The Visual Computer

# A simple linear algorithm for intersecting convex polygons

Godfried T. Toussaint

School of Computer Science,
McGill University,
805 Sherbrooke Street West,
Montreal, Quebec H3A 2K6,
Canada

Let $P$ and $Q$ be two convex polygons with $m$ and $n$ vertices, respectively, which are specified by their cartesian coordinates in order. A simple $O(m+n)$ algorithm is presented for computing the intersection of $P$ and $Q$. Unlike previous algorithms, the new algorithm consists of a two-step combination of two simple algorithms for finding convex hulls and triangulations of polygons.

**Key words:** Algorithms – Complexity – Computational geometry – Convex polygons – Intersection

Let $P=\{p_1, p_2, \ldots, p_m\}$ and $Q=\{q_1, q_2, \ldots, q_n\}$ be two convex polygons whose vertices are specified by their cartesian coordinates in clockwise order. It is assumed that the polygons are in *standard* form, i.e., the vertices of each polygon are distinct and no three consecutive vertices are colinear [7]. Several linear time algorithms have recently been proposed for computing the intersection of $P$ and $Q$, which is itself another convex polgon of at most $m+n$ vertices [6]–[9].The algorithms in [7]–[9] are relatively cumbersome to program due to the large number of cases that arise when intersecting the trapezoids that result with the "slab" method employed in [7]. The simplest and most elegant of the above algorithms is the one due to O'Rourke et al. [6]. Here two "bugs", one on the boundary of $P$ and the other on the boundary of $Q$, "chase" each other in an alternating fashion as each tries to cross the "forward line of sight" of the other.

In this paper we present a new simple algorithm for constructing the intersection of $P$ and $Q$ in $O(m+n)$ time in the worst case. Unlike the previous algorithm of [6]–[9] the new algorithm is a combination of existing simple procedures for computing convex hulls and triangulations of polygons. Because of this it may be a little slower in practice than the algorithm of O'Rourke et al. [6], depending on which convex hull algorithm is employed. On the other hand little new specialized code is needed if the convex hull and triangulation codes are already available. Furthermore, the algorithm presented here is conceptually transparently clear and affords an easy proof of correctness.

## Preliminary results

In this section we present an informal description of the algorithm and some preliminary results. A detailed description of the algorithm and a proof of correctness is included in section three. We assume in this paper that the interiors of the polygons $P$ and $Q$ intersect. If this is not true there is no intersection polygon to construct and the intersection then is either a line segment, a point, or the empty set. In any case, determining whether the interiors of $P$ and $Q$ intersect can be easily performed in $O(\log(m+n))$ time [1], [2]. In order to simplify the description of the algorithm and to prevent the essential aspects from drowning in a sea of detail we further assume that no three vertices in $P \cup Q$ are colinear and all vertices in $P \cup Q$ are distinct. This implies that if $P$ and $Q$ intersect then so do their interiors. It also implies



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that if the boundaries of $P$ and $Q$ intersect the intersection points do not coincide with vertices of $P$ or $Q$. Special case tests can be included for the "singularities" that arise when this assumption is not made and these are similar for all the algorithms outlined above [6]–[9]. A very clear exposition on handling these cases is given by O'Rourke et al. [6].

Consider then two polygons $P$ and $Q$ whose boundaries intersect and construct the convex hull of their union (Fig. 1). Let the boundaries of $P$ and $Q$ intersect at $k$ intersection points $I_1, I_2, \ldots, I_k$ indexed in clockwise order. The boundaries of $P$, $Q$, and the convex hull of $P$ and $Q$, $CH(P \cup Q)$, partition the plane into $2k+1$ bounded regions: the convex intersection region $(I_1, \ldots, I_2, \ldots, I_k, \ldots)$, $k$ regions where $P$ and $Q$ lie outside $P \cap Q$ ($P_{st}$ associated with $I_s$ and $I_t$, and $Q_{uv}$ associated with $I_u$ and $I_v$) and $k$ "pockets" $K_1, K_2, \ldots, K_k$ where a pocket $K_v$ is associated with $I_v$ and is in the region inside $CH(P \cup Q)$ but outside $P \cup Q$. With each pocket $K_v$ we associate a bridge which is an edge of $CH(P \cup Q)$, denoted by $B_v(p_{i_v}, q_{j_v})$, and which joins vertex $p_{i_v}$ of $P$ with vertex $q_{j_v}$ of $Q$. The algorithm for computing $P \cap Q$ can now be described informally as the following three-step procedure: first construct the convex hull of $P \cup Q$, then for each bridge $B_i$ find its corresponding intersection point $I_i$, and finally "merge" the corresponding polygonal chains that connect adjacent intersection points.

We now prove some lemmas that we will need to prove the correctness of the algorithm described in section three. Let $L(u, v)$ denote the directed line through $u$, and $v$ in the direction $u, v$ and let $RH(u, v)$ denote the closed half-plane to the right of $L(u, v)$.

The following lemma has been proved by Guibas et al. [4] using a powerful new framework involving convolutions (a special case of fiber products) of polygons. We include an alternate elementary proof here for completeness.

**Lemma 1.** *If $P$ and $Q$ intersect there exists a unique mutual one-to-one correspondence between the* bridges *of $CH(P \cup Q)$ and the* intersection points *of $P \cap Q$.*

*Proof.* Let $B(p_i, q_j)$ be a bridge and refer to Fig. 2. $L(p_i, q_j)$ must be a line of support for both $P$ and $Q$. Furthermore $P$ and $Q$ must both lie in $RH(p_i, q_j)$. Trace $P$ in a clockwise manner starting

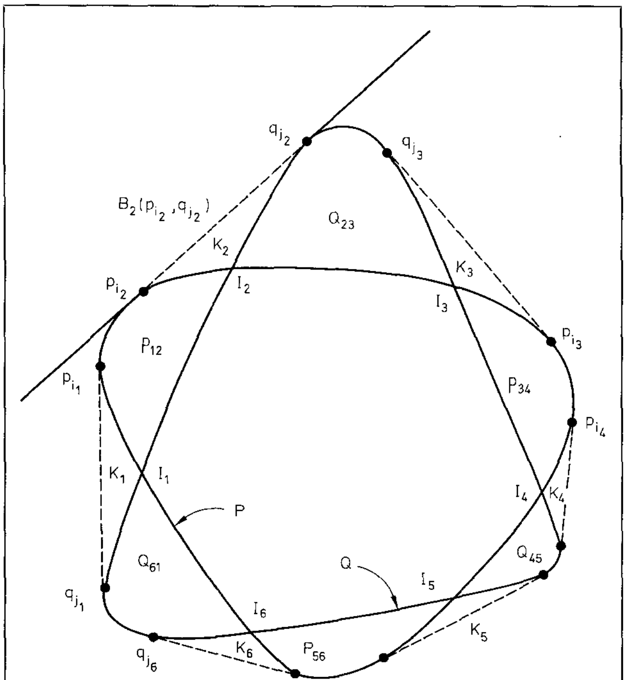


**Fig. 1**

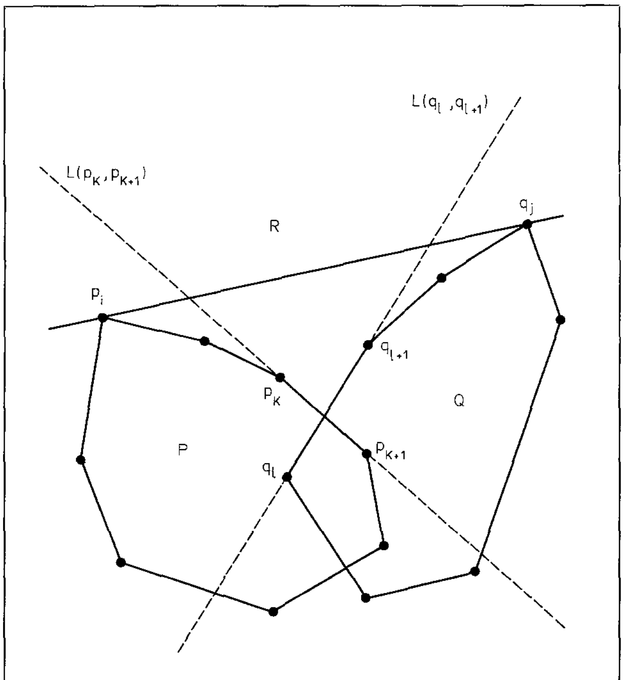


**Fig. 2**

at $p_i$ until an edge of $P$ intersects an edge of $Q$ at $I$. Similarly trace $Q$ in a counter-clockwise manner starting at $q_j$ until an edge of $Q$ intersects an edge of $P$. From convexity it follows that this intersection point is also $I$ and thus $I$ corresponds to $B(p_i, q_j)$. On the other hand assume that $I$ is some intersection point between edge $\overline{p_k p_{k+1}} \in P$ and $\overline{q_l q_{l+1}} \in Q$. Since $P \in RH(p_k, p_{k+1})$ and $Q \in RH(q_l, q_{l+1})$ it follows that no edge of $P$ or $Q$ other than $\overline{p_k p_{k+1}}$ and $\overline{q_l q_{l+1}}$ may intersect the region $R = RH(p_{k+1}, p_k) \cap RH(q_{l+1}, q_l)$. Furthermore, since angle $p_k I q_{l+1} < 180°$ it follows that there must exist an edge $p_i q_j \in CH(P \cup Q)$ that intersects $R$ and this is the bridge corresponding to $I$. Q.E.D.

We now define a restricted class of simple polygons and establish some results concerning their triangulation. While we are not explicitly interested in triangulating these polygons these results will be useful in understanding, and proving the correctness of, the algorithm. A polygonal chain $C(p_i, p_{i+1}, \ldots, p_j)$ is a portion of consecutive vertices and edges of a simple polygon. If all turns are *right* (convex angles) we have a *convex chain.* If all turns are *left* (reflex angles) we have a *concave chain.*

*Definition.* A *sail* polygon $P_s$ is one that contains an edge $\overline{p_i p_{i+1}}$ called the *mast* of $P$ and a vertex $p_j$ called the *sail tip* of $P$ such that $p_j$ is connected to $p_i$ and $p_{i+1}$ by *concave chains* (Fig. 3) Note that $P_s$ must be completely in $RH(p_i, p_{i+1})$.

*Definition.* A line segment, lying in $P$, that connects two non-adjacent vertices of $P$ is a *diagonal* of $P$.

*Definition.* Three consecutive vertices $p_i\, p_{i+1}\, p_{i+2}$ are said to form an *ear* of $P$ at $p_{i+1}$ if the diagonal joining $p_i$ and $p_{i+2}$ lies in $P$.

*Definition.* Two ears are *non-overlapping* if their interior regions are disjoint.

Meisters [5] proves the following "two-ears" theorem.

**Lemma 2.** *Every polygon of n sides* ($n > 3$) *has at least two non-overlapping ears.*

This theorem leads Meisters to propose an $O(n^3)$ algorithm for triangulating simple polygons by finding ears and "cutting them off". *Sail* polygons on the other hand have enough structure that we can "cut off all the ears" in $O(n)$ time. Note that, by definition, only *convex* vertices can be *ears.* Also, a sail polygon has the property that only $p_i$, $p_{i+1}$, and $p_j$ are *convex*, and thus candidates for *ears*. We thus have the following results.

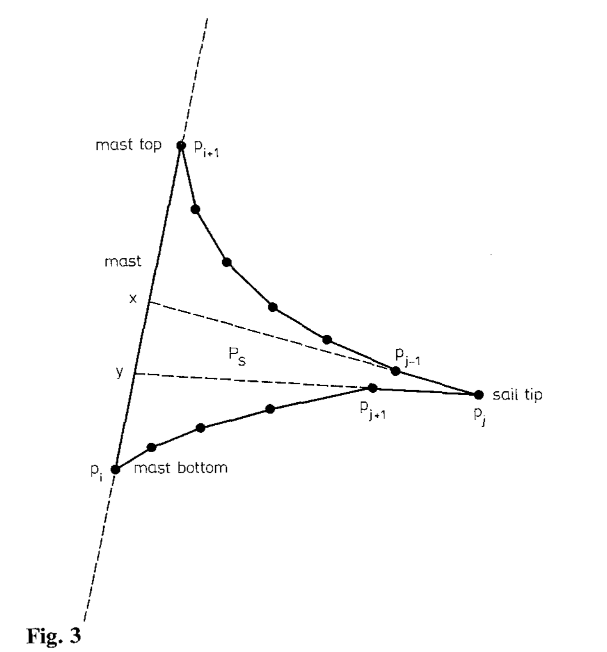


Fig. 3

**Lemma 3.** *The* tip *of a* sail *polygon is an* ear.

*Proof.* Extend $\overline{p_j p_{j-1}}$ and $\overline{p_j p_{j+1}}$ to intersect $L(p_i, p_{i+1})$ at $x$ and $y$, respectively, (Fig. 3) Point $x$ must lie on $p_i\, p_{i+1}$ or else $p_j$ could not be joined to $p_i$ with a *concave* chain. The same argument holds for $y$. By construction $p_j\, p_{j-1}\; x\; y\; p_{j+1}\, p_j$ forms a triangle and by convexity it lies completely in $P_s$. Therefore the diagonal $\overline{p_{j-1}\, p_{j+1}}$ lies in $P_s$. Q.E.D.

**Lemma 4.** *Either the* mast top *or the* mast bottom *of a* sail *polygon is an* ear.

*Proof.* Only $p_i$, $p_{i+1}$, and $p_j$ in $P_s$ can be ears. By Lemma 3 $p_j$ must be an ear. By Lemma 2 $P_s$ must have at least two ears. Therefore either $p_i$ or $p_{i+1}$ must be an ear. Q.E.D.

Lemma 4 allows us to triangulate $P_s$ in $O(n)$ time by "wrapping the sail around the mast" until only the sail tip remains. In other words, starting at the mast we cut off either the top ear or the bottom ear and proceed to the polygon remaining. The correctness of the algorithm follows from the induction hypothesis that, at each step, the polygon remaining is a *sail* polygon. The proof of this induction hypothesis is left as an easy exercise for the reader. The linearity follows from the fact that at each step which takes constant time $P_s$ contains one less vertex. Note that other linear time algorithms could be used for triangulating $P_s$. For example $P_s$ is *edge-visible* from the mast and thus the algorithm of [13] can be used. Alternately, $P_s$ is *monotonic* in the direction perpendicular to the *mast* and therefore the algorithm of Garey et al. [3] applies. The advantages of the algorithm presented here are that, first, unlike those of [13] and [3] it does not incorporate backtracking and is thus simpler, and second, the last diagonal to be added is $\overline{p_{j-1} p_{j+1}}$. This latter property is crucial for solving the polygon intersection problem. The "ear-cutting" algorithm is in essence a trimmed version of the algorithm of Garey et al. [3] that exploits the added structure that *sail* polygons have over *monotone* polygons.

## The algorithm

Before describing the complete algorithm we present PROCEDURE STEPDOWN which receives as input a bridge $B_k(p_i, p_j)$ of $CH(P \cup Q)$ and exits with the corresponding pair of edges that determine the intersection point $I_k$. Without loss of generality assume $p_1$ and $q_1$ form the bridge, $Q$ is given in counter-clockwise order, and $\overline{p_s p_{s+1}} \cap \overline{q_t q_{t+1}}$ determines the intersection point $I$. (Fig. 4.) A convenient data structure for $P$ and $Q$ here is a doubly-linked circular list so that we can scan in either direction and set up pointers between the vertices of $P$ and those of $Q$. Procedure STEPDOWN finds the two vertices $p_s$ and $q_t$ that can then be used to compute $I$. The variables $p_i$ and $q_j$ are the "current" vertices under consideration and are a tentative solution. When the algorithm stops $p_i = p_s$ and $q_j = q_t$. The boolean variable "finished" indicates when $p_s$ and $q_t$ are reached by taking on the value "true" after an execution of the "repeat" loop.

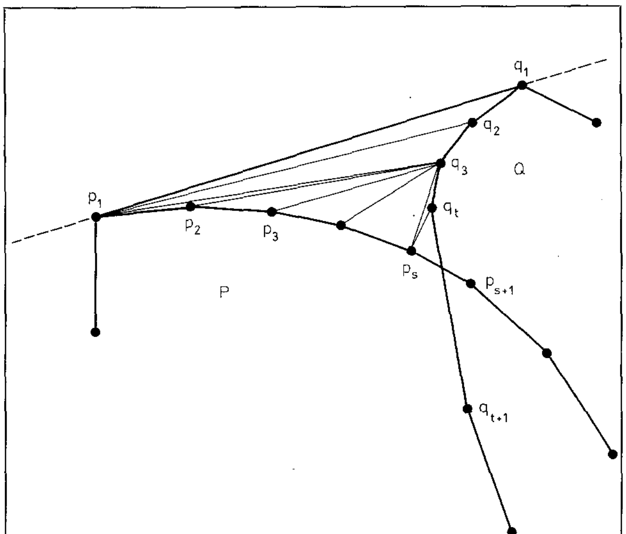


**Fig. 4**

```
PROCEDURE STEPDOWN
{initialization} i←1; j←1
   repeat
      finished ← true
      while (p_i p_{i+1} q_{j+1}) left do
      begin
            j←j+1
            finished ← false
      end
      while (q_j q_{j+1} p_{i+1}) right do
      begin
            i←i+1
            finished ← false
      end
   until finished
p_s←p_i; q_t←q_j
END STEPDOWN
```

**Lemma 5.** *Procedure STEPDOWN correctly computes the intersection point corresponding to a bridge in $O(n)$ time.*

*Proof.* The proof follows essentially from the realization that STEPDOWN is an implementation of the "ear-cutting" triangulation algorithm for *sail* polygons given in the previous section. Note that $(p_1, q_1, q_2, \ldots, q_t, I, p_s, p_{s-1}, \ldots, p_2)$ would be a *sail* polygon if $I$ were a vertex connected to $p_s$ and $q_t$. Thus the "ear-cutting" algorithm must eventually arrive at $\overline{p_s q_t}$. Now in a true *sail* polygon the algorithm automatically stops here because $p_{s+1} = q_{t+1}$. However, in this situation this is not the case since $p_{s+1}$ and $q_{t+1}$ belong to different polygonal chains. The tests for left and right turns in the inner WHILE loops of STEPDOWN not only prevent the algorithm from continuing past $p_s$ and $q_t$, but also determine an ordering for "ear-cutting", by invoking Lemma 4. Q.E.D.

We now describe the algorithm for computing the intersection of two intersecting convex polygons $P$ and $Q$. The portions of the boundaries of $P$ and $Q$ outside $P \cap Q$ will be referred to as *outer chains*, those portions inside $P \cup Q$ as *inner chains*.

ALGORITHM INTERCONPOL
**Begin**
*Step 1.* Find the convex hull of the union of $P$ and $Q$, $CH(P \cup Q)$.
If $CH(P \cup Q) = P$(or $Q$)
then Exit with $Q$(or $P$) as
the intersection; Else continue.
*Step 2.* For each *bridge* of $CH(P \cup Q)$ call procedure STEPDOWN to compute the intersection points of $P \cap Q$.
*Step 3.* Merge the *inner chains* of $P$ and $Q$ determined by the intersection points found in step 2.
**End**

Theorem. *Algorithm INTERCONPOL correctly computes the intersection polygon of two intersecting convex polygons P and Q in* $O(m+n)$ *time.*

*Proof.* The correctness of the algorithm follows from Lemmas 1 and 5. Thus we turn to its complexity. Finding the convex hull of two intersecting convex polygons in step 1 can be done in $O(m+n)$ time with several algorithms [7], [10], [11]. The simplest of all the algorithms is the "rotating caliper" method [11] which, unlike those of [7] and [10], does not involve backtracking and at the same time can answer the question of whether $CH(P \cup Q) = P$ or $Q$. If there are $k$ *bridges* on $CH(P \cup Q)$ then STEPDOWN is called $k$ times in step 2. Each call requires time linear in the number of vertices processed and the total number of these vertices is the sum total of the vertices on all the *outer chains* of $P$ and $Q$. Thus step 2 runs in $O(m+n)$ time. Finally, if we leave pointers from the intersection points to the *inner* and *outer* chains in both directions, as we find them in step 2, then the merge step of the inner chains in step 3 can be done in linear time by a mere traversal of the two lists for $P$ and $Q$. Q.E.D.

## Concluding remarks

As a final remark we mention that the "ear-cutting" triangulation algorithm for *sail* polygons presented in section two can be applied to the problem of triangulating a set of $n$ points on the plane in $O(n \log n)$ time via divide-and-conquer. Here, if the points have been presorted, at each step we must merge two triangulations $T_1$ and $T_2$ which are linearly separable triangulated convex polygons (Fig. 5). The merge step consists of triangulating the *hourglass* polygon "in between" $T_1$ and $T_2$. This region lies outside $T_1$ and $T_2$ but inside $CH(T_1 \cup T_2)$. An *hourglass* polygon is a polygon consisting of two edges called the top (bridge $p_i, p_j$) and the bottom (bridge $p_k, p_l$) such that $p_i$ and $p_l$ (as well as $p_k, p_j$) are joined by *concave* chains and $(p_i, p_j, p_k, p_l)$ forms a *convex* quadrilateral. Now consider a *critical line of support* between $T_1$ and $T_2$ at $p_u$ and $p_v$. This line decomposes the *hourglass* polygon into two *sail* polygons $P_{s_1}$ and $P_{s_2}$. Finding the bridges and the edge $p_u p_v$ can be done in linear time with the rotating calipers [11]. Triangulating the *sail* polygons will thus solve the merge of $T_1$ and $T_2$ in linear time which is sufficient to obtain the overall $O(n \log n)$ performance. Note that the triangulation algorithms of [13] and [3] *cannot* be used here since an *hourglass* polygon need be neither *edge-visible* nor *monotone*. Finally, we remark that this algo-

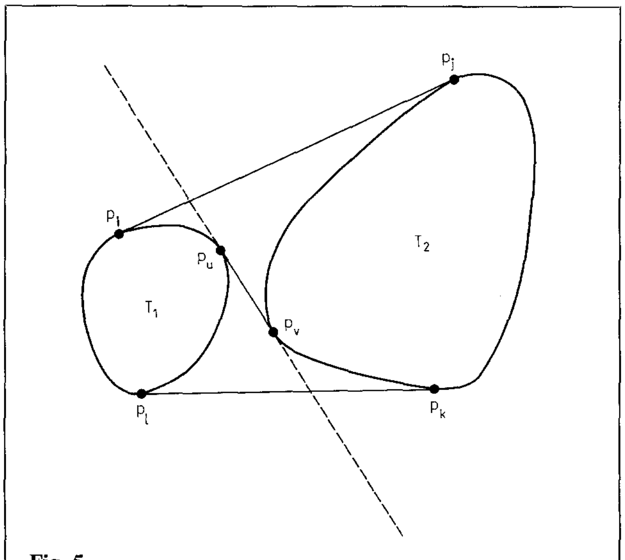


**Fig. 5**

rithm can be applied to the problem of computing distances between crossing convex polygons [12].

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