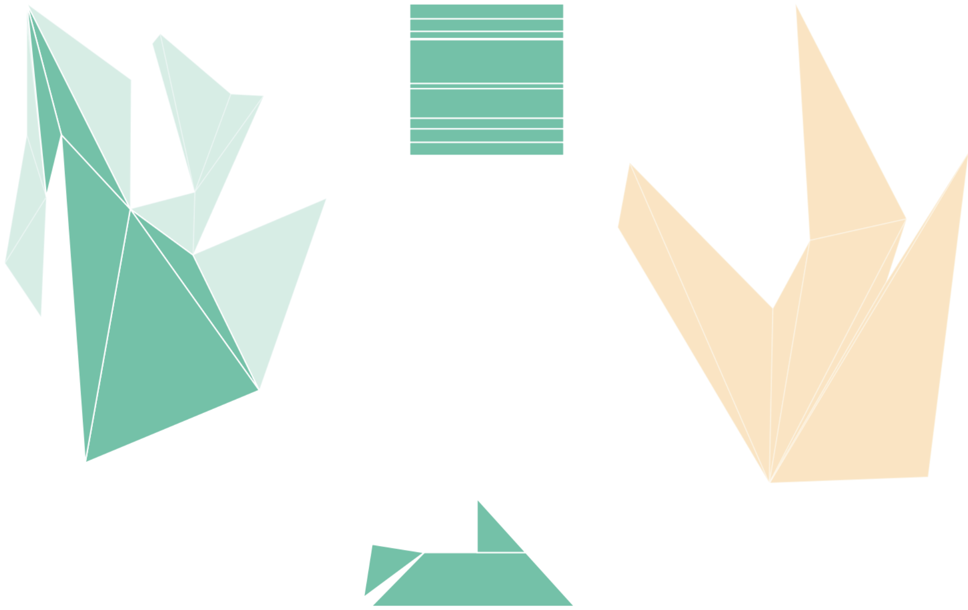

Figure 2 Screenshot of the application performing step 2 of the algorithm: a triangle of the triangulation of the initial polygon (in sea foam green) is being equidecomposed into a rectangle that will eventually be stacked (in the middle) on its way to forming the triangulation of the terminal polygon (in apricot orange).

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