Journal of Global Optimization (2020) 76:189–209
https://doi.org/10.1007/s10898-019-00834-6

Solving *k*-center problems involving sets based on optimization techniques

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Received: 24 June 2018 / Accepted: 25 September 2019 / Published online: 23 October 2019

Abstract
The continuous k-center problem aims at finding k balls with the smallest radius to cover a finite number of given points in $\mathbb{R}^n$. In this paper, we propose and study the following generalized version of the k-center problem: Given a finite number of nonempty closed convex sets in $\mathbb{R}^n$, find k balls with the smallest radius such that their union intersects all of the sets. Because of its nonsmoothness and nonconvexity, this problem is very challenging. Based on nonsmooth optimization techniques, we first derive some qualitative properties of the problem and then propose new algorithms to solve the problem. Numerical experiments are also provided to show the effectiveness of the proposed algorithms.

Keywords k-center problem · Multifacility location problem · Majorization-minimization principle · Difference of convex functions

AMS Subject Classification Primary: 49J52 · 49J53; Secondary: 90C30

1 Introduction

Proposed by the English mathematician James Joseph Sylvester (1814–1897) in the nineteenth century, the *smallest enclosing circle problem* or the *1-center problem* asks for the smallest circle enclosing a finite number of points in the plane; see [33]. This problem and its version in higher dimensions have received great attention from many researchers because of their important applications to clustering, nearest neighbor search, data classification, facility location, collision detection, computer graphics, and military operations. The read-

N. T. An and X. Qin: Research of the first author was supported by the National Natural Science Foundation of China under Grant No.11950410503, the China Postdoctoral Science Foundation under Grant No.2017M622991 and the Vietnam National Foundation for Science and Technology Development under Grant No.101.01-2017.325. N. M. Nam: Research of the second author was partly supported by the National Science Foundation under Grant DMS-1716057.

This article was supported by the National Natural Science Foundation of China under Grant No.11401152.

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ers are referred to [1,6,12,17,31,36–38,40] and the references therein for more information involving the smallest enclosing circle problem and its generalizations from both theoretical and numerical standpoints. Note that the smallest enclosing circle problem is significantly different from the *Fermat–Torricelli problem* of finding a point that minimizes the sum of the distances to a finite number of points in the plane; see, e.g., [16]. The smallest enclosing circle problem and the Fermat–Torricelli problem are important examples of facility location problems in which a single facility/service center is chosen to provide service to customers. The distance from the center to the farthest customer is minimized in the case of the smallest enclosing circle problem, and the total transportation cost is minimized in the case of the Fermat–Torricelli problem. It is certainly of great interest to deal with multifacility location versions of these problems with the motivation coming from building a chain of service centers instead of one.

Given a finite number of data points $a^1, \ldots, a^m$ in a Euclidean space and a fixed positive integer k, we seek k centers $x^1, \ldots, x^k$ and assign each a^i for $i = 1, \ldots, m$ to its nearest center in a way that the largest distance $r \geq 0$ from the centers to their assigned data points is minimal. Then the union of the closed balls with k centers $x^1, \ldots, x^k$ and the same radius r would cover all of the given data points. Observe that after k centers have been chosen, the data points are divided into k clusters, and the radius of each cluster is determined by the distance from the center to its farthest assigned data point. Thus, the objective of the *k-center problem* is to find k balls to cover all of the target points such that the radius of the largest balls is minimized; see [14,15] and the references therein. Depending on the way of choosing the centers of the balls, there are two cases of the k-center problem in the literature. In the discrete case, the centers must be chosen among the target points, while in the continuous case these points can be located anywhere in $\mathbb{R}^n$. In a similar way, the multifacility versions of the Fermat–Torricelli problem are called the k-median problem; see, e.g., [20,28] and the references therein.

In the classical models of facility location, each location is identified with a data point. A question arises as follows: Can we deal with facility location problems in which given data points are replaced by given sets? Following the developments from [5,21,23,24,27,30,32] and related references for set-facility location versions of the smallest enclosing circle problem and the Fermat–Torricelli location problem, in this paper we study a generalized version of the k-center problem: Given a finite number of nonempty closed convex sets in $\mathbb{R}^n$, find k balls of the same minimal radius such that their union intersects all of the sets. Besides the mathematical motivation, this question appears in more complicated models of facility location in which the sizes of the locations are not negligible, as in bilevel transportation problems. For instance, people would like to deliver supplies to different islands from k aircraft carriers, and suppose that the supplies delivered anywhere on each island can be distributed using local transportation. Then it is necessary to find a position for the aircraft carriers such that the maximum distance from an aircraft carrier to the nearest island is minimal. In this situation, without neglecting the sizes of the islands, we need to deal with a model of facility location in which points are replaced by sets.

The main challenge in solving the generalized k-center problem comes from the intrinsic nondifferentiability and nonconvexity of the models. Our main goal in this paper is to study these problems from both theoretical and numerical viewpoints. We focus particularly on developing numerical algorithms for solving them using a heuristic method and a new method based on the DCA, an algorithm for minimizing differences of convex functions introduced by Tao and An; see [34,35].

The paper is organized as follows. In Sect. 2, we formulate the set model of the classical k-center problem and introduce necessary notations and definitions. Basic tools of convex analysis and the DCA are introduced in Sect. 3. Section 4 is devoted to some qualitative properties of the generalized k-center problem. Numerical algorithms for solving this problem are the main topics of Sect. 5. Finally, we provide numerical examples in Sect. 6.

2 Notations and problem formulations

Throughout the paper, we use the matrix $\mathbf{X} = (x^1, \ldots, x^k)^\top \in \mathbb{R}^{k\times n}$, whose ℓth row is $(x^\ell)^\top$, to store k centers to be found. Consider a finite number of nonempty closed and convex subsets $\Lambda^1, \ldots, \Lambda^m$ of $\mathbb{R}^n$. There always exist nonnegative real numbers $R_1, \ldots, R_k$ satisfying

$$\Lambda^i \cap \left[\bigcup_{\ell=1}^{k} \mathbb{B}(x^\ell; R_\ell)\right] \neq \emptyset \quad \text{for all } i = 1, \ldots, m. \tag{2.1}$$

The objective of the *generalized k-center problem* generated by m target sets $\{\Lambda^i\}_{i=1}^m$ is to find k balls such that property (2.1) holds and the largest radius $R_{\max} := \max\{R_1, \ldots, R_k\}$ is minimal. In what follows we use the following index sets:

$$I := \{1, \ldots, m\} \text{ and} J := \{1, \ldots, k\}.$$

For each $i \in I$, the distance from the target set Λ^i to its closest center is defined by

$$d_i(\mathbf{X}) := \min\{d(x^\ell; \Lambda^i) \mid \ell \in J\}, \tag{2.2}$$

where the *Euclidean distance function* associated with a subset Λ of $\mathbb{R}^n$ is defined by

$$d(x; \Lambda) := \inf\{\|x - a\| \mid a \in \Lambda\}, \quad x \in \mathbb{R}^n.$$

For each $\ell \in J$, the *cluster* associated with the center x^ℓ is defined by

$$\mathcal{A}(x^\ell) := \{\Lambda^i \mid i \in I(x^\ell)\},$$

where $I(x^\ell) := \{i \in I \mid d(x^\ell; \Lambda^i) = d_i(\mathbf{X})\}$. We can see that for any $\mathbf{X} \in \mathbb{R}^{k\times n}$, $\bigcup_{\ell\in J} I(x^\ell) = I$. Thus, $\mathcal{A}(x^\ell)$ contains all the target sets that are closer to x^ℓ than other centers. It is possible that $\mathcal{A}(x^\ell) = \emptyset$ for some $\ell \in J$. The *radius of cluster* $\mathcal{A}(x^\ell)$ is defined by

$$R(x^\ell) := \begin{cases} \max\{d_i(\mathbf{X}) \mid i \in I(x^\ell)\}, & \text{if } I(x^\ell) \neq \emptyset \\ 0, & \text{if } I(x^\ell) = \emptyset. \end{cases}$$

From the definition, we can show that the ball $\mathbb{B}(x^\ell; R(x^\ell))$ intersects all the target sets in the cluster $\mathcal{A}(x^\ell)$, i.e., for any $i \in I(x^\ell)$,

$$\Lambda^i \cap \mathbb{B}(x^\ell; R(x^\ell)) \neq \emptyset. \tag{2.3}$$

To ensure that each of the target sets is intersected by at least such a ball, we define the radius

$$R(\mathbf{X}) := \max\left\{R(x^\ell) \mid \ell \in J\right\}$$

and observe from these definitions that, for each $i \in I$, there exists $\ell \in J$ such that $d(x^\ell;\Lambda^i) = d_i(\mathbf{X}) \le R(x^\ell) \le R(\mathbf{X})$ and therefore $\Lambda^i \cap \mathbb{B}(x^\ell; R(\mathbf{X})) \ne \emptyset$. It is obvious that

$$R(\mathbf{X}) = \max\{d_i(\mathbf{X}) \mid i \in I\}.$$

The goal of the *generalized k-center problem* is to minimize $R(\mathbf{X})$ by solving the following optimization problem:

$$\begin{aligned} &\text{minimize } \mathcal{F}_k(x^1,\ldots,x^k) := \max_{i\in I}\min_{\ell\in J} d(x^\ell;\Lambda^i)\\ &x^1,\ldots,x^k \in \mathbb{R}^n. \end{aligned} \tag{GkC}$$

Using $\mathbf{X}$ as a variable, we can rewrite problem (GkC) as

$$\text{minimize } \mathcal{F}_k(\mathbf{X}) := \max_{i\in I} d_i(\mathbf{X}),\ \mathbf{X} \in \mathbb{R}^{k\times n},$$

where $d_i(\mathbf{X})$ is defined in (2.2). Note that the nonconvexity and nondifferentiability of the objective function $\mathcal{F}_k$ require nonsmooth optimization techniques beyond convexity.

If $k \ge m$, by choosing arbitrarily $x^\ell \in \Lambda^\ell$ for $\ell = 1,\ldots,m$ and $x^\ell \in \mathbb{R}^n$ for $\ell = m+1,\ldots,k$, it is obvious that $\mathbf{X} = (x^1,\ldots,x^k)^\top$ is a solution with the optimal value $\mathcal{F}_k(\mathbf{X}) = 0$. We henceforth assume that $k < m$ as our standing assumption in this paper.

In the case where all the target sets are singletons with $\Lambda^i = \{a^i\}$ for all $i \in I$, problem (GkC) reduces to the classical continuous k-center problem (see, e.g., [9,11]):

$$\begin{aligned} &\text{minimize } f_k(x^1,\ldots,x^k) := \max_{i\in I}\min_{\ell\in J} \|x^\ell - a^i\|,\\ &x^1,\ldots,x^k \in \mathbb{R}^n. \end{aligned} \tag{kC}$$

If $k = 1$, problem (GkC) reduces to the *smallest intersecting ball problem* considered in [3,5,26]:

$$\text{minimize } \mathcal{F}(x) := \max_{i\in I} d(x;\Lambda^i),\ x \in \mathbb{R}^n. \tag{SIB}$$

If x_* is a solution of problem (SIB) in which $\mathcal{F}(x) = \max_{i\in I(x^\ell)} d(x;\Lambda^i)$, then the ball $\mathbb{B}(x_*;\mathcal{F}(x_*))$ is called the *smallest intersecting ball* of the cluster $\mathcal{A}(x^\ell)$.

When all the target sets are singletons with $\Lambda^i = \{a^i\}$ for all $i \in I$, problem (SIB) reduces to the *smallest enclosing ball problem* (see, e.g., [8,33,38]):

$$\text{minimize } f(x) := \max_{i\in I} \|x - a^i\|,\ x \in \mathbb{R}^n. \tag{SEB}$$

3 Tools of convex analysis and optimization

In this section, we introduce some basic concepts of convex analysis and optimization tools used to solve the generalized k-center problem (GkC).

Given a convex function $f\colon \mathbb{R}^n \to (-\infty,\infty]$ with $\bar{x} \in \operatorname{dom}(f) := \{x \in \mathbb{R}^n \mid f(x) < \infty\}$, the *subdifferential in the sense of convex analysis* of f at $\bar{x}$ is defined by

$$\partial f(\bar{x}) := \left\{v \in \mathbb{R}^n \mid \langle v, x-\bar{x}\rangle \le f(x) - f(\bar{x}) \text{ for all } x \in \mathbb{R}^n\right\}.$$

It follows directly from the definition that $\bar{x}$ is an absolute minimizer of f on $\mathbb{R}^n$ if and only if

$$0 \in \partial f(\bar{x}). \tag{3.1}$$

This generalized differentiation notion satisfies comprehensive calculus rules in both finite and infinite dimensions. In particular, if $f_i : \mathbb{R}^n \to (-\infty, \infty)$ for $i = 1, \ldots, m$ are *real-valued* convex functions on $\mathbb{R}^n$, then

$$\partial(f_1 + \ldots + f_m)(\bar{x}) = \partial f_1(\bar{x}) + \ldots + \partial f_m(\bar{x}) \ \text{ for all } \bar{x} \in \mathbb{R}^n. \tag{3.2}$$

Define $f := \max_{i=1\ldots,m} f_i$ and consider the *active index set* of f at $\bar{x} \in \mathbb{R}^n$ given by

$$\widetilde{I}(\bar{x}) := \{i \in \{1, \ldots, m\} \mid f_i(\bar{x}) = f(\bar{x})\}.$$

Then we have the following *subdifferential maximum rule* (see, e.g., [25, Proposition 2.54]):

$$\partial f(\bar{x}) = \operatorname{conv}\left(\bigcup_{i \in \widetilde{I}(\bar{x})} \partial f_i(\bar{x})\right), \tag{3.3}$$

where $\operatorname{conv}(A)$ stands for the *convex hull* of a set A in $\mathbb{R}^n$.

Given a subset Λ of $\mathbb{R}^n$, the *Euclidean projection* from a point $x \in \mathbb{R}^n$ to Λ is defined by

$$P(x; \Lambda) := \big\{w \in \Lambda \;\big|\; d(x; \Lambda) = \|x - w\|\big\}.$$

Note that if Λ is a nonempty closed convex set, then the projection $P(x; \Lambda)$ is a singleton. In addition, the *squared distance function* $g(x) := d(x; \Lambda)^2$ for $x \in \mathbb{R}^n$ is Fréchet differentiable on $\mathbb{R}^n$ with $\nabla g(x) = 2[x - P(x; \Lambda)]$; see [18, p. 137].

Given a convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty)$, the *Fenchel conjugate* of φ is defined by

$$\varphi^*(y) := \sup\{\langle y, x\rangle - \varphi(x) : x \in \mathbb{R}^n\}, \ y \in \mathbb{R}^n.$$

Note that $\varphi^* : \mathbb{R}^n \to (-\infty, \infty]$ is also a convex function and $v \in \partial\varphi^*(y)$ if and only if $y \in \partial\varphi(v)$; see, e.g., [18].

We are now ready to introduce an algorithm used broadly to cope with both nonconvexity and nondifferentiability in the case where the objective function is representable as a difference of two convex functions. Consider the problem

$$\text{minimize } f(x) := g(x) - h(x), \ x \in \mathbb{R}^n, \tag{3.4}$$

where $g : \mathbb{R}^n \to (-\infty, \infty)$ and $h : \mathbb{R}^n \to (-\infty, \infty)$ are convex functions. The function f in (3.4) is called a *DC function* and $g - h$ is called a *DC decomposition* of f. Introduced by Tao and An [34,35], the DCA is a simple but effective optimization scheme for minimizing differences of convex functions. Although the algorithm is used for nonconvex optimization problems, the convexity of the functions involved still plays a crucial role as elements of convex analysis such as subgradients and Fenchel conjugates are involved. The algorithm is summarized below, as applied to (3.4).

It is well-known that $v \in \partial g^*(y)$ if and only if

$$v \in \operatorname{argmin}\big\{g(x) - \langle y, x\rangle \mid x \in \mathbb{R}^n\big\}.$$

Moreover, $w \in \partial h(x)$ if and only if

$$w \in \operatorname{argmin}\big\{h^*(y) - \langle y, x\rangle \mid y \in \mathbb{R}^n\big\}.$$

Thus, in the case where we cannot find x_p or y_p exactly in Algorithm 1, we can find them approximately by solving convex optimization problems.

Algorithm 1 : DCA for minimizing $f(x) = g(x) - h(x)$

INPUT: $x_1 \in \mathbb{R}^n$, $N \in \mathbb{N}$.
for $p = 1, \ldots, N$ **do**
Find $y_p \in \partial h(x_p)$.
Find $x_{p+1} \in \partial g^*(y_p)$.
end for
OUTPUT: x_{N+1}.

4 Some properties of the generalized *k*-center problem

In this section, we study some qualitative properties of the generalized k-center problem. In particular, we study sufficient conditions for the existence of optimal solutions and necessary optimality conditions with illustrative examples.

According to [26, Proposition 3.2], the smallest intersecting ball problem (**SIB**) admits a solution if one of the target set is bounded. However, in the general case of problem (**G*k*C**) the boundedness of one target set is not enough to guarantee the solution existence.

Example 4.1 Consider problem (**G*k*C**) in $\mathbb{R}^2$ with $k = 2$ and four target sets given by

$$\Lambda^1 := \left\{(x, y) \in \mathbb{R}^2 \mid x \geq 1, y \geq \frac{1}{x}\right\}, \quad \Lambda^2 := \left\{(x, y) \in \mathbb{R}^2 \mid x \geq 1, y \leq 0\right\},$$

Λ^3 and Λ^4 are two closed and bounded subsets in the region $\{(x, y) \in \mathbb{R}^2 \mid x < 0\}$ with nonempty intersection. This problem has no solution.

Theorem 4.2 below shows that if all the target sets are bounded, then the set of global solutions of the generalized k-center problem (**G*k*C**) is nonempty. Moreover, we can find a global solution in a compact domain.

Theorem 4.2 *Assume that all the target sets Λ^i are bounded. Then problem* (**G*k*C**) *admits a global optimal solution.*

Proof Consider the smallest intersecting ball problem (**SIB**) generated by $\{\Lambda^i\}_{i=1}^m$. By [26, Proposition 3.2], this problem has a solution x_* with optimal value $r := \mathcal{F}(x_*)$. Using the assumption on the boundedness of all target sets, we can choose $\delta > r$ such that

$$\bigcup_{i \in I} \Lambda^i \subset \mathbb{B}(x_*; \delta).$$

Let $\mathbf{X} = (x^1, \ldots, x^k)^\top \in \mathbb{R}^{k \times n}$. Without lost of generality, we can assume that there exists a positive integer p with $1 \leq p \leq k$ such that

$$\|x^\ell - x_*\| > 2\delta \quad \text{for all } \ell = 1, \ldots, p,$$

and

$$\|x^\ell - x_*\| \leq 2\delta \quad \text{for all } \ell = p + 1, \ldots, k.$$

For any $\ell = 1, \ldots, p$ and $i = 1, \ldots, m$, we have

$$d(x^\ell; \Lambda^i) \geq d(x^\ell; \mathbb{B}(x_*; \delta)) = \|x^\ell - x_*\| - \delta > 2\delta - \delta > r \geq d(x_*; \Lambda^i). \tag{4.1}$$

Define

$$\Omega := \left\{\mathbf{X} = (x^1, \ldots, x^k)^\top \in \mathbb{R}^{k \times n} \mid \|x^j - x_*\| \leq 2\delta \text{ for all } j \in J\right\}.$$

Then Ω is a compact subset of $\mathbb{R}^{k\times n}$. It follows from (4.1) that

$$\begin{aligned}\mathcal{F}_k\left(x^1,\ldots,x^k\right) &\geq \mathcal{F}_k\left(x_*,\ldots,x_*,x^{p+1},\ldots,x^k\right)\\ &\geq \inf\left\{\mathcal{F}_k\left(x^1,\ldots,x^k\right) \mid (x^1,\ldots,x^k)^\top \in \Omega\right\},\end{aligned}$$

where the last inequality is due to $(x_*,\ldots,x_*,x^{p+1},\ldots,x^k)^\top \in \Omega$. We thus have

$$\inf\left\{\mathcal{F}_k(\mathbf{X}) \;\mid\; \mathbf{X}\in\mathbb{R}^{k\times n}\right\} = \inf\left\{\mathcal{F}_k(\mathbf{X}) \;\mid\; \mathbf{X}\in\Omega\right\}.$$

Since $\mathcal{F}_k$ is a continuous function and Ω is a compact set, $\mathcal{F}_k$ has a global minimum on Ω by the classical Weierstrass theorem. The proof is now complete. □

It is well-known that, the smallest enclosing ball problem (**SEB**) always has a unique solution; see [26, Proposition 3.9]. This result is still valid for the smallest intersecting ball problem (**SIB**) under the additional assumption that all the target sets Λ^i are strictly convex; see [3, Theorem 3.1]. The example below shows that this property no longer holds for the case of k-center problems with $k \geq 2$. The latter usually has infinitely many solutions, the solution set may be unbounded, and there may be some centers having empty clusters.

Example 4.3 Consider the 3-center problem (k**C**) in $\mathbb{R}^2$ generated by four points $\{a^i\}_{i=1}^4$ corresponding to the columns of the matrix

$$\begin{pmatrix}-4 & -2 & 2 & 4\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Employing the collinear property of these target points, we can see that the optimal value of this problem is $R_* = 1$, and the solution set of this problem contains the unbounded set $S = \{(-3,0)\}\times\{(3,0)\}\times\mathbb{R}^2$. Moreover, if a solution $\mathbf{X}_*\in S$ has the third component $x^3 \notin \bigcup_{i=1}^4 \mathbb{B}(a^i;1)$, then $\mathcal{A}(x^3)=\emptyset$.

Remark 4.4 (a) Let $\mathbf{X}\in\mathbb{R}^{k\times n}$ be the matrix whose ℓth row is $(x^\ell)^\top$. We can easily see that

$$\mathcal{F}_k(\mathbf{X}) = R(\mathbf{X}) = \min\left\{t\geq 0 \mid \Lambda^i \cap \left[\bigcup_{j\in J}\mathbb{B}(x^j;t)\right]\neq\emptyset \;\text{ for all } i\in I\right\}.$$

(b) We also observe that solving problem (Gk**C**) reduces to finding k balls with the same smallest radius such that each of the target sets can be intersected by at least one of the balls. In fact, let $\mathbf{X}_* = (x_*^1,\ldots,x_*^k)^\top \in \mathbb{R}^{k\times n}$ and let $R_* := \mathcal{F}_k(\mathbf{X}_*)$. Then $\mathbf{X}_*$ is an optimal solution of problem (Gk**C**) if and only if the following conditions hold:

(i) For each $i\in I$, there exists $\ell\in J$ satisfying $\Lambda^i\cap\mathbb{B}(x_*^\ell;R_*)\neq\emptyset$.
(ii) For any $\mathbf{X}=(x^1,\ldots,x^k)^\top\in\mathbb{R}^{k\times n}$ and any $R\geq 0$ such that $\Lambda^i\cap\left[\bigcup_{\ell\in J}\mathbb{B}(x^\ell;R)\right]\neq\emptyset$ for all $i\in I$, we have $R_*\leq R$.

(c) Let $\mathbf{X}_* = (x_*^1,\ldots,x_*^k)^\top$ be an optimal solution of problem (Gk**C**). Then $\mathcal{F}_k(\mathbf{X}_*)=0$ if and only if $R(x_*^\ell)=0$ for all $\ell\in J$. This is equivalent to

$$x_*^\ell \in \bigcap_{i\in I(x_*^\ell)}\Lambda^i \quad \text{for all } \ell\in J.$$

This means that each center x_*^ℓ is one of the common points of all target sets in its corresponding cluster $\mathcal{A}(x_*^\ell)$. If $\mathcal{F}_k(\mathbf{X}_*)>0$, then there exists a cluster $\mathcal{A}(x_*^\ell)$ such that

$\bigcap_{i\in I(x_*^\ell)} \Lambda^i = \emptyset$ and therefore the smallest intersecting ball of this cluster has a positive radius.

An important property of problem (**SEB**) is that the solution must belong to the convex hull of all target points lying on the boundary of the smallest ball; see, e.g., [8, Theorem 3.6]. A generalization of this property for problem (**SIB**) is given in [26, Corollary 3.11]. We now extend these results and give a necessary optimality condition for the generalized k-center problem (**GkC**).

Theorem 4.5 *If $\mathbf{X}_* = (x_*^1, \ldots, x_*^k)^\top$ is an optimal solution of problem (**GkC**) with the optimal value $R_* = \mathcal{F}_k(\mathbf{X}_*) > 0$, then there exists a component x_*^ℓ satisfying the following conditions:*

(i) $\mathbb{B}(x_*^\ell; R_*)$ *is the smallest intersecting ball of the corresponding cluster* $\mathcal{A}(x_*^\ell)$,
(ii) $|I(x_*^\ell)| \geq 2$ *and* $R(x_*^\ell) = R_*$,
(iii) $x_*^\ell \in \operatorname{conv}\{P(x_*^\ell; \Lambda^i) \mid i \in \widetilde{I}(x_*^\ell)\}$, *where* $\widetilde{I}(x_*^\ell) := \{i \in I(x_*^\ell) \mid d(x_*^\ell; \Lambda^i) = R_*\}$.

Proof Let $\mathbf{X}_* = (x_*^1, \ldots, x_*^k)^\top$ be an optimal solution of problem (**GkC**) with $R_* = \mathcal{F}_k(\mathbf{X}_*)$. Since $R_* = \mathcal{F}_k(\mathbf{X}_*) = \max\{R(x_*^\ell) \mid \ell \in J\}$, we have

$$\Lambda^i \cap \mathbb{B}(x_*^\ell; R_*) \neq \emptyset \ \text{ for all } i \in I(x_*^\ell).$$

Thus, each ball $\mathbb{B}(x_*^\ell; R_*)$ intersects all the target sets in the corresponding cluster $\mathcal{A}(x_*^\ell)$. By contradiction, suppose that the ball $\mathbb{B}(x_*^\ell; R_*)$ is not the smallest intersecting ball of $\mathcal{A}(x_*^\ell)$. Then there exist $r_1, \ldots, r_k$ belonging to the interval $[0, R_*)$ and $y_*^1, \ldots, y_*^k$ such that $\mathbb{B}(y_*^\ell; r_\ell)$ is the smallest intersecting ball of $\mathcal{A}(x_*^\ell)$ for $\ell \in J$. Thus, for each $\ell \in J$, we have

$$\Lambda^i \cap \mathbb{B}(y_*^\ell; r_\ell) \neq \emptyset \ \text{ for all } i \in I(x_*^\ell).$$

Let $r := \max\{r_1, \ldots, r_k\}$. Since $\mathcal{F}_k(\mathbf{X}_*) > 0$, by Remark 4.4, $0 < r < R_*$. Moreover, since $\bigcup_{\ell\in J} I(x_*^\ell) = I$, we have

$$\Lambda^i \cap \left[\bigcup_{\ell\in J} \mathbb{B}(y_*^\ell; r)\right] \neq \emptyset \ \text{ for all } i \in I. \tag{4.2}$$

Since $\mathbf{X}_*$ is a solution, by Remark 4.4(b), we have from (4.2) that $R_* \leq r$. This is a contradiction and thus there must exist a component x_*^ℓ such that $\mathbb{B}(x_*^\ell; R_*)$ is the smallest intersecting ball of its cluster $\mathcal{A}(x_*^\ell)$.

Now assume that $\mathbb{B}(x_*^\ell; R_*)$ is the smallest intersecting ball of $\mathcal{A}(x_*^\ell)$. Using (2.3) and the minimality of R_*, we have $R(x_*^\ell) = R_*$. Moreover, since $R_* > 0$ by the assumption, cluster $\mathcal{A}(x_*^\ell)$ contains at least two sets; i.e., $|I(x_*^\ell)| \geq 2$. Since $\mathbb{B}(x_*^\ell; R_*)$ is the smallest intersecting ball of $\mathcal{A}(x_*^\ell)$, x_*^ℓ is the solution of the following problem:

$$\text{minimize} \varphi(x) := \max_{i\in I(x_*^\ell)} d(x; \Lambda^i)^2, \ x \in \mathbb{R}^n. \tag{4.3}$$

The optimal value of this problem is $\varphi(x_*^\ell) = R_*^2$. Since (4.3) is a convex problem, using optimality condition (3.1) and the subdifferential maximum rule (3.3), we have

$$0 \in \partial\varphi(x_*^\ell) = \operatorname{conv}\left\{2(x_*^\ell - P(x_*^\ell; \Lambda^j)) \mid j \in \widetilde{I}(x_*^\ell)\right\},$$

in which $\widetilde{I}(x_*^\ell) = \{j \in I(x_*^\ell) \mid d(x_*^\ell; \Lambda^j) = R_*\}$. Therefore, there exist $\lambda_j \geq 0$ for $j \in \widetilde{I}(x_*^\ell)$ with $\sum_{j\in\widetilde{I}(x_*^\ell)} \lambda_j = 1$ satisfying

$$0 = \sum_{j\in\widetilde{I}(x_*^\ell)} 2\lambda_j (x_*^\ell - P(x_*^\ell; \Lambda^j)).$$

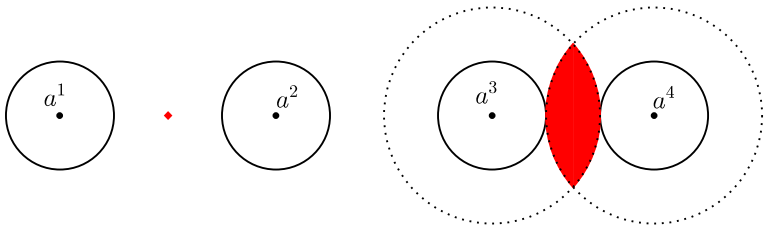

Fig. 1 Let $a^1 = (0,0)$, $a^2 = (4,0)$, $a^3 = (8,0)$, $a^4 = (11,0)$ be four points in $\mathbb{R}^2$. The solution set of the 2-center problem (**G*k*C**) generated by four balls with the same radius $r = 1$ coincides with the solution set of the 2-center problem (***k*C**) generated by four centers $\{a^i\}_{i=1}^4$. This solution set is given by $S := \left\{(x^1, x^2) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x^1 = \frac{a^1+a^2}{2}, \|x^2 - a^3\| \le 2, \|x^2 - a^4\| \le 2\right\}$

This is equivalent to $x_*^\ell = \sum_{j\in\widetilde{I}(x_*^\ell)} \lambda_j P(x_*^\ell; \Lambda^j)$. Therefore, $x_*^\ell \in \operatorname{conv}\{P(x_*^\ell; \Lambda^j) \mid j \in \widetilde{I}(x_*^\ell)\}$. The proof is now complete. □

We now consider an example of the generalized k-center problem where each target set Λ^i is a ball $\mathbb{B}(a^i, r_i)$ for $i = 1, \ldots, m$. Note that in this example, these balls have different radii.

Example 4.6 In $\mathbb{R}^2$, the 1-center problem (**SIB**) generated by two balls $\mathbb{B}((0,0); 2)$ and $\mathbb{B}((5,0); 1)$ has a unique solution $x_* = (3,0)$, while the 1-center problem (***k*C**) generated by two points $a^1 = (0,0)$ and $a^2 = (5,0)$ has a unique solution $\bar{x} = (\frac{5}{2}, 0)$.

In the next example, we consider the generalized k-center problem where the target sets Λ^i for $i = 1, \ldots, m$ are the balls $\mathbb{B}(a^i, r)$ for $i = 1, \ldots, m$ with the same radius; see Fig. 1.

Remark 4.7 Let $\{\Lambda^i := \mathbb{B}(a^i; r)\}_{i=1}^m$ be m pairwise disjoint balls with the same radius r in $\mathbb{R}^n$. Then the solution set of the k-center problem (***k*C**) generated by m points $\{a^i\}_{i=1}^m$ coincides with the solution set of the generalized k-center problem (**G*k*C**) generated by m balls $\{\Lambda^i\}_{i=1}^m$.

Indeed, the objective function of problem (**G*k*C**) with m target sets $\{\Lambda^i\}_{i=1}^m$ is given by $\mathcal{F}_k(\mathbf{X}) = \max_{i\in I} d_i(\mathbf{X})$, where

$$\begin{aligned} d_i(\mathbf{X}) &= \min\{d(x^1; \Lambda^i), \ldots, d(x^k; \Lambda^i)\} \\ &= \min\left\{\max\{\|x^\ell - a^i\| - r, 0\}, \ldots, \max\{\|x^k - a^i\| - r, 0\}\right\}. \end{aligned}$$

Given $\mathbf{X} \in \mathbb{R}^{k\times n}$, we define $\widehat{I}(\mathbf{X}) := \{i \in I \mid d_i(\mathbf{X}) > 0\}$. Let us first claim that $\mathcal{F}_k(\mathbf{X}) > 0$ for all $\mathbf{X} \in \mathbb{R}^{k\times n}$ or, equivalently, $\widehat{I}(\mathbf{X}) \neq \emptyset$ for all $\mathbf{X} \in \mathbb{R}^{k\times n}$. Indeed, if this is not the case, then there exists $\mathbf{X}$ such that $d_i(\mathbf{X}) = 0$ for all $i \in I$. Since $d_i(\mathbf{X}) = \min\{d(x^\ell; \Lambda^i) \mid \ell \in J\}$, this implies that each target set Λ^i contains at least one component x^ℓ of $\mathbf{X}$. By our standing assumption $k < m$, there exists a component belonging to at least two target sets. This contradicts the assumption that all Λ^i are pairwise disjoint. From this fact, we can rewrite the cost function as

$$\begin{aligned}\mathcal{F}_k(\mathbf{X}) &= \max_{i\in\widehat{I}(\mathbf{X})} d_i(\mathbf{X})\\ &= \max_{i\in\widehat{I}(\mathbf{X})} \min\{\|x^1 - a^i\| - r, \ldots, \|x^k - a^i\| - r\}\\ &= \max_{i\in\widehat{I}(\mathbf{X})} \min\{\|x^1 - a^i\|, \ldots, \|x^k - a^i\|\} - r\\ &= \max_{i\in I}\min_{\ell\in J} \|x^\ell - a^i\| - r = f_k(\mathbf{X}) - r.\end{aligned}$$

The last equality is satisfied since $\widehat{I}(\mathbf{X}) \neq \emptyset$ and $\min\{\|x^1 - a^i\|, \ldots, \|x^k - a^i\|\} \leq r$ for every $i \notin \widehat{I}(\mathbf{X})$. Thus, the difference between the objective function of the k-center problem (k**C**) generated by m points $\{a^i\}_{i=1}^m$ and the objective function of the generalized k-center problem (**G**k**C**) generated by m balls $\{\Lambda^i\}_{i=1}^m$ is just a constant r. Therefore, they have the same solution set.

5 Numerical solution methods

In this section, we propose two descent methods for solving the generalized k-center problem (**G**k**C**). The first method is based on the DCA and the second one is a heuristic method motivated by Theorem 4.5.

5.1 The DCA-based method for (**G**k**C**)

As previously mentioned, problem (**G**k**C**) is a continuous optimization problem with a nonsmooth and nonconvex objective function. Among available schemes, the DCA is perhaps the most suitable option for solving our problem. The DCA is a descent method without linesearch and has a rigorous convergence guarantee; see [34,35]. It is nowadays an effective tool for solving nonconvex optimization problems. We refer the reader to the comprehensive survey [2] for many applications of the DCA in different fields.

In what follows, we propose a DC representation for the objective function of problem (**G**k**C**) which is favorable for applying the DCA. We first observe that

$$d_i(\mathbf{X})^2 = \min_{1\leq\ell\leq k} d(x^\ell;\Lambda^i)^2 = \sum_{\ell=1}^k d(x^\ell;\Lambda^i)^2 - \max_{1\leq r\leq k}\sum_{\ell=1,\ell\neq r}^k d(x^\ell;\Lambda^i)^2. \tag{5.1}$$

Since $\mathcal{F}_k(\mathbf{X}) \geq 0$ for all $\mathbf{X}\in\mathbb{R}^{k\times n}$, $\mathbf{X}_*$ is a minimizer of $\mathcal{F}_k$ if and only if it is a minimizer of $\frac{1}{2}\mathcal{F}_k^2$. From (5.1), we choose the following DC decomposition for the function $\frac{1}{2}\mathcal{F}_k^2$:

$$\frac{1}{2}\mathcal{F}_k^2(\mathbf{X}) = \left[\mathcal{G}(\mathbf{X}) + \frac{\lambda}{2}\|\mathbf{X}\|_F^2\right] - \left[\mathcal{H}(\mathbf{X}) + \frac{\lambda}{2}\|\mathbf{X}\|_F^2\right],$$

where $\lambda > 0$ is a fixed parameter, $\|\mathbf{X}\|_F = \left(\sum_{\ell=1}^k \|x^\ell\|^2\right)^{\frac{1}{2}}$ is the Frobenius norm of $\mathbf{X}$ and

$$\begin{aligned}\mathcal{G}(\mathbf{X}) &:= \frac{1}{2}\max_{1\leq i\leq m}\left\{\sum_{\ell=1}^k d(x^\ell;\Lambda^i)^2 + \sum_{j=1,j\neq i}^m \max_{1\leq r\leq k}\sum_{\ell=1,\ell\neq r}^k d(x^\ell;\Lambda^j)^2\right\},\\ \mathcal{H}(\mathbf{X}) &:= \frac{1}{2}\sum_{i=1}^m \max_{1\leq r\leq k}\sum_{\ell=1,\ell\neq r}^k d(x^\ell;\Lambda^i)^2.\end{aligned}$$

Recall that to minimize the DC function $\frac{1}{2}\mathcal{F}_k^2$ associated with the above decomposition, the DCA repeatedly solves a sequence of the following subproblems:

$$\mathbf{X}_{p+1} \in \operatorname*{argmin}_{\mathbf{Y}\in\mathbb{R}^{k\times n}} \left\{ \mathcal{G}(\mathbf{Y}) + \frac{\lambda}{2}\|\mathbf{Y}\|_F^2 - \langle V_p + \lambda \mathbf{X}_p, \mathbf{Y}\rangle \right\}, \tag{5.2}$$

where $V_p \in \partial\mathcal{H}(\mathbf{X}_p)$. To proceed further, for each index $i \in I$ we define

$$\mathcal{G}_i(\mathbf{X}) := \frac{1}{2}\sum_{\ell=1}^{k} d(x^\ell; \Lambda^i)^2,$$
$$\mathcal{H}_i(\mathbf{X}) := \max_{1\le r\le k} \mathcal{H}_{i,r}(\mathbf{X}),$$

where $\mathcal{H}_{i,r}(\mathbf{X}) := \frac{1}{2}\sum_{\ell=1,\ell\neq r}^{k} d(x^\ell; \Lambda^i)^2$. From the definition, we can rewrite

$$\mathcal{G}(\mathbf{X}) = \max_{1\le i\le m} \left\{ \mathcal{G}_i(\mathbf{X}) + \sum_{j=1, j\neq i}^{m} \mathcal{H}_j(\mathbf{X}) \right\},$$
$$\mathcal{H}(\mathbf{X}) = \sum_{i=1}^{m} \mathcal{H}_i(\mathbf{X}).$$

The functions $\mathcal{G}_i$ and $\mathcal{H}_{i,r}$ are Fréchet differentiable on $\mathbb{R}^{k\times n}$ with gradients respectively given by

$$\nabla\mathcal{G}_i(\mathbf{X}) = \mathbf{X} - \mathbf{P}_i(\mathbf{X}),$$
$$\nabla\mathcal{H}_{i,r}(\mathbf{X}) = \mathbf{X} - \mathbf{P}_i(\mathbf{X}) - \mathbf{e}_r(x^r - P(x^r; \Lambda^i))^\top,$$

for $\mathbf{X} = (x^1, \ldots, x^k)^\top \in \mathbb{R}^{k\times n}$, where $\mathbf{P}_i(\mathbf{X})$ is the $k \times n$ matrix whose ℓth row is $\left[P(x^\ell; \Lambda^i)\right]^\top$ for $\ell \in J$, and $\mathbf{e}_r$ is the $k \times 1$ column vector with a one in the rth position and zeros elsewhere.

For each $i \in I$, the active index set of function $\mathcal{H}_i$ at a point $\mathbf{X}$ is defined by

$$J_i(\mathbf{X}) := \{r \in J \mid \mathcal{H}_{i,r}(\mathbf{X}) = \mathcal{H}_i(\mathbf{X})\}.$$

From the definition of $\mathcal{H}_i$ and $\mathcal{H}$, we have

$$J_i(\mathbf{X}) = \left\{ r \in J \mid d(x^r; \Lambda^i) = d_i(\mathbf{X}) \right\} = \left\{ r \in J \mid \Lambda^i \in \mathcal{A}(x^r) \right\}.$$

Employing (3.3), we have $\partial\mathcal{H}_i(\mathbf{X}) = \operatorname{conv}\{\partial\mathcal{H}_{i,r}(\mathbf{X}) \mid r \in J_i(\mathbf{X})\}$. For each $i \in I$, we pick an index $r(i)$ such that $\Lambda^i \in \mathcal{A}(x^{r(i)})$ and then we can take

$$V = \sum_{i=1}^{m} \left[\mathbf{X} - \mathbf{P}_i(\mathbf{X}) - \mathbf{e}_{r(i)}(x^{r(i)} - P(x^{r(i)}; \Lambda^i)^\top \right],$$

as an explicit subgradient of $\mathcal{H}$ at $\mathbf{X}$ by subdifferential sum rule (3.2).

To apply the DCA, we need to solve subproblem (5.2). We rewrite the objective function of this subproblem as $\Phi(\mathbf{Y}) := \max_{1\le i\le m} \Phi_i(\mathbf{Y})$, where

$$\Phi_i(\mathbf{Y}) := \mathcal{G}_i(\mathbf{Y}) + \sum_{j=1, j\neq i}^{m} \mathcal{H}_j(\mathbf{Y}) + \frac{\lambda}{2}\|\mathbf{Y}\|_F^2 - \langle V_p + \lambda \mathbf{X}_p, \mathbf{Y}\rangle.$$

Subproblem (5.2) has a unique solution because its objective function is strongly convex. However, this problem is nonsmooth and is very hard to apply some available smoothing techniques. Subgradient-based methods seem to be an appropriate choice for us. In its simplest form, the classical subgradient method is given by

$$\mathbf{Y}_{t+1} = \mathbf{Y}_t - \alpha_t W_t,$$

where $W_t \in \partial\Phi(\mathbf{Y}_t)$. It remains to show how to find a subgradient of Φ at a given point $\mathbf{Y}$. From the definition of two functions Φ and $\mathcal{G}$, we have

$$\begin{aligned}\{i \mid \Phi_i(\mathbf{Y}) = \Phi(\mathbf{Y})\} &= \{i \mid \mathcal{G}_i(\mathbf{Y}) - \mathcal{H}_i(\mathbf{Y}) = \mathcal{G}(\mathbf{Y}) - \mathcal{H}(\mathbf{Y})\}\\ &= \{i \mid d_i(\mathbf{Y}) = \mathcal{F}_k(\mathbf{Y})\}.\end{aligned}$$

Let $i \in I$ be an arbitrary index such that $d_i(\mathbf{Y}) = \mathcal{F}_k(\mathbf{Y})$. By the subdifferential maximum rule (3.3), we can choose $W \in \partial\Phi(\mathbf{Y})$ as follows

$$W = \mathbf{Y} - \mathbf{P}_i(\mathbf{Y}) + \sum_{j=1, j\neq i}^{m} \left[\mathbf{Y} - \mathbf{P}_j(\mathbf{Y}) - \mathbf{e}_{r(j)}(y^{r(j)} - P(y^{r(j)}; \Lambda^j)^\top\right] + \lambda\mathbf{Y} - (V_p + \lambda\mathbf{X}_p).$$

5.2 The k-center algorithm for (GkC)

Theorem 4.5 shows that a necessary condition for $\mathbf{X} = (x^1, \ldots, x^k)^\top \in \mathbb{R}^{k\times n}$ to be a solution of problem (**GkC**) is that there exists a component x^ℓ such that $\mathbb{B}(x^\ell; R(\mathbf{X}))$ is the smallest intersecting ball of the corresponding cluster $\mathcal{A}(x^\ell)$. This observation is the motivation for us to propose the following algorithm called the *k*-center algorithm.

The algorithm starts by selecting an initial guess $\mathbf{X} = (x^1, \ldots, x^k)^\top \in \mathbb{R}^{k\times n}$. At the current iteration $\mathbf{X}$, one computes k clusters $\mathcal{A}(x^1), \ldots, \mathcal{A}(x^k)$ associated with $\mathbf{X}$ by finding the index sets

$$I(x^\ell) = \{i \in I \mid d(x^\ell; \Lambda^i) = d_i(\mathbf{X})\} \quad \text{for } \ell \in J,$$

where $d_i(\mathbf{X})$ is defined in (2.2). After that, the next iteration $\mathbf{X}_+ = (x_+^1, \ldots, x_+^k)^\top$ is computed by reassigning each component x^ℓ with $I(x^\ell) \neq \emptyset$ to a solution of problem (**SIB**) generated by sets in $\mathcal{A}(x^\ell)$ and keeping the other components unchanged. It means that we update by the following rule:

$$\begin{aligned} x_+^\ell &\longleftarrow \operatorname*{argmin}_{x\in\mathbb{R}^n} \mathcal{F}(x) := \max_{i\in I(x^\ell)} d(x; \Lambda^i), && \text{if } I(x^\ell) \neq \emptyset,\\ x_+^\ell &\longleftarrow x^\ell, && \text{if } I(x^\ell) = \emptyset.\end{aligned}$$

One repeats this process until there is no difference between two successive iterations, i.e., $\mathbf{X}_+ = \mathbf{X}$.

Algorithm 2 : k-center algorithm for solving (GkC)

INPUT: Target sets Λ^i for $i \in I$ and the number of centers to be found k.
INITIALIZATION: Initial guess $\mathbf{X} = (x^1, \ldots, x^k)^\top \in \mathbb{R}^{k\times n}$ and tolerance $\varepsilon > 0$.
For $p = 1, \ldots, \infty$ **do**
1. Find the clusters $\mathcal{A}(x^\ell)$ for $\ell = 1, \ldots, k$.
2. Update centers:
For $\ell = 1, \ldots, k$ **do**
If $I(x^\ell) \neq \emptyset$, **then** $x_+^\ell \longleftarrow \operatorname{argmin}_{x\in\mathbb{R}^n} \mathcal{F}(x) = \max_{i\in I(x^\ell)} d(x; \Lambda^i)$
else $x_+^\ell \longleftarrow x^\ell$.
end
3. Check the termination condition and reassign: **If** $\|\mathbf{X}_+ - \mathbf{X}\|_F < \varepsilon$, **then** break
else $\mathbf{X} \longleftarrow \mathbf{X}_+$.
end
OUTPUT: $\mathbf{X}_+$

The main step in the k-center algorithm is solving the smallest intersecting ball problem (**SIB**) generated by sets in the cluster $\mathcal{A}(x^\ell)$:

$$\text{minimize } \mathcal{F}(x) := \max_{i\in I(x^\ell)} d(x; \Lambda^i),\ x \in \mathbb{R}^n, \tag{5.3}$$

for each ℓ with $I(x^\ell) \neq \emptyset$. This convex subproblem is nonsmooth and cannot be solved in a closed form. To overcome the slow convergence rate of the classical subgradient method, we use a scheme proposed in [3]. In the first step, one approximates the objective function $\mathcal{F}$ by the *log-exponential smoothing function* given by

$$\mathcal{F}_\mu(x) := \mu \ln \sum_{i\in I(x^\ell)} \exp\left(\frac{\sqrt{d(x; \Lambda^i)^2 + \mu^2}}{\mu}\right)$$

where $\mu > 0$ is a smoothing parameter. Then we try to minimize $\mathcal{F}_\mu$ with a sufficient small value of μ by employing the *majorization-minimization algorithm* [19,22].

Given a starting point x_0, the *MM sequence* $\{x_p\}_{p=1}^\infty$ is recursively defined by

$$x_{p+1} := \operatorname*{argmin}_{x\in\mathbb{R}^n} \mathcal{G}_\mu(x, x_p),$$

where

$$\mathcal{G}_\mu(x, x_p) := \mu \ln \sum_{i\in I(x^\ell)} \exp\left(\frac{\sqrt{\|x - P(x_p; \Lambda^i)\|^2 + \mu^2}}{\mu}\right)$$

is a *surrogate function* of $\mathcal{F}_\mu(x)$ at x_p that satisfies the following two conditions:

$$\mathcal{F}_\mu(x_p) = \mathcal{G}_\mu(x_p, x_p) \quad \text{and} \quad \mathcal{F}_\mu(x) \le \mathcal{G}_\mu(x, x_p) \quad \text{for all } x \in \mathbb{R}^n.$$

For fixed μ and x_p, the function $\mathcal{G}_\mu(x, x_p)$ with variable x is a convex differentiable function on $\mathbb{R}^n$ with

$$\nabla_x \mathcal{G}_\mu(x, x_p) = \sum_{i\in I(x^\ell)} \frac{\lambda_\mu^i(x, x_p)}{\mathcal{G}_\mu^i(x, x_p)} \left(x - P(x_p; \Lambda^i)\right),$$

where

$$\mathcal{G}_\mu^i(x, x_p) := \sqrt{\|x - P(x_p; \Lambda^i)\|^2 + \mu^2},$$
$$\lambda_\mu^i(x, x_p) := \frac{\exp\left(\mathcal{G}_\mu^i(x, x_p)/\mu\right)}{\sum_{i \in I(x^\ell)} \exp\left(\mathcal{G}_\mu^i(x, x_p)/\mu\right)}.$$

Moreover, the gradient $\nabla_x \mathcal{G}_\mu(x, x_p)$ is Lipschitz continuous with constant $L = \frac{2}{\mu}$; see [39, Proposition 2]. For solving each subproblem (5.3), one can apply Nesterov's accelerated gradient method [29]. The final algorithm is outlined in Algorithm 3; see [3] for more details.

Algorithm 3 : MM algorithm for solving subproblem (5.3)

INPUT: Target sets Λ^i for $i \in I(x^\ell)$.
INITIALIZATION: $x_0 \in \mathbb{R}^n$, $\mu_0 > \mu_\infty > 0$, $0 < \sigma < 1$, $\varepsilon > 0$.
Set $y \longleftarrow x_0$, $\mu \longleftarrow \mu_0$
Repeat
1. Use Nesterov's accelerated gradient method to solve approximately
$$y \longleftarrow \operatorname{argmin}_{x \in \mathbb{R}^n} \mathcal{G}_\mu(x, y)$$
with stopping criteria $\|\nabla_x \mathcal{G}_\mu(x, y)\| < \varepsilon$.
2. Update $\mu \longleftarrow \sigma\mu$.

until $\mu < \mu_\infty$.
OUTPUT: y

Proposition 5.1 *Let $\{\mathbf{X}_p\}_{p=1}^\infty$ be the iterative sequence generated by the k-center algorithm. The sequence of objective function values $\{\mathcal{F}_k(\mathbf{X}_p)\}_{p=1}^\infty$ is convergent.*

Proof Let $\mathbf{X}$ be the current iteration point of the k-center algorithm. We first show that each step of the algorithm drives the objective function downhill, i.e., $\mathcal{F}_k(\mathbf{X}_+) \le \mathcal{F}_k(\mathbf{X})$. For a fixed $i \in I$, there exists $\ell \in J$ such that Λ^i is closest to the center x^ℓ. This means $d(x^\ell; \Lambda^i) = d_i(\mathbf{X}) = \min\{d(x^j; \Lambda^i) \mid j \in J\}$ or, equivalently, $i \in I(x^\ell)$. We now consider two cases:

1. If $I(x^\ell) = \emptyset$, then $x_+^\ell = x^\ell$. It follows that $d(x_+^\ell; \Lambda^i) = d(x^\ell; \Lambda^i) \le R(x^\ell) \le R(\mathbf{X}) = \mathcal{F}_k(\mathbf{X})$.
2. If $I(x^\ell) \ne \emptyset$, then x_+^ℓ is a solution of the subproblem (5.3) with the optimal value denoted by $\mathcal{F}_+^\ell := \max_{i \in I(x^\ell)} d(x_+^\ell; \Lambda^i)$. This means that $\mathbb{B}(x_+^\ell; \mathcal{F}_+^\ell)$ is the smallest ball that intersects all of the sets in $\mathcal{A}(x^\ell)$. Combining this fact with (2.3), we can conclude that $\mathcal{F}_+^\ell \le R(x^\ell) \le R(\mathbf{X}) = \mathcal{F}_k(\mathbf{X})$. Moreover, since $\Lambda^i \cap \mathbb{B}(x_+^\ell; \mathcal{F}_+^\ell) \ne \emptyset$, we also have $d(x_+^\ell; \Lambda^i) \le \mathcal{F}_+^\ell$.

Thus, in both cases we always have $d(x_+^\ell; \Lambda^i) \le \mathcal{F}_k(\mathbf{X})$. This implies that

$$d_i(\mathbf{X}_+) = \min\left\{d(x_+^j; \Lambda^i) \mid j \in J\right\} \le d(x_+^\ell; \Lambda^i) \le \mathcal{F}_k(\mathbf{X}).$$

As i is arbitrarily taken in I, we conclude that $\mathcal{F}_k(\mathbf{X}_+) = \max\{d_i(\mathbf{X}_+) \mid i \in I\} \le \mathcal{F}_k(\mathbf{X})$. Thus, the sequence $\{\mathcal{F}_k(\mathbf{X}_p)\}_{p=1}^\infty$ is monotone decreasing and bounded from below by 0. Therefore, it is convergent. The proof is now complete. □

Algorithm 4 : Farthest-first traversal algorithm

INPUT: Target sets Λ^i for $i \in I$ and number of centers k.
Find $c \longleftarrow \operatorname{argmin}_{x\in\mathbb{R}^n} \mathcal{F}(x) = \max_{i\in I} d(x;\Lambda^i)$
Pick randomly $r \in I$ and set $x^1 := P(c;\Lambda^r)$
Set $j := 2$
While $j \le k$ **do**
Find $t \longleftarrow \operatorname*{argmax}_{i\in I} \min\left\{d(x^1;\Lambda^i),\ldots,d(x^{j-1};\Lambda^i)\right\}$
Set $x^j := P(c;\Lambda^t)$
Set $j := j+1$
end
OUTPUT: $\mathbf{X} = (x^1,\ldots,x^k)^\top \in \mathbb{R}^{k\times n}$

The performance of the k-center algorithm is quite sensitive to the choice of the initial guess. For our problem, we use an adapted version of the *farthest-first traversal algorithm* proposed by Gonzalez [13]. One starts by finding the center c of problem (**SIB**) generated by all the target sets and takes the first center as the projection of c onto an arbitrary target set. Then one finds the set that is farthest from the centers chosen so far and adds the projection of c onto this set as the next center. One repeats this process until k centers are found. The pseudocode is shown in Algorithm 4.

6 Numerical experiments

In this section, we implement proposed algorithms to solve the generalized k-center problem in a number of examples. In each of the following examples, we set $\lambda = 0.1$ for the DCA and use the termination condition $\|\mathbf{X}_+ - \mathbf{X}\|_F < 10^{-10}$ for Algorithm 2. All the tests are implemented by MATLAB R2016b on a personal computer with an Intel Core i5 CPU 1.6 GHz and 4G of RAM.

Example 6.1 Let us first consider the 4-center problem in $\mathbb{R}^3$ generated by 10 cubes with their centers given by the columns of the following matrix

$$\begin{pmatrix} 29 & 42 & 37 & 79 & 14 & 91 & 29 & 50 & 79 & 68 \\ 6 & 28 & 6 & 38 & 36 & 21 & 28 & 40 & 6 & 31 \\ 34 & 32 & 8 & 18 & 1 & 14 & 6 & 19 & 23 & 32 \end{pmatrix}$$

and the respective radii given by $[2, 3, 4, 4, 3, 4, 1, 4, 2, 1]$. We run the DCA and the k-center algorithm (Algorithm 2) using different starting points generated by Algorithm 4. Both algorithms yield the same approximate solution

$$\mathbf{X}_* = \begin{pmatrix} 64.500 & 35.965 & 22.0569 \\ 35.692 & 14.875 & 23.1380 \\ 22.500 & 31.000 & 4.5010 \\ 84.000 & 12.500 & 19.500 \end{pmatrix}$$

with the optimal value $\mathcal{F}_k(\mathbf{X}_*) = 12.158$; see Fig. 2.

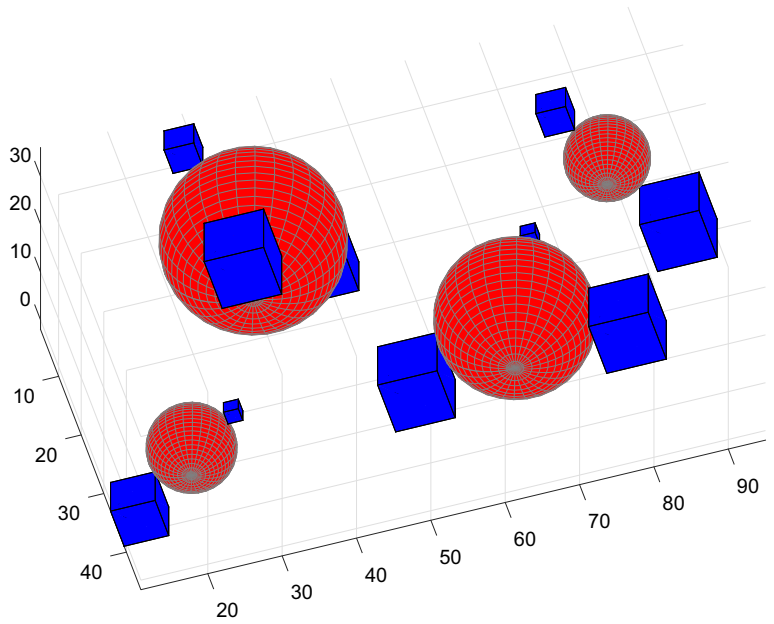

Fig. 2 A generalized 4-center problem generated by 10 cubes in $\mathbb{R}^3$

Example 6.2 In this example, Algorithm 2 is applied to solve the classical k-center problem ($k\mathbf{C}$) on 6 datasets from the TSP-Library ($m = 76, 101, 130, 195, 535, 575$) using value of k ranging from k=10–50 with an increment of 10. All of these datasets are available at http://elib.zib.de/pub/mp-testdata/tsp/tsplib/tsp/index.html. For each dataset and for each k, the best optimal radius $\mathcal{F}_k^{\mathbf{best}}$ with starting points generated by Algorithm 4, the running time in seconds and the number of iterations when $\mathcal{F}_k^{\mathbf{best}}$ was found are reported in Table 1.

Example 6.3 We now consider the latitudes and longitudes of the 50 most populous US cities taken from 2014 United States Census Bureau data.[1] Each city is approximated by a ball with radius $0.1\sqrt{A/\pi}$, where A is the city's reported area in square miles. We solve the generalized k-center problem generated by these 50 balls by the k-center algorithm and the DCA. For each k from 4 to 20 with an incremental step of 4, we run both algorithms using 100 different random starting points with coordinates in $(-122.65, -71.0202) \times (25.7752, 47.6205)$ (the range of the geographical coordinates of the cities). We terminate the DCA whenever the relative error $\|\mathbf{X}_+ - \mathbf{X}\|_F$ is smaller than 10^{-3}. The overall best objective value $\mathcal{F}_k^{\mathbf{best}}$, the average of objective values $\mathcal{F}_k^{\mathbf{mean}}$ for the 100 runs and the number of iterations when $\mathcal{F}_k^{\mathbf{best}}$ was found are reported in Table 2.

The best approximate solution obtained by the k-center algorithm when $k = 8$ is plotted in Fig. 3 using a *plate Carrée projection*[2]. This solution is given below with an approximate objective value $\mathcal{F}_k(\mathbf{X}_*) = 4.639280$:

[1] https://en.wikipedia.org/wiki/List_of_United_States_cities_by_population.

[2] https://www.mathworks.com/help/map/pcarree.html.

Table 1 Performance of the k-center algorithm on TSP-Lib datasets

Data	$k = 10$			$k = 20$			$k = 30$			$k = 40$			$k = 50$		
	$\mathcal{F}_k^{\mathbf{best}}$	Iter.	Time	$\mathcal{F}_k^{\mathbf{best}}$	Iter.	Time	$\mathcal{F}_k^{\mathbf{best}}$	Iter.	Time	$\mathcal{F}_k^{\mathbf{best}}$	Iter.	Time	$\mathcal{F}_k^{\mathbf{best}}$	Iter.	Time
eil76	11.8761	5	1.152	8.6708	4	0.907	6.5000	3	0.420	4.6098	3	0.301	3.8079	2	0.118
eil101	12.8062	7	2.283	8.8168	5	1.562	7.2364	5	1.374	6.0207	3	0.650	5.4083	3	0.537
ch130	127.5371	7	3.596	87.3589	4	1.807	59.5482	5	2.137	52.8589	4	1.419	40.0935	4	1.157
rat195	37.3913	7	4.266	25.3743	9	5.965	20.8626	8	5.582	17.8885	6	4.058	15.8828	4	2.439
ali535	30.1219	10	19.920	19.0517	9	17.447	15.3754	8	14.353	12.3938	8	8.390	10.2945	7	12.365
rat575	70.1082	9	12.414	48.1006	14	30.533	37.9789	7	14.928	33.6341	8	10.498	30.3568	9	11.737

Table 2 Performance of DCA and k-center algorithm

k	DCA			k-center algorithm		
	$\mathcal{F}_k^{\mathbf{best}}$	$\mathcal{F}_k^{\mathbf{mean}}$	# Iterations	$\mathcal{F}_k^{\mathbf{best}}$	$\mathcal{F}_k^{\mathbf{mean}}$	# Iterations
4	9.645034	11.490791	117	8.947238	9.594627	7
8	4.781154	5.868804	168	4.639280	5.162225	6
12	3.175647	4.156655	109	3.174503	3.621956	4
16	2.096450	2.720009	215	2.064151	2.515924	7
20	1.598951	2.085603	190	1.596295	1.974087	8

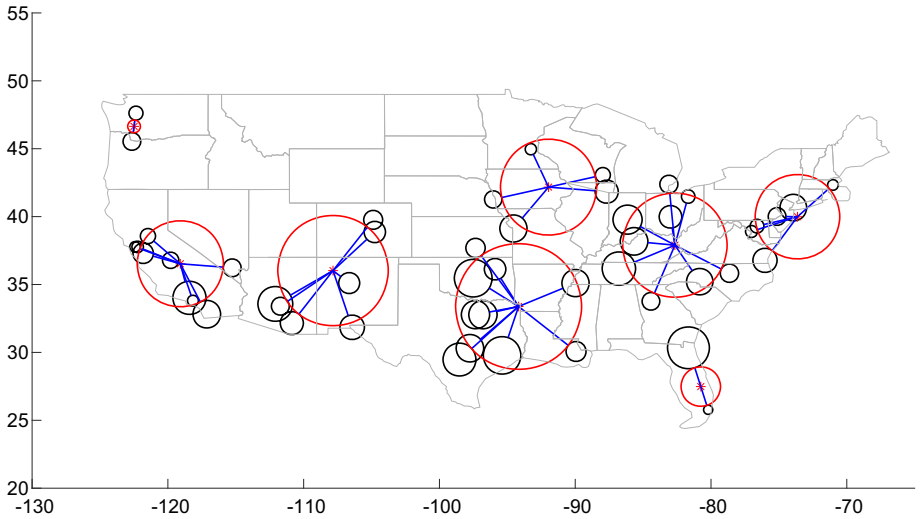

Fig. 3 A generalized 8-center problem for 50 most populous US cities

$$\mathbf{X}_* = \begin{pmatrix} -91.963 & 42.171 \\ -119.084 & 36.526 \\ -80.752 & 27.482 \\ -107.838 & 36.024 \\ -73.632 & 39.992 \\ -94.171 & 33.380 \\ -82.658 & 37.896 \\ -122.491 & 46.645 \end{pmatrix}.$$

Table 2 shows that in almost all cases (except when $k = 4$), the best values $\mathcal{F}_k^{\mathbf{best}}$ obtained by two methods for 100 runs are almost the same. However, the average value $\mathcal{F}_k^{\mathbf{mean}}$ and the number of required iterations of the k-center algorithm is much smaller than that of the DCA.

7 Conclusion

In this work we proposed a generalized version of the continuous k-center problem. The intrinsic nonsmoothness and nonconvexity of its optimization model make the problem very difficult to solve. We provided our first effort in studying this problem from both theoretical and numerical aspects. We particularly focused on numerical algorithms for solving the problem using mathematical programming and heuristic optimization approaches. The first approach involves a DCA-based method with mathematical programming formulation. In the second approach, we introduced a heuristic algorithm based on available algorithms for solving the smallest intersecting ball problems. The DCA-based method is slow due to the fact that a subproblem must be solved in each iteration of the DCA using the subgradient method, which is known to be slow. Meanwhile, the heuristic algorithm does not guarantee to find a global optimal solution. We would like overcome these shortcomings in our future research by developing more effective algorithms for solving the generalized k-center problem with the use of more tools such as mixed integer programming, accelerated versions of the DCA (see [4]), and the combination of the DCA and derivative-free methods. As suggested by one of the referees, it would be interesting to apply the numerical methods developed in this paper for the k-center problems on the sphere with geodesic distances, see [7,10] and the references therein. The challenge in this new research direction would be the possibility that the distortion of real-world distances could get worse near the poles.

Acknowledgements The authors are very grateful to the two anonymous referees for taking their valuable time to read and give us invaluable comments that allowed us to improve the presentation and the content of the paper.

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