Computational Geometry
Chapter 4: Voronoi Diagrams

Prof. Dr. Sándor Fekete

Algorithms Division
Department of Computer Science
TU Braunschweig

Technische Universität Braunschweig

1. Introduction and Motivation
2. Definitions
3. Representing planar partitions
4. Properties
5. Fortune's algorithm
6. Variations
7. The Voronoi game
8. Summary and conclusions

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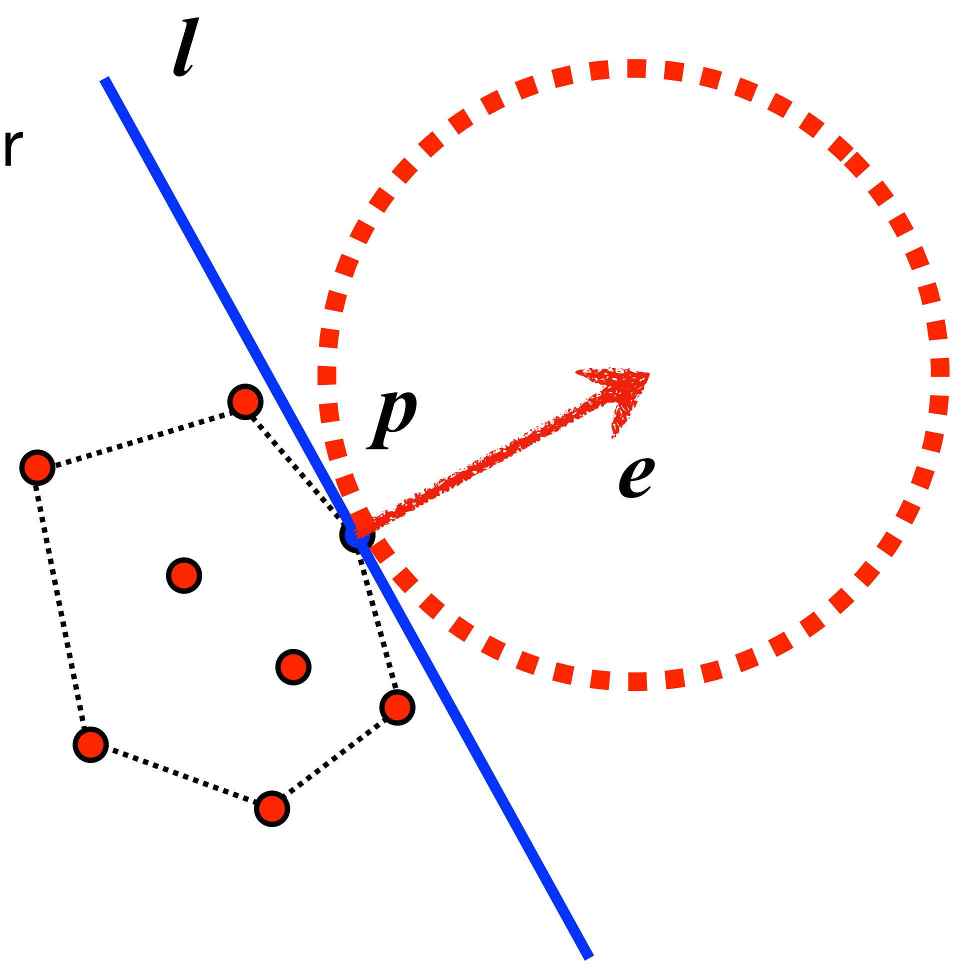
Lemma 4.13

$p \in \mathcal{P}$ lies on boundary of $\text{conv}(\mathcal{P}) \Leftrightarrow V(p)$ unbounded.

Corollary 4.14:

Computing the Voronoi diagram for n points has a lower bound of $\Omega(n \log n)$.



VIRONOI MAN

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Approach:

- Consider a moving „frontier“ between resolved and unresolved part.

Crucial issue:

- $p \in \mathcal{P}$ below ℓ can influence $Vor(p)$ above ℓ.

Observation:

- The separation between resolved and unresolved part for a point p and line ℓ is a curve consisting of points that have equal distance from p and ℓ.

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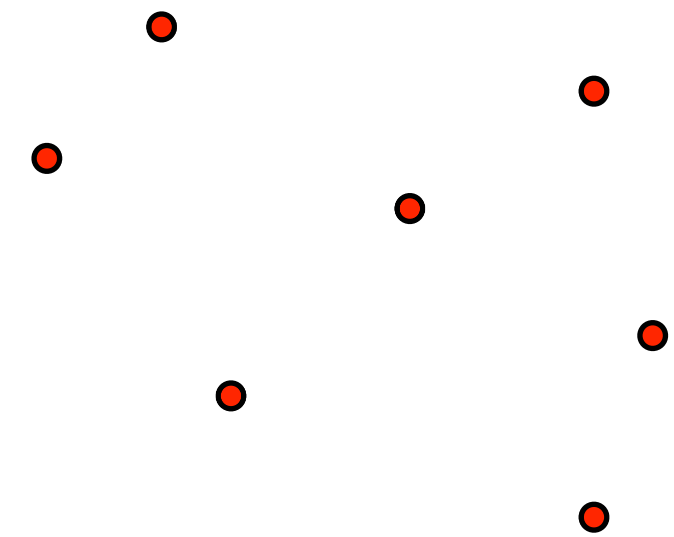

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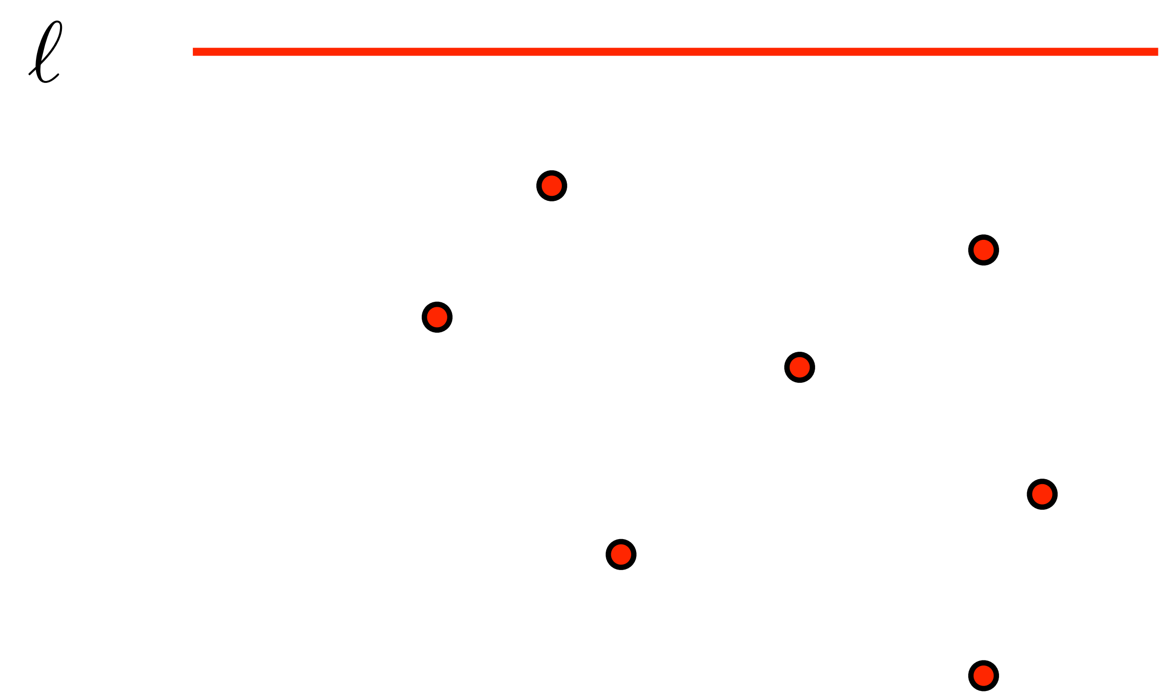

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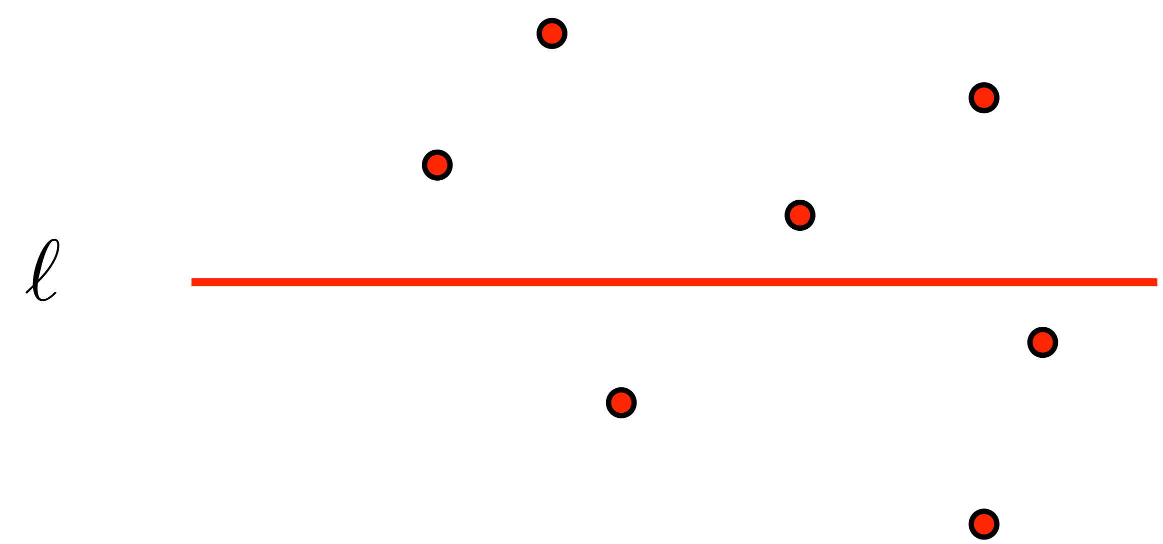

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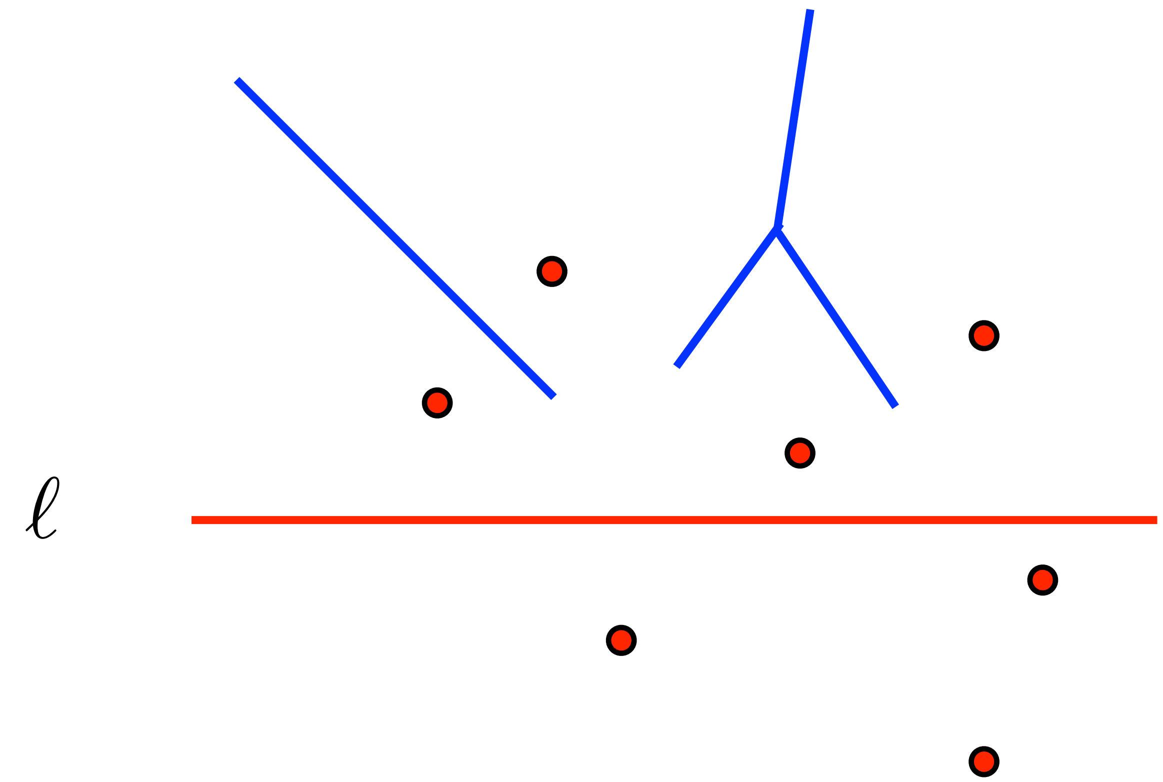

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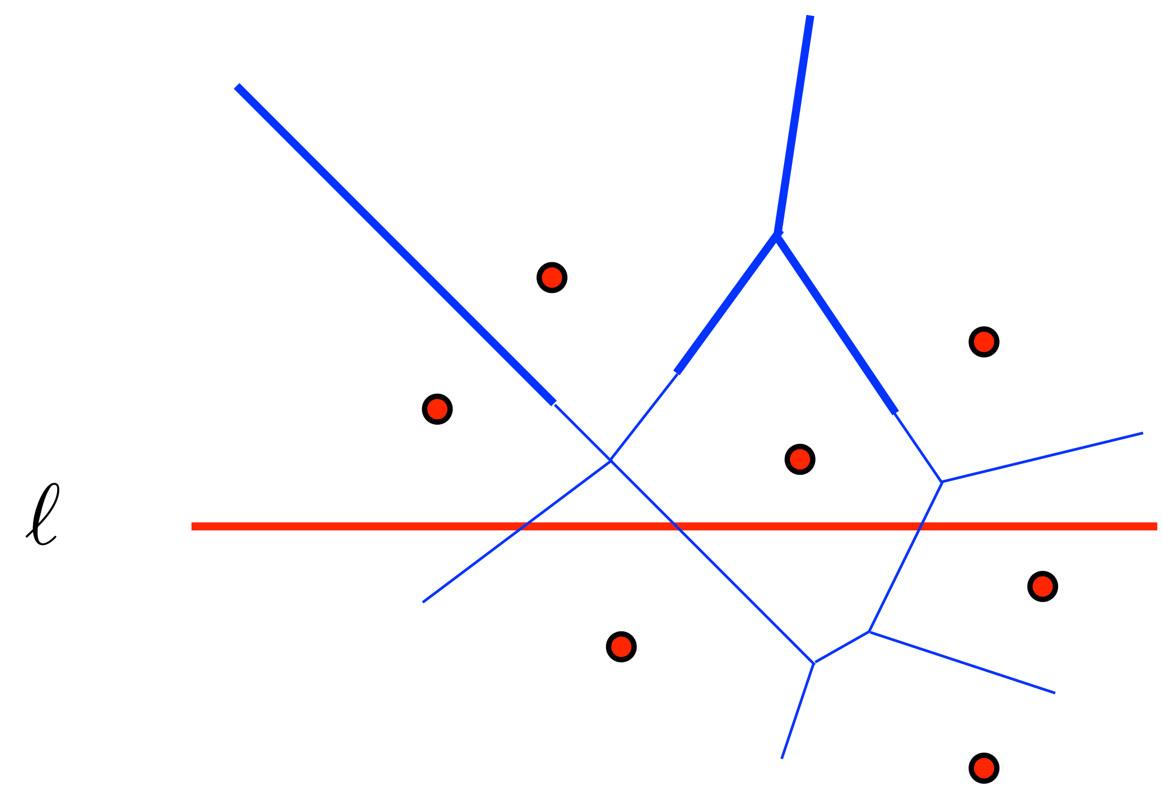

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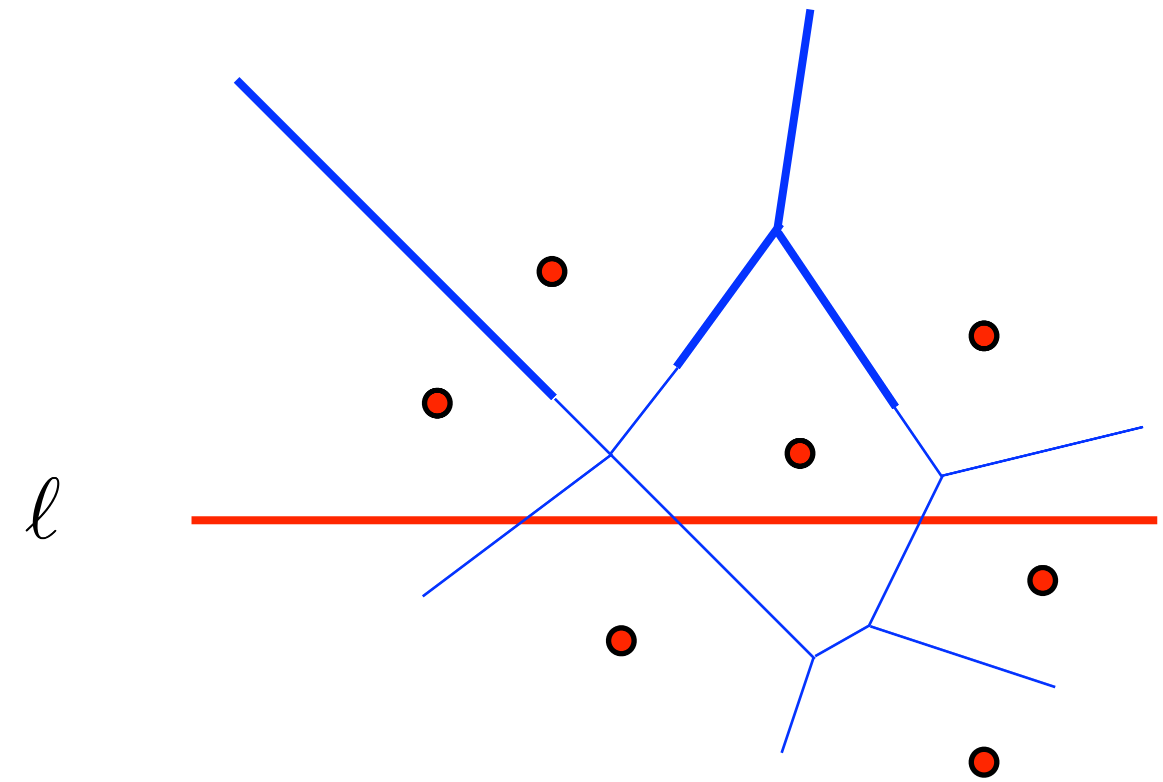

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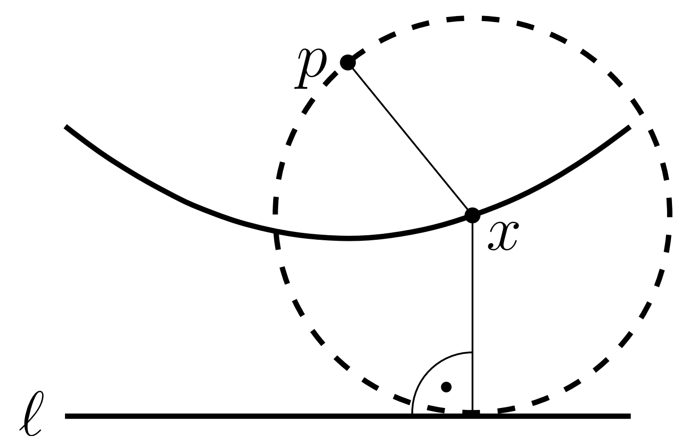

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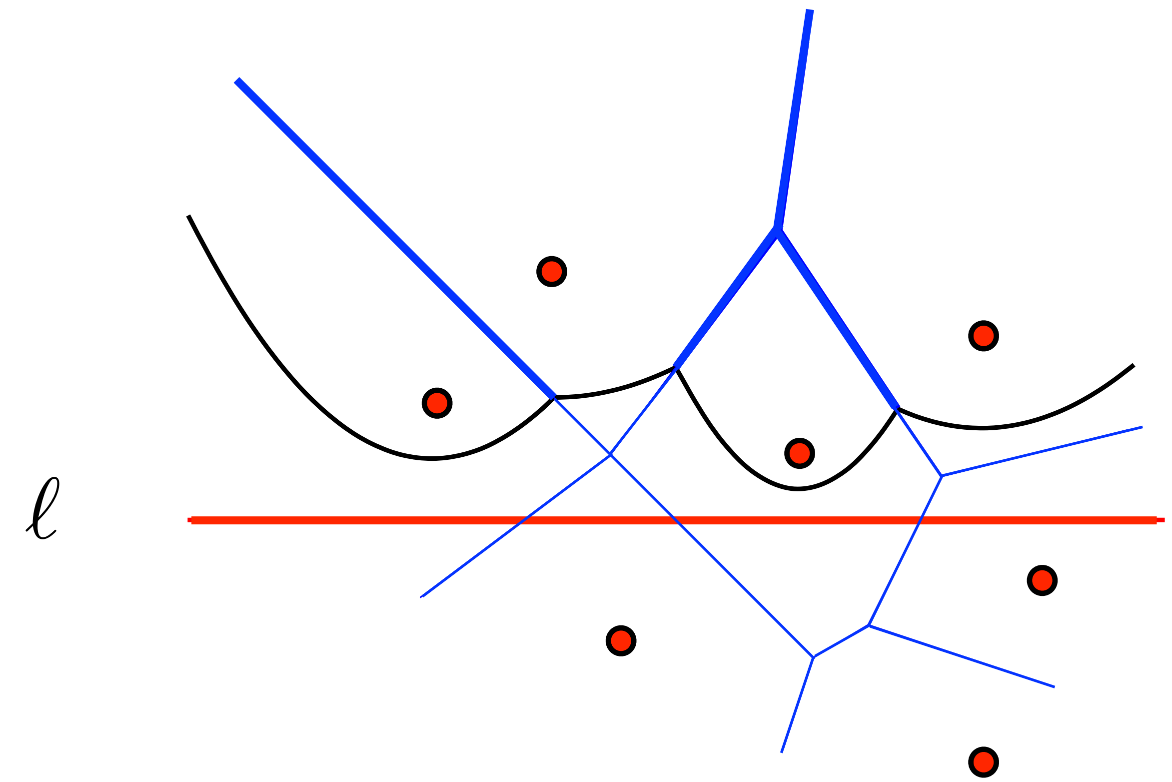

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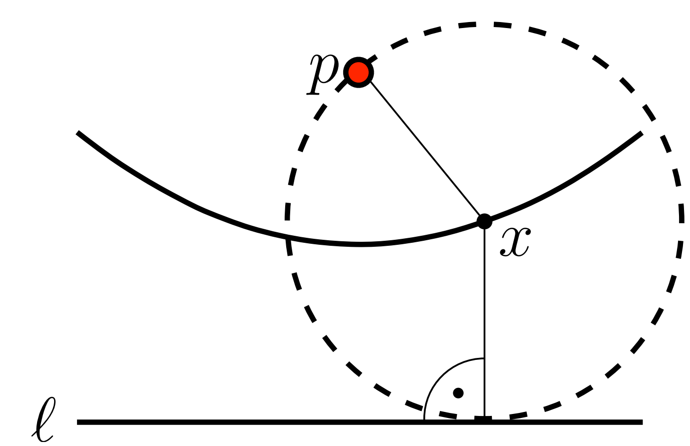

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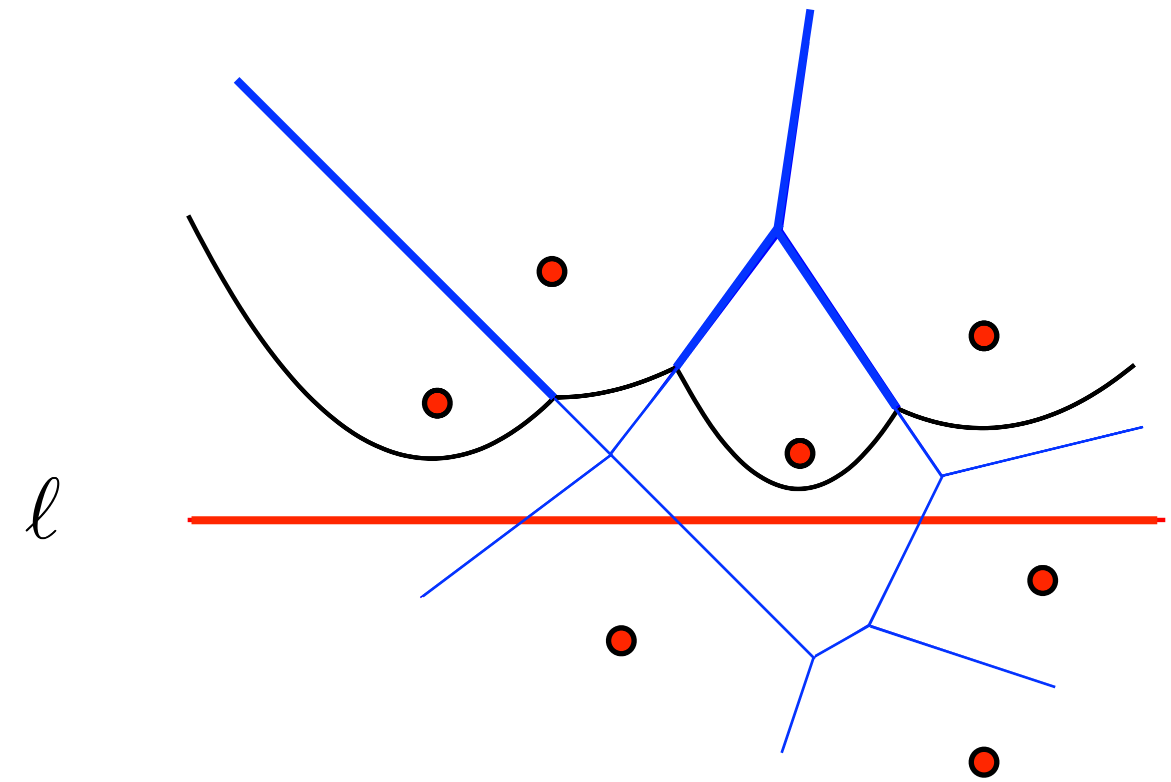

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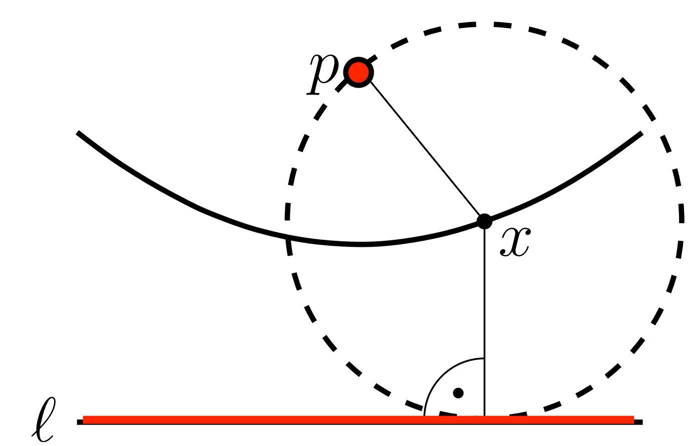

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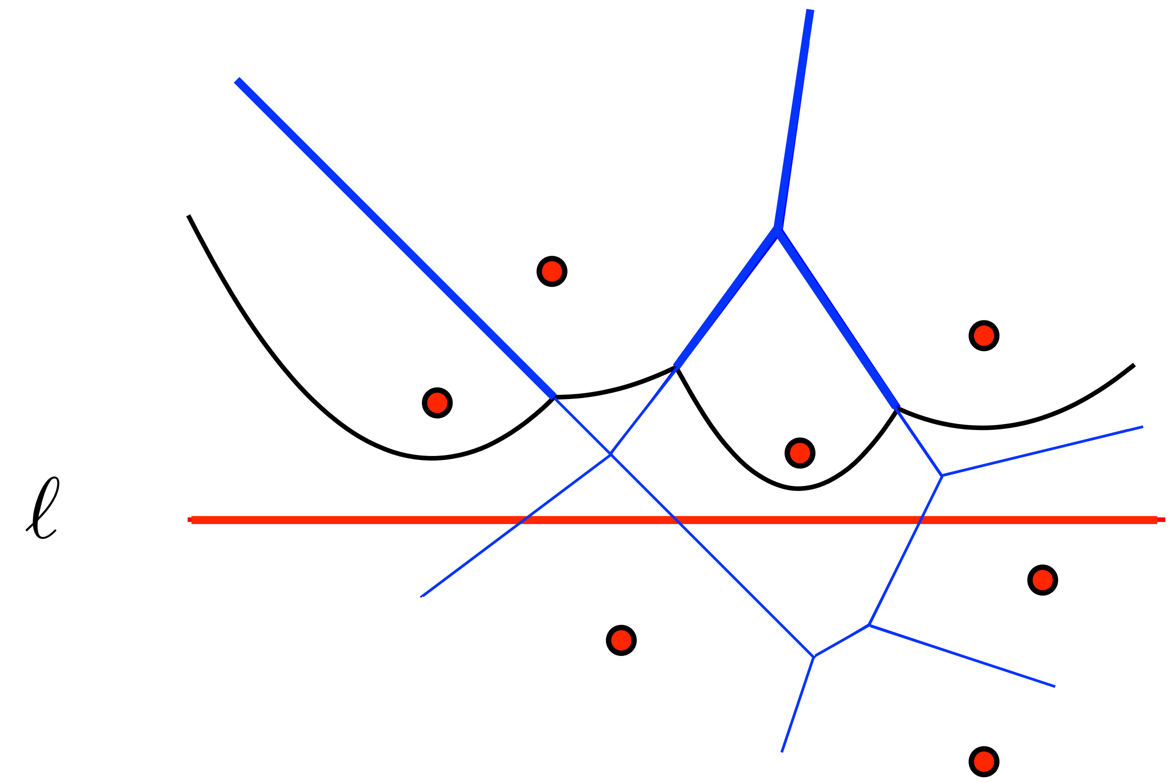

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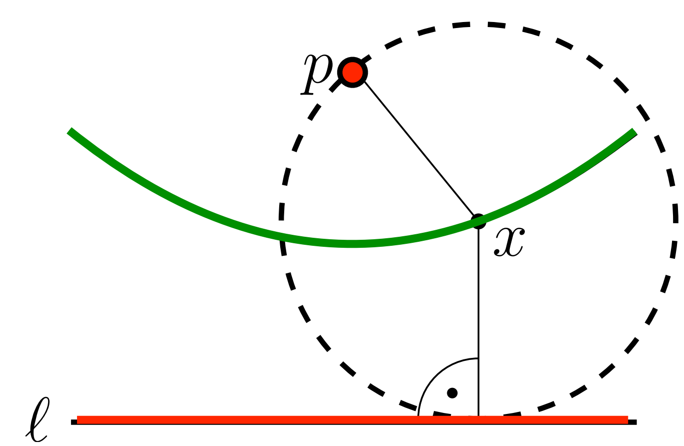

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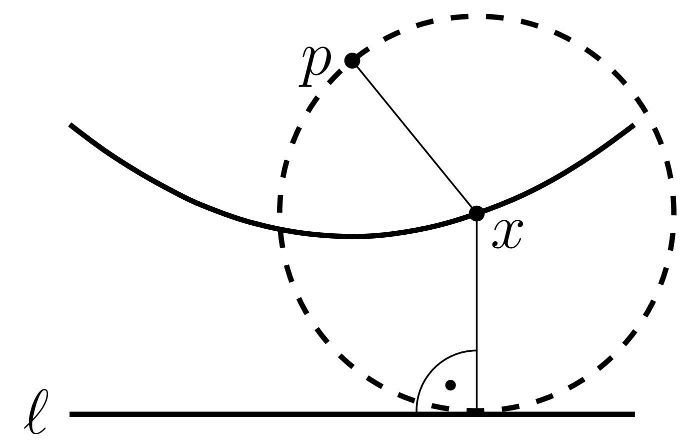
$$\{x \in \mathbb{R}^2 \mid d(x,p) = d(x,\ell)\}$$



Theorem 4.15:

The curve is a parabola (with *focus* p and *directrix* ℓ).

Proof:

Consider $p=(0,s)$ and $X=(x,0)$.
Then $C=(x,y)$ with

$$d_1^2 = x^2 + (y-s)^2$$
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So

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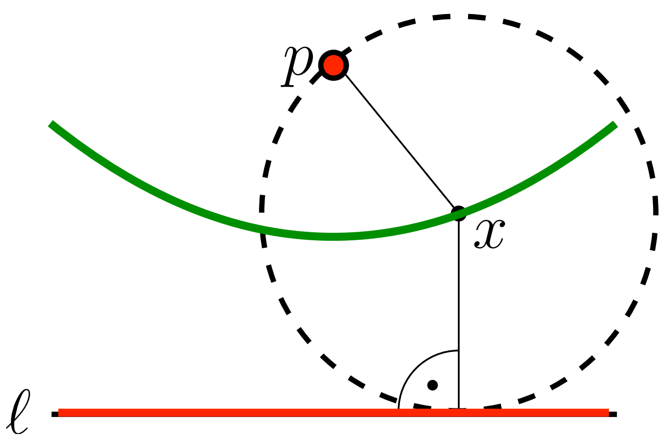

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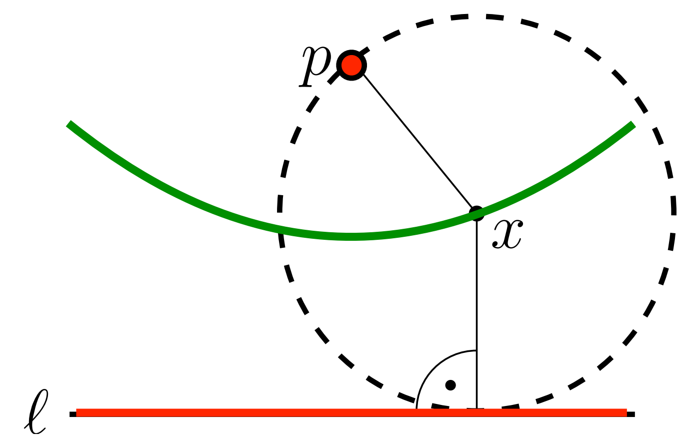

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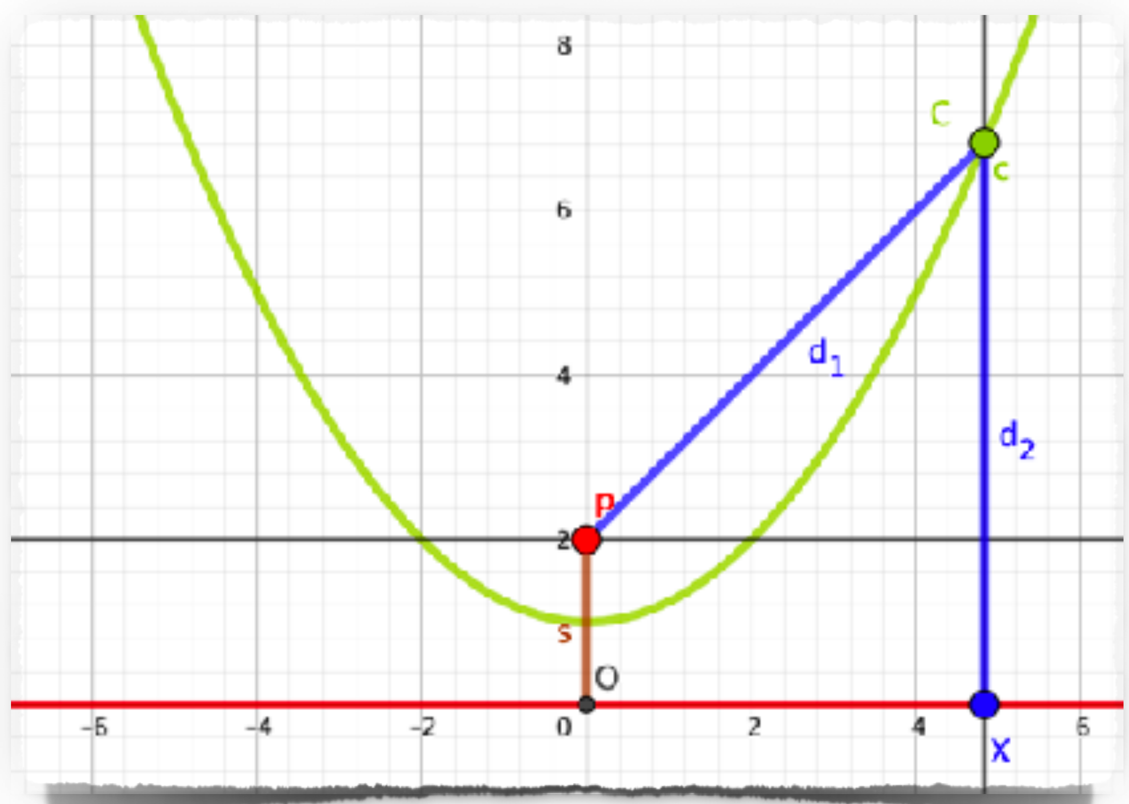

The Separating Curve

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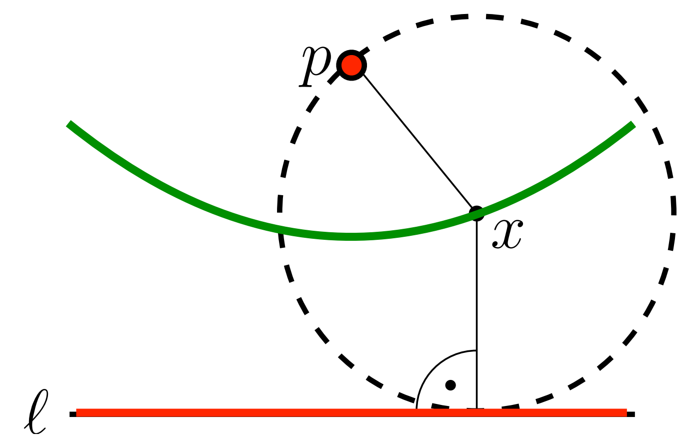

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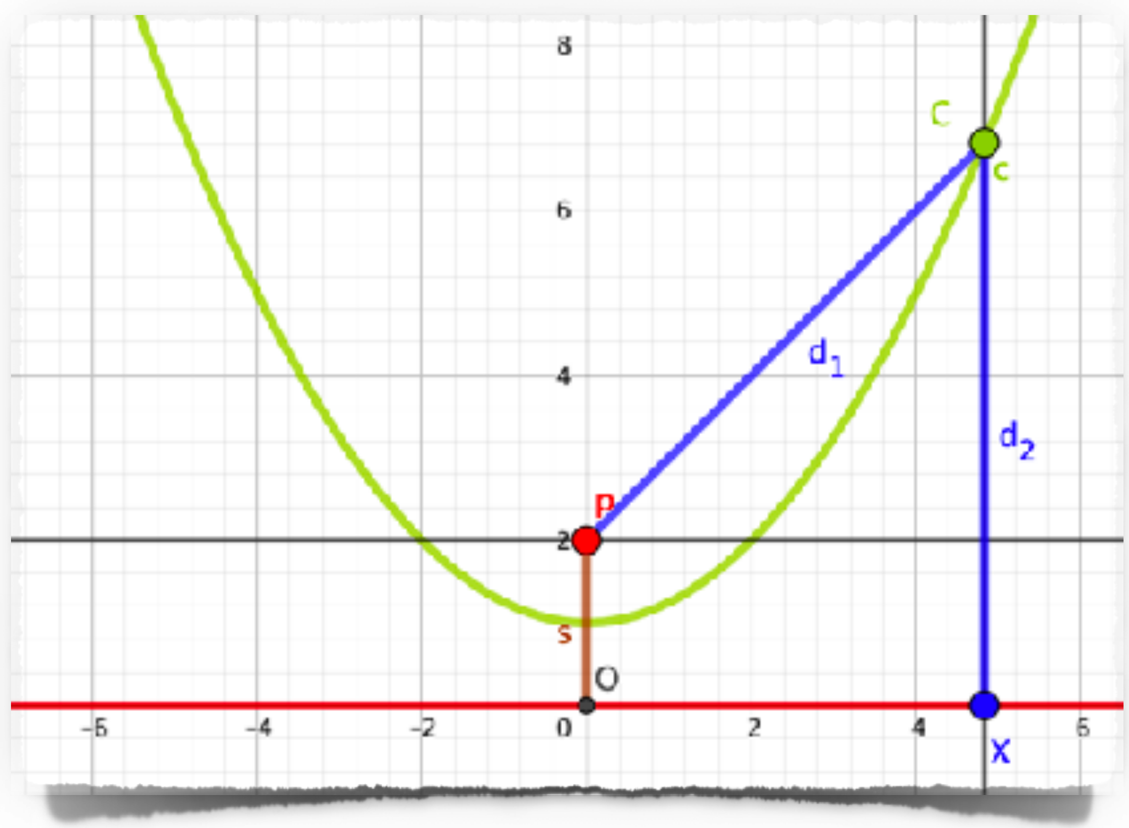

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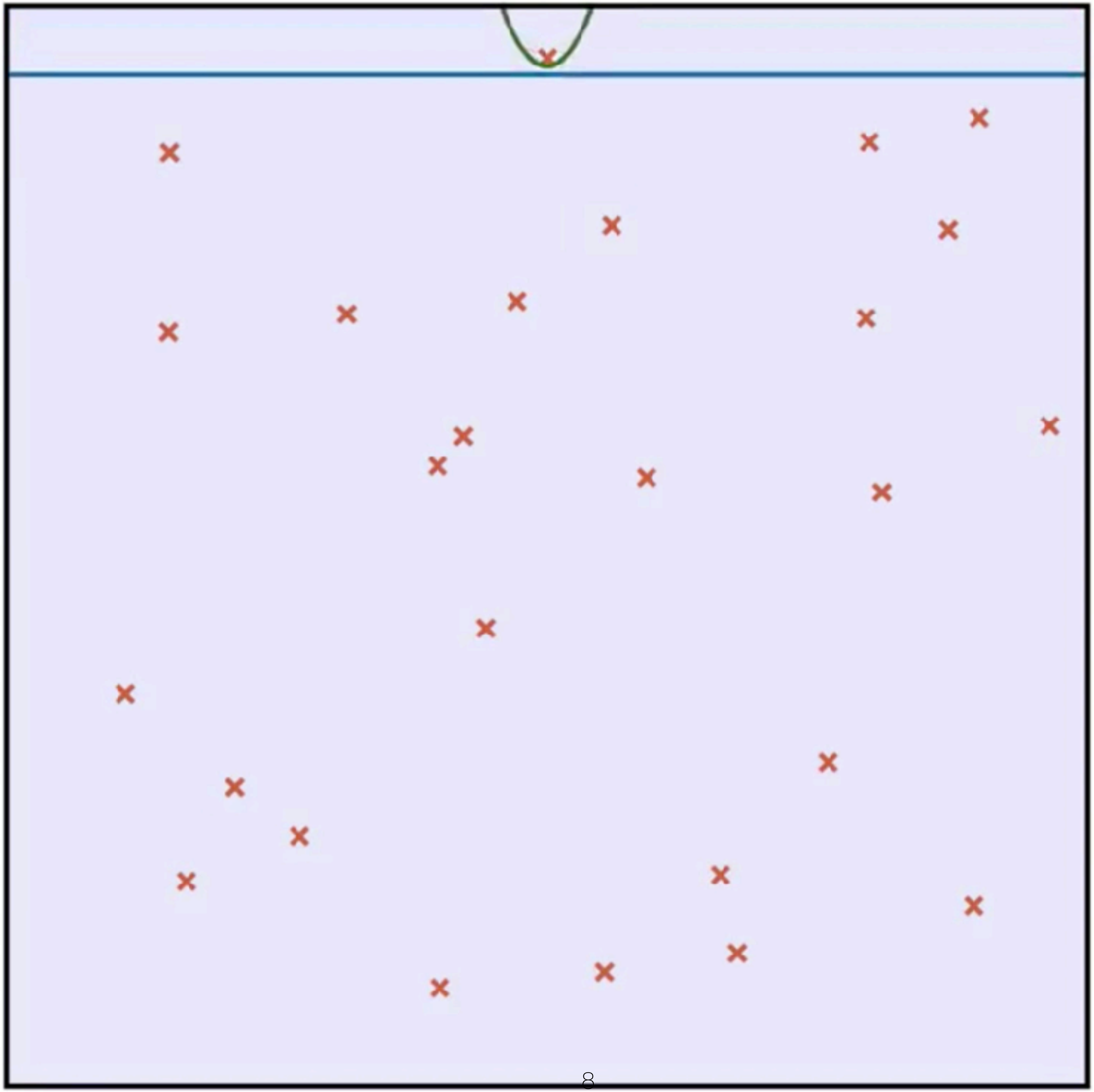

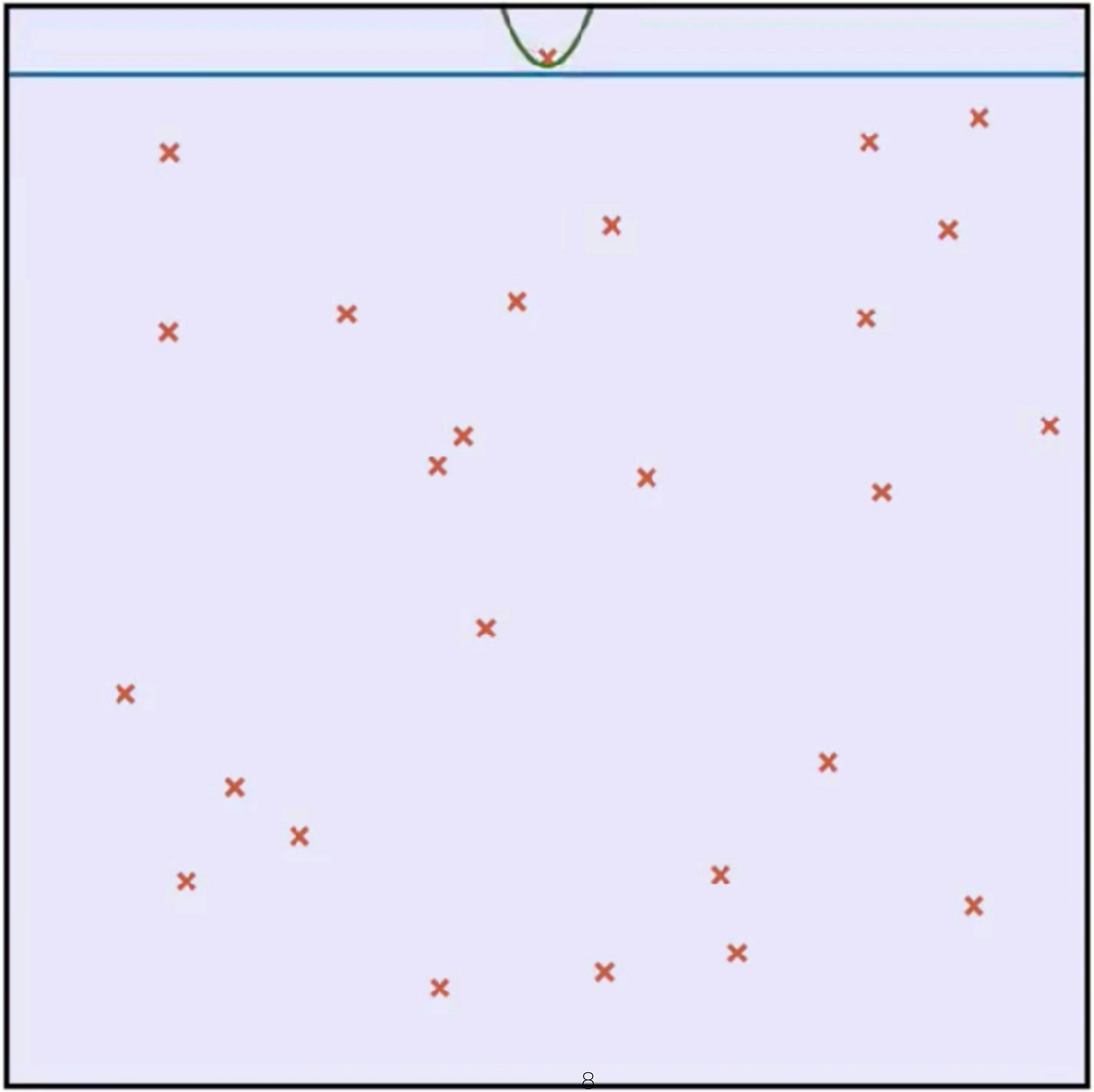

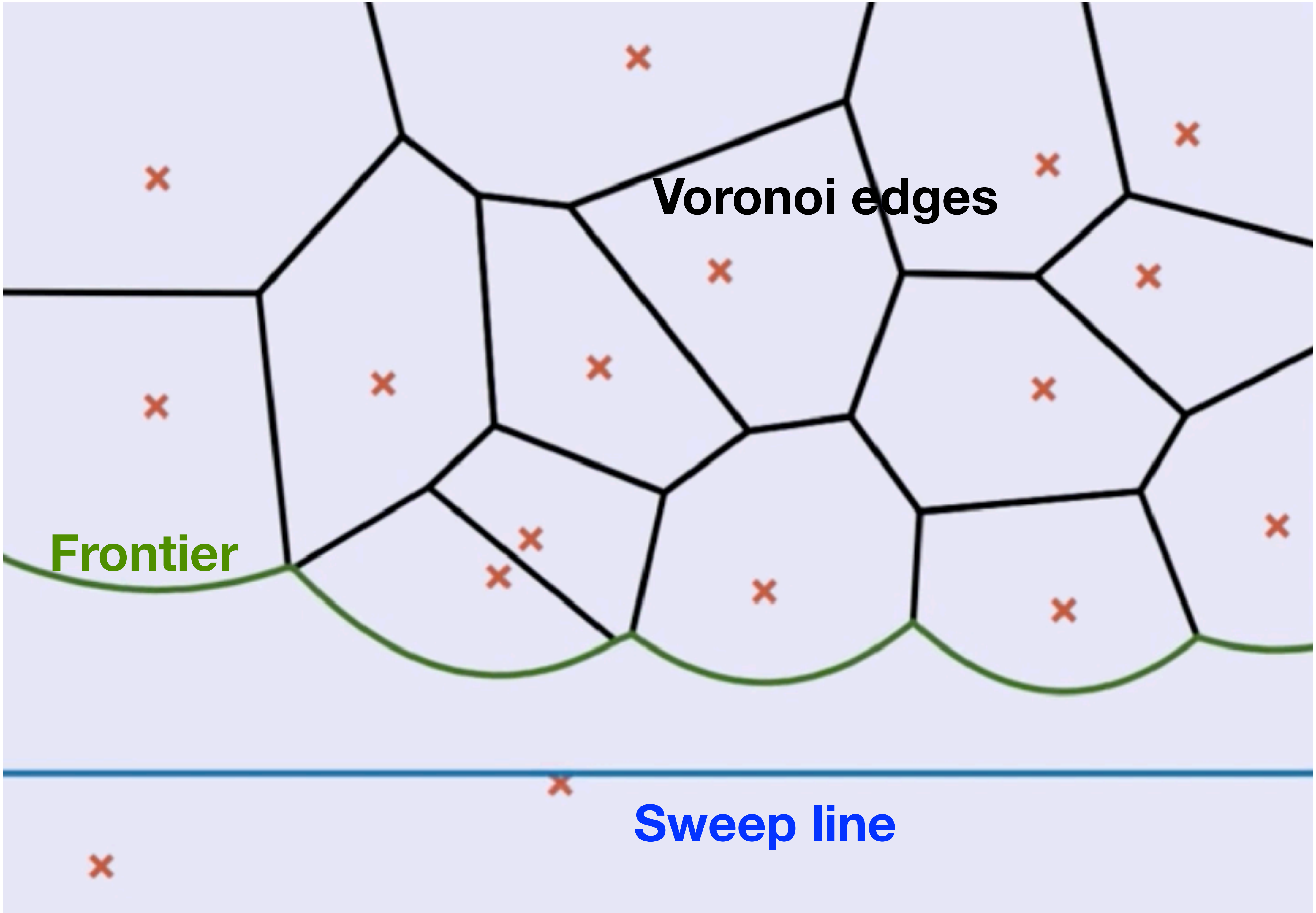
Voronoi edges
Frontier
Sweep line


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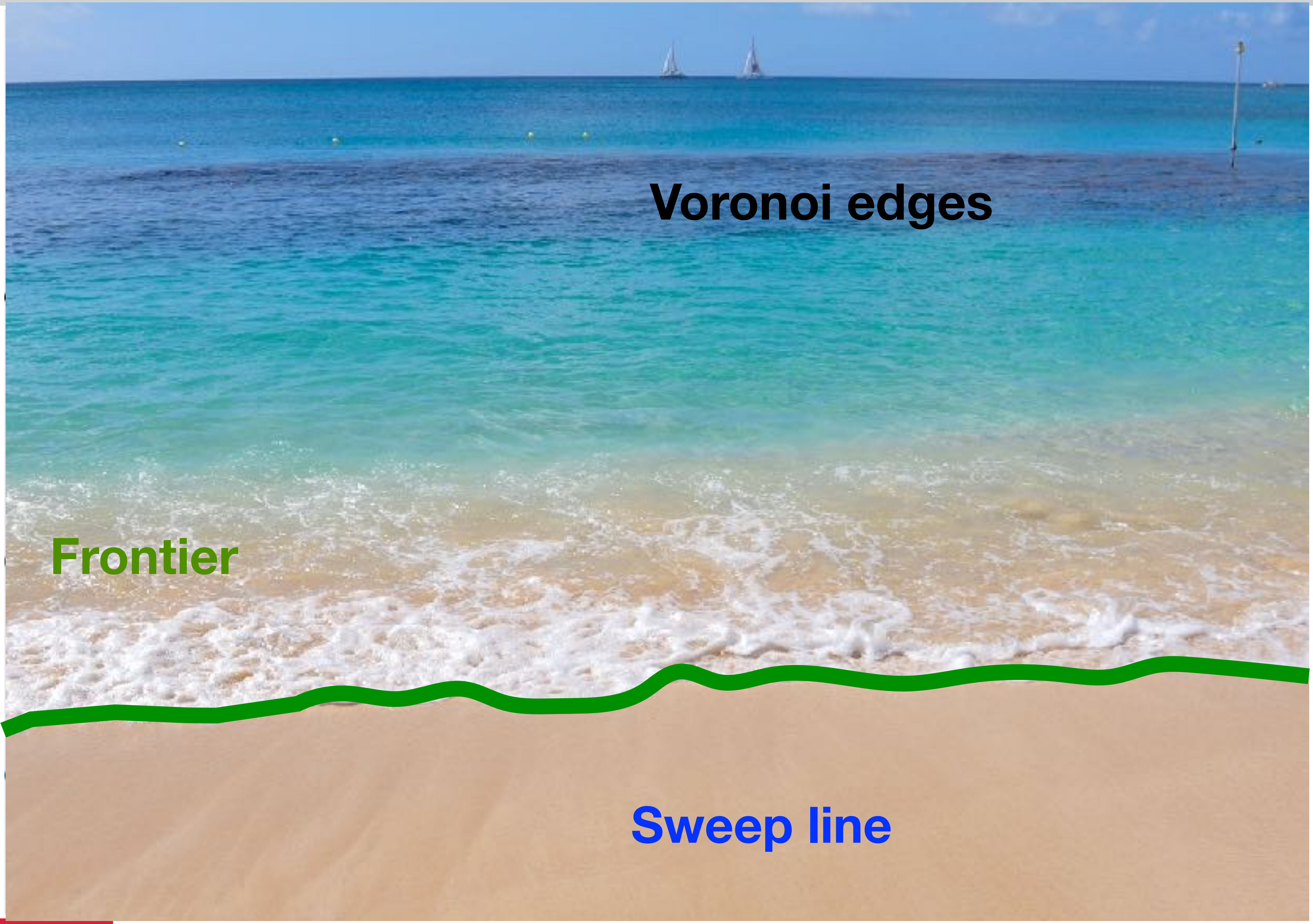

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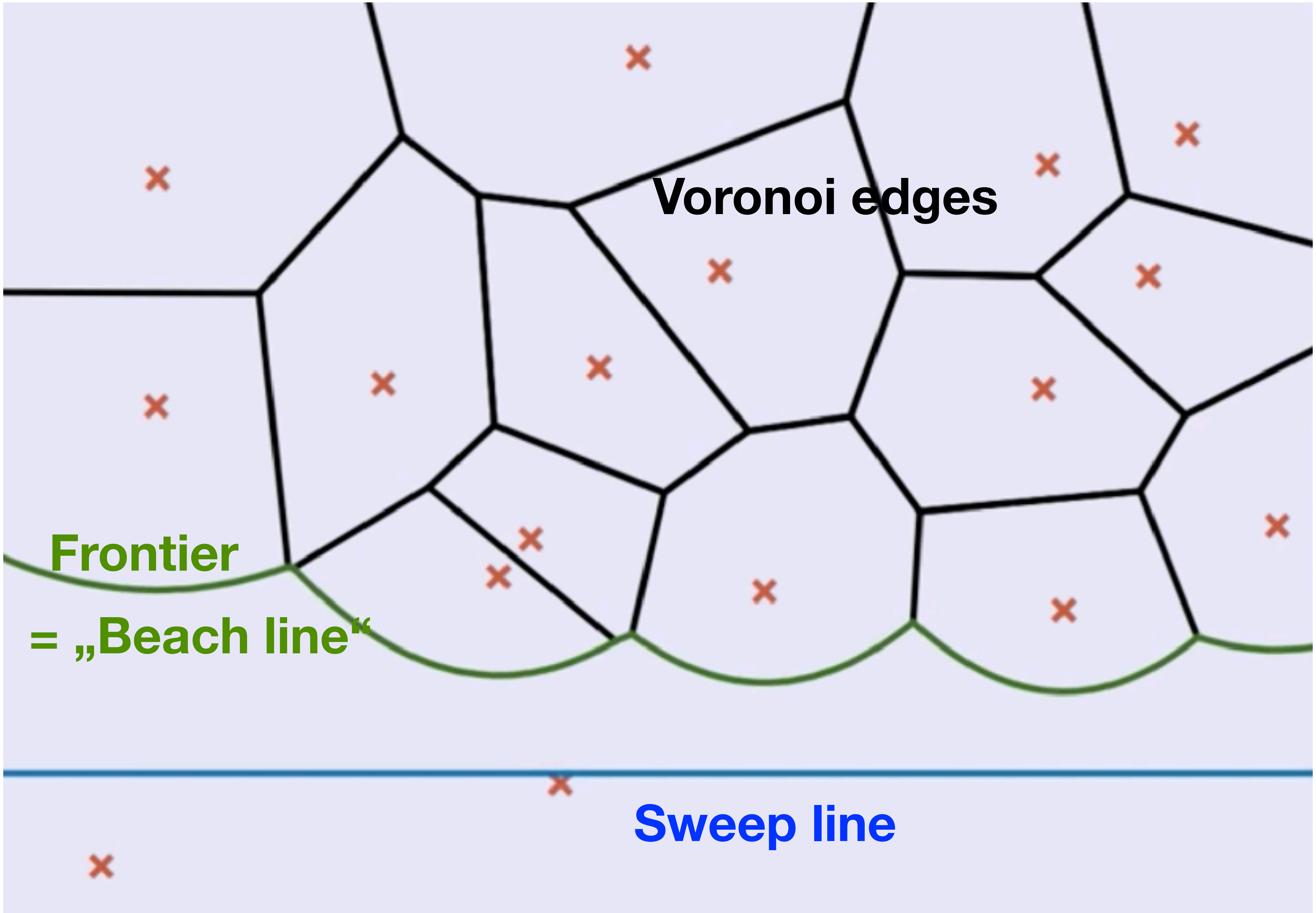
Voronoi edges
Frontier
= „Beach line“
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- Let $q \in \mathcal{P}$ be nearest site for x.
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- $\{x \in \mathbb{R}^2 \mid d(x,q) \leq d(x,\ell)\}$
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Consequence:

- $\exists$ parabola β: x above β
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Beach line:

- $p_1, \ldots, p_k$ above ℓ $\rightarrow$ parabolas $\beta_1, \ldots, \beta_k$
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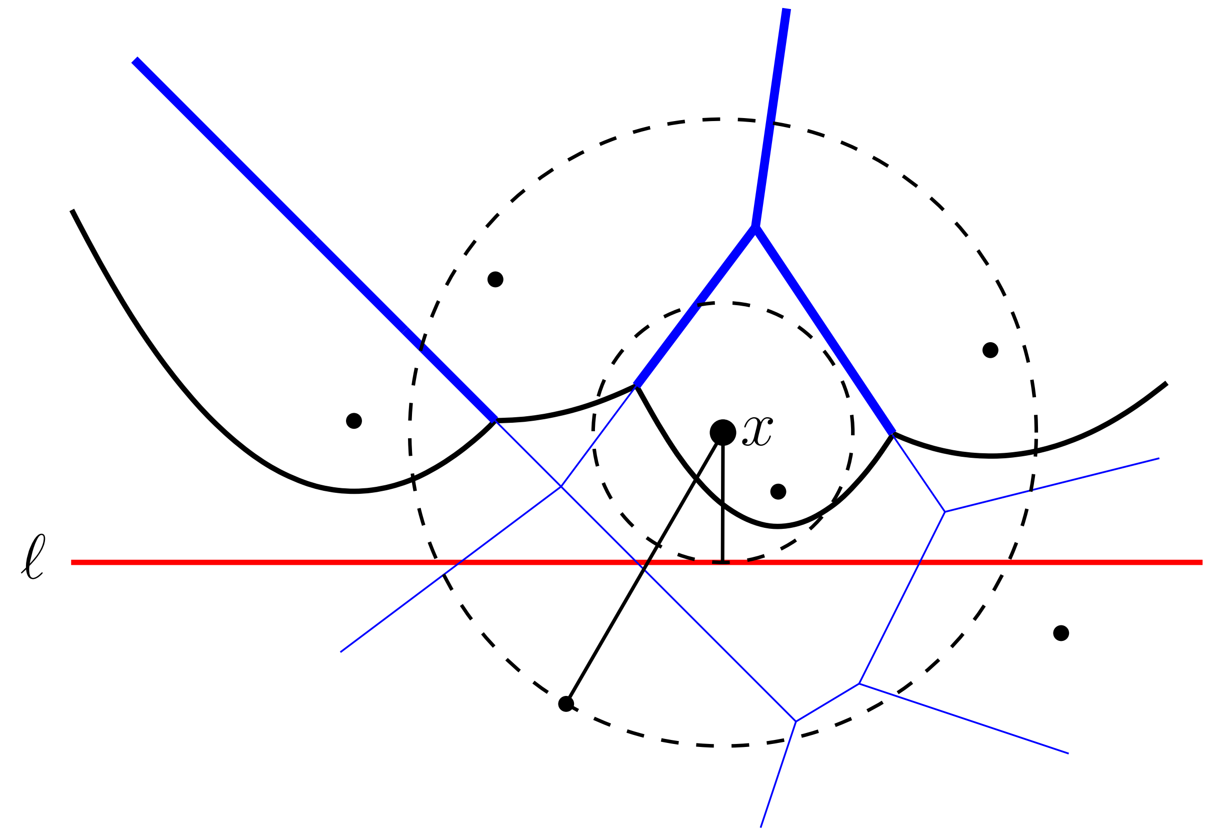

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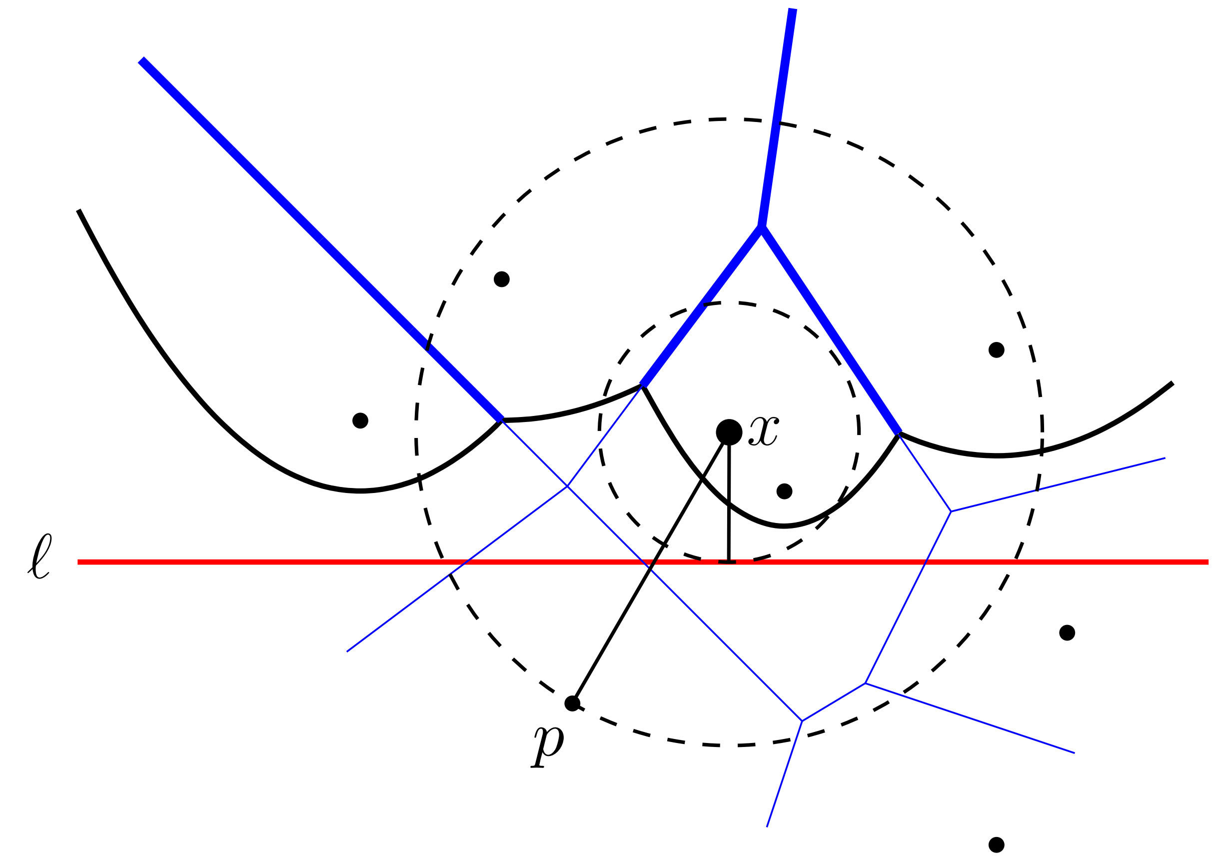

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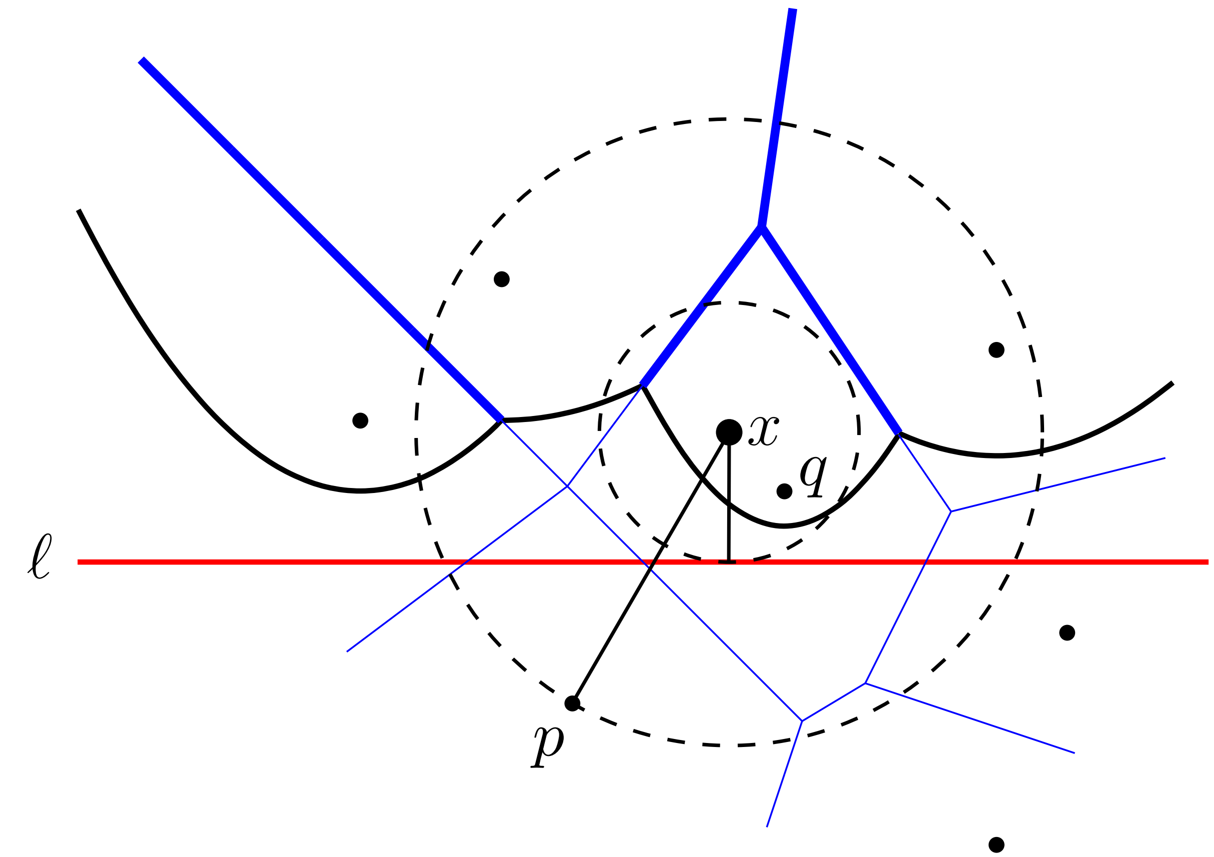

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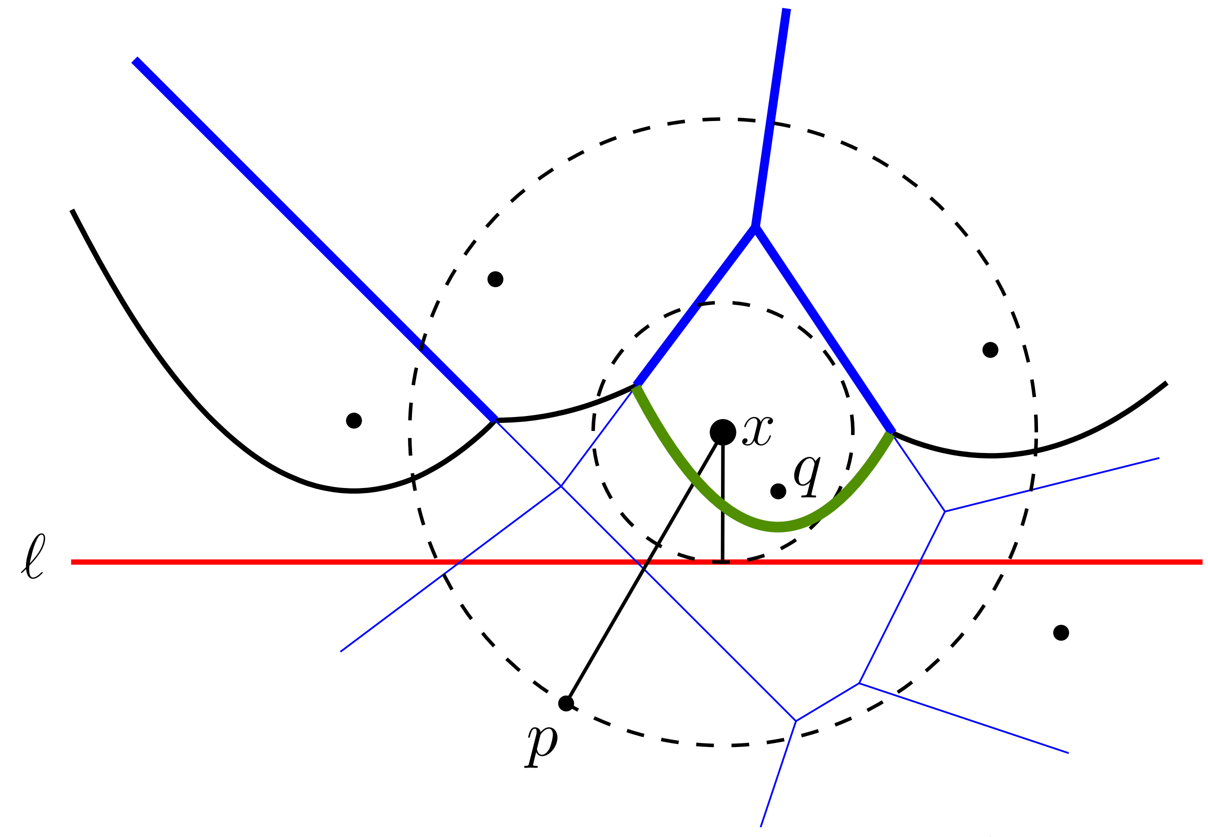

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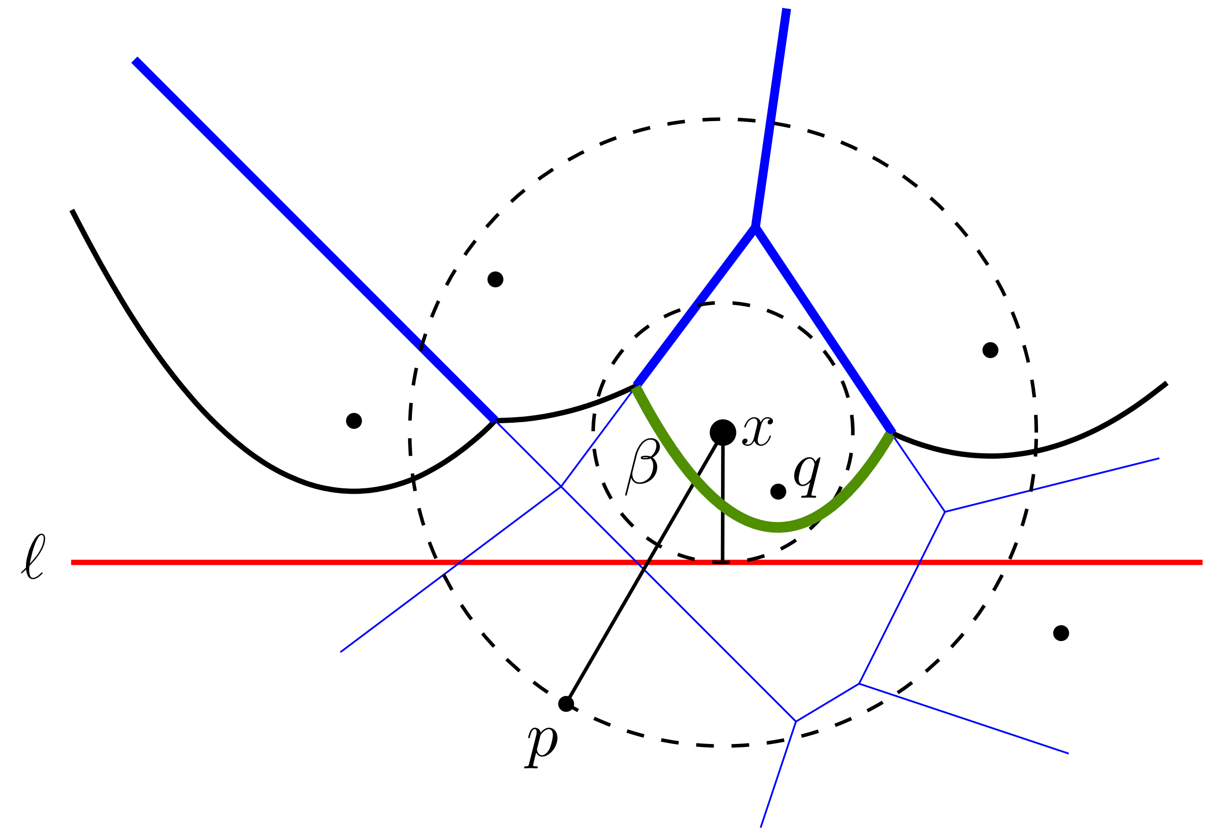

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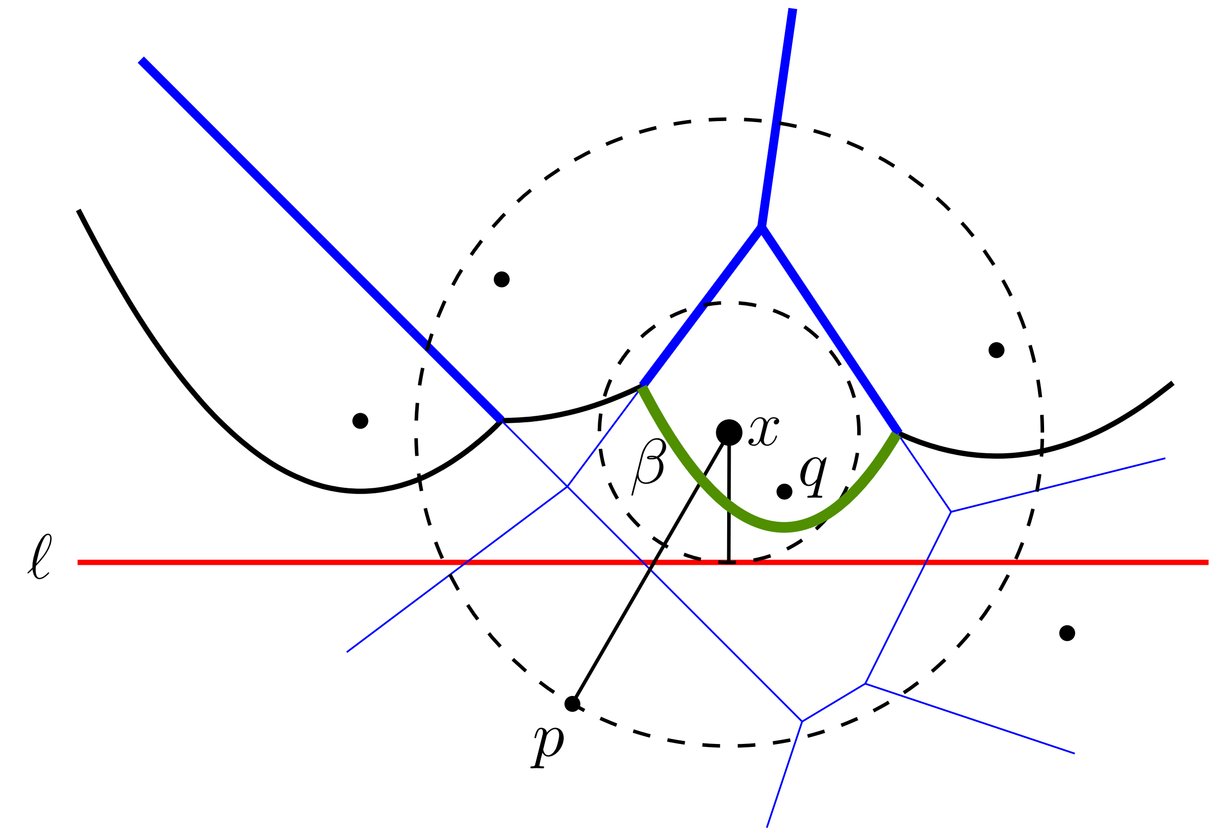

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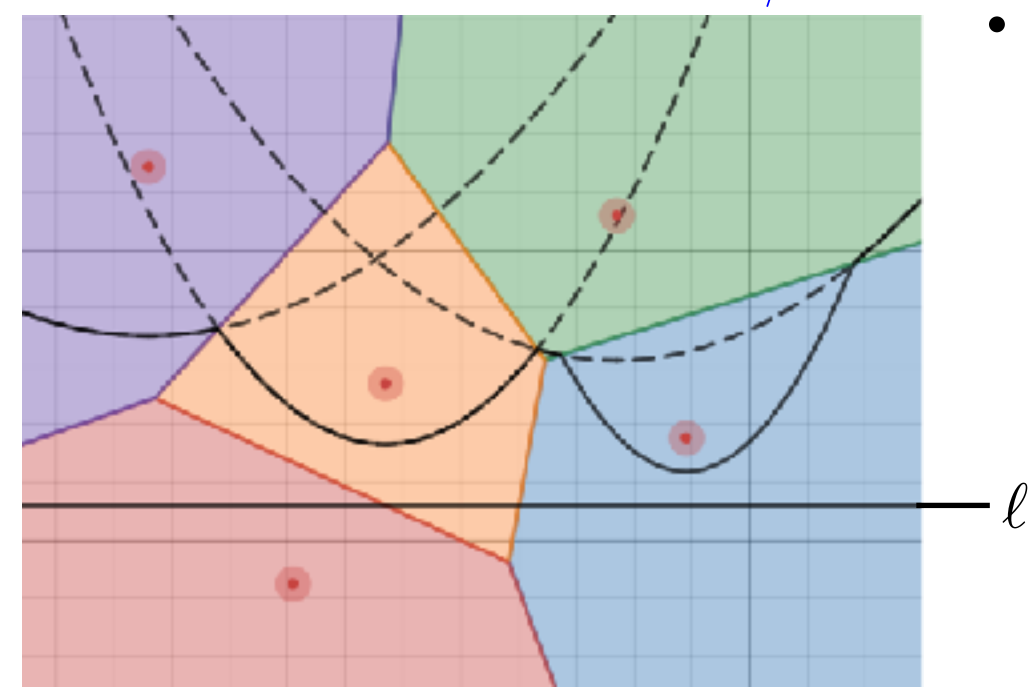

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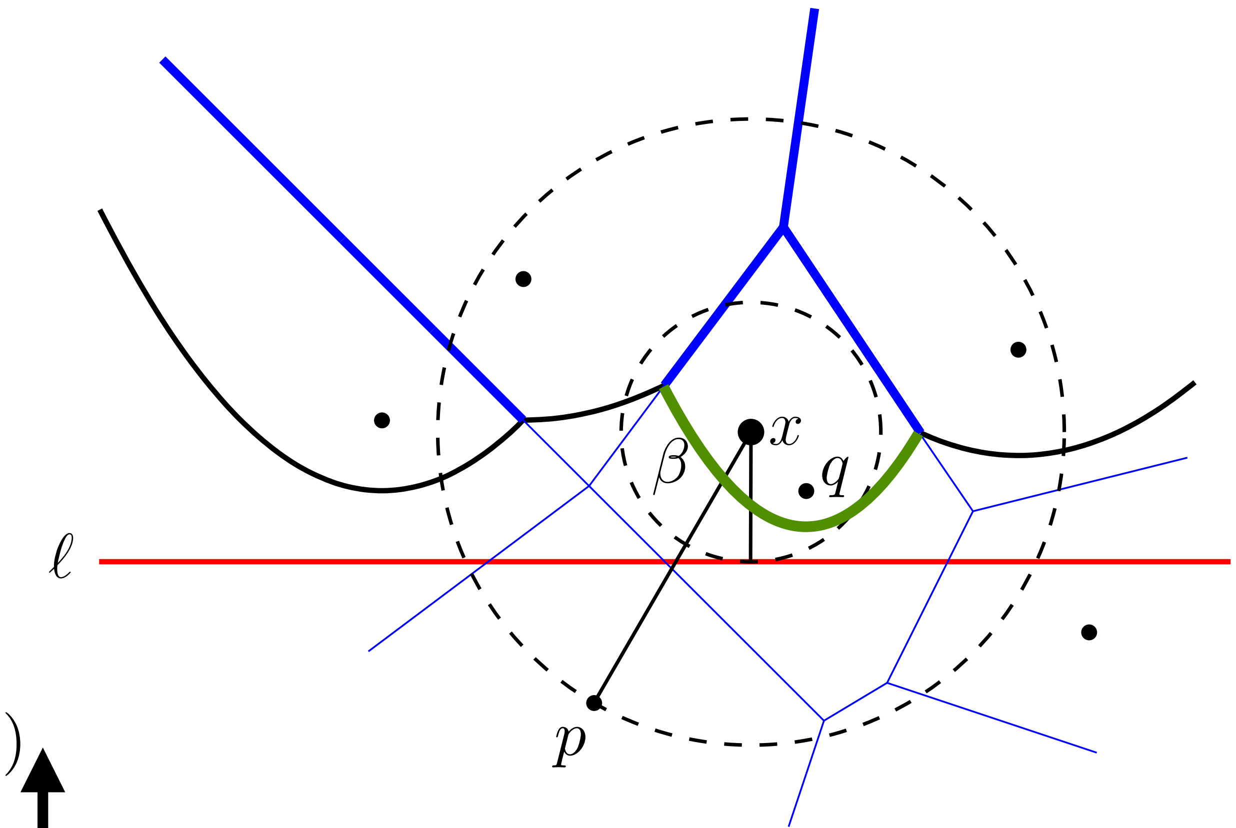

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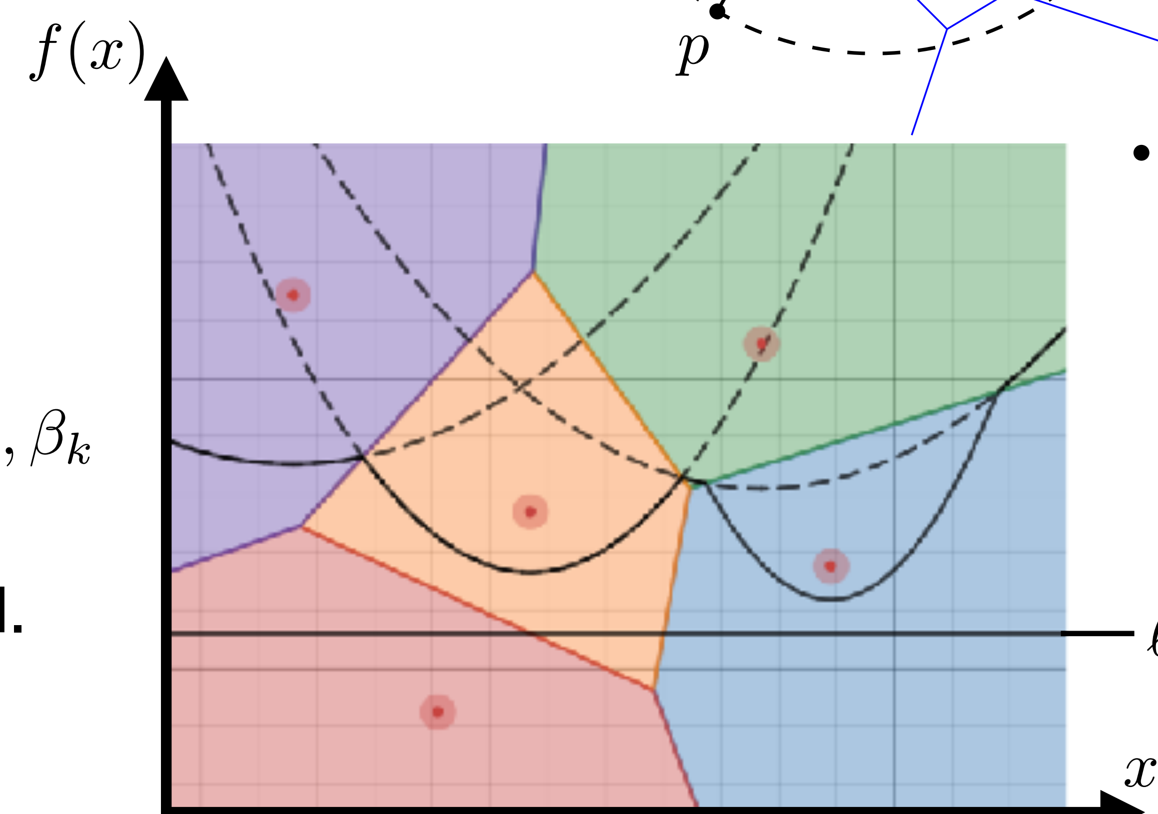

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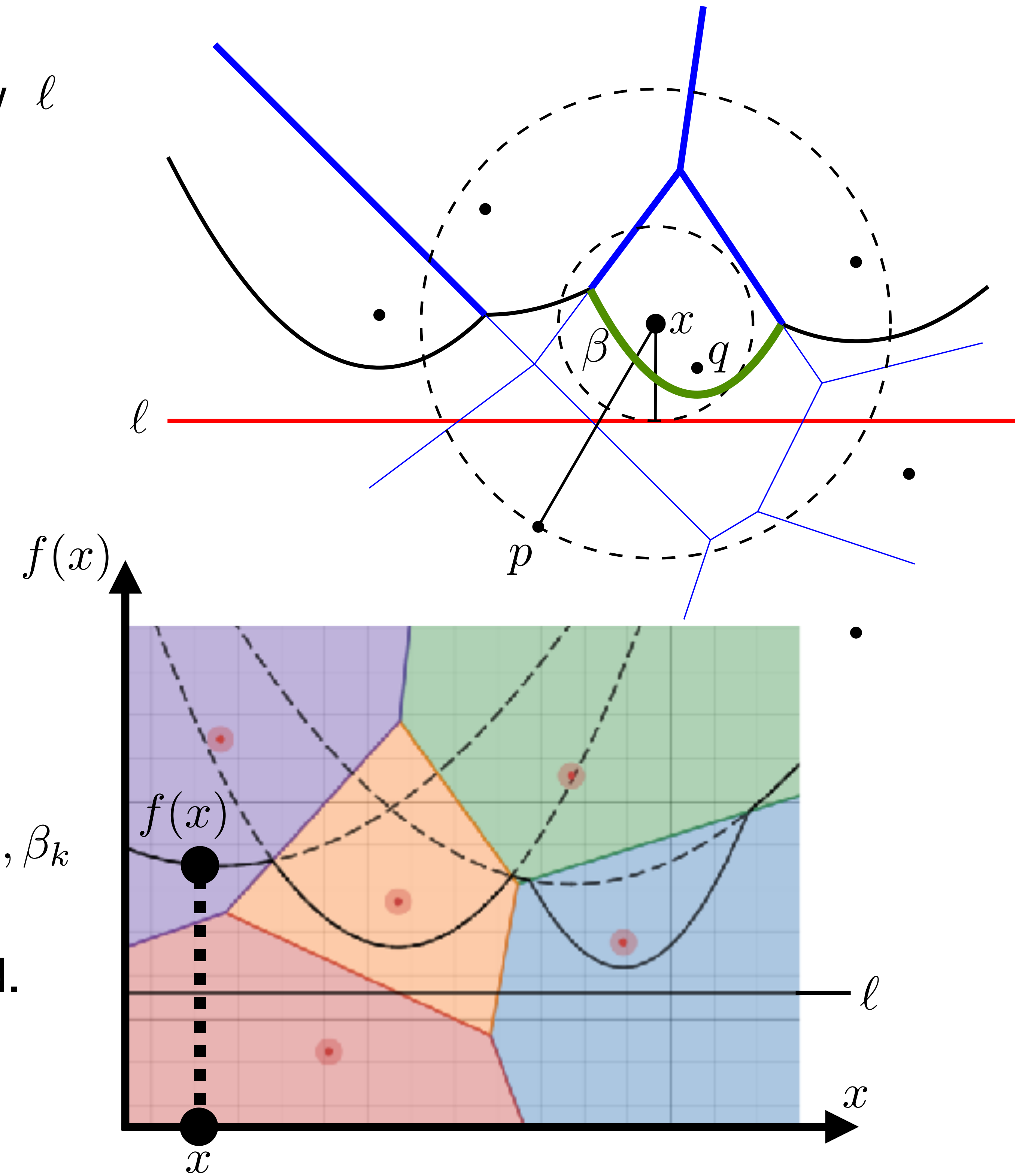

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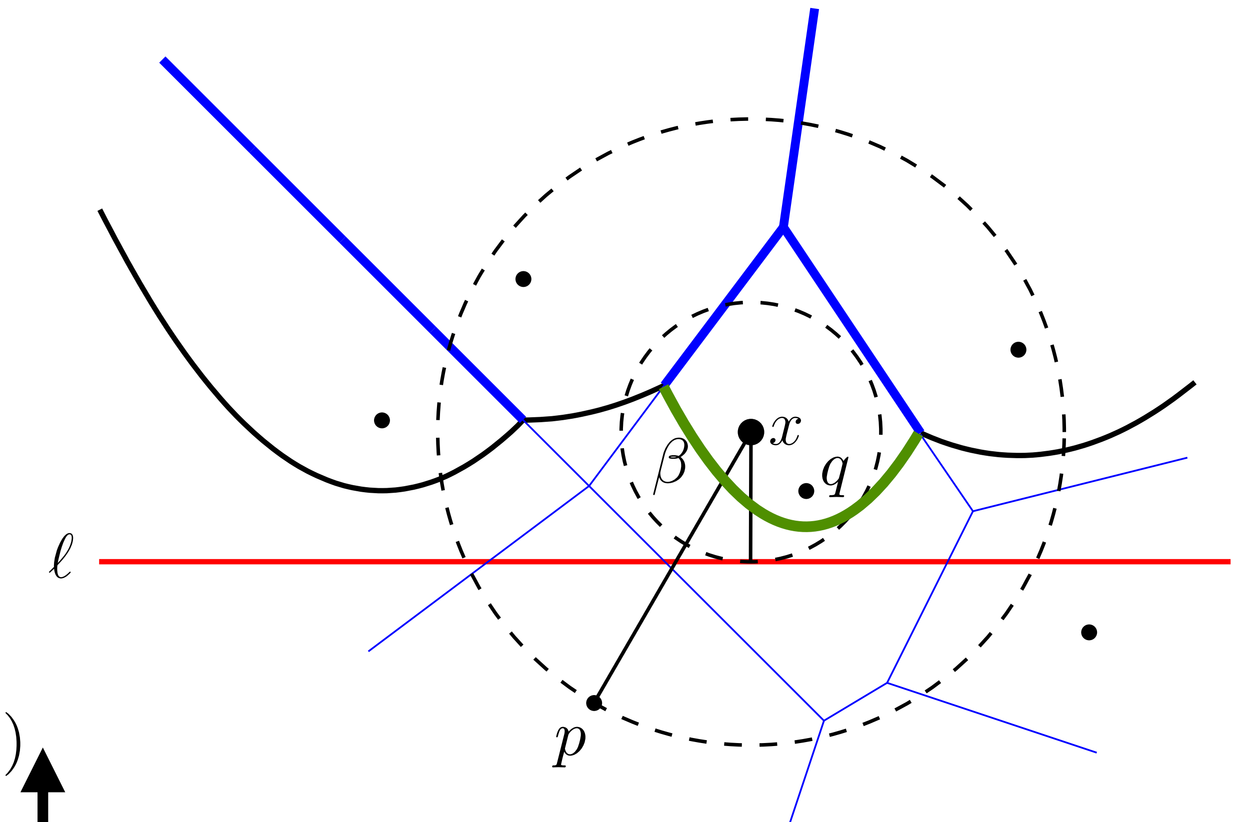

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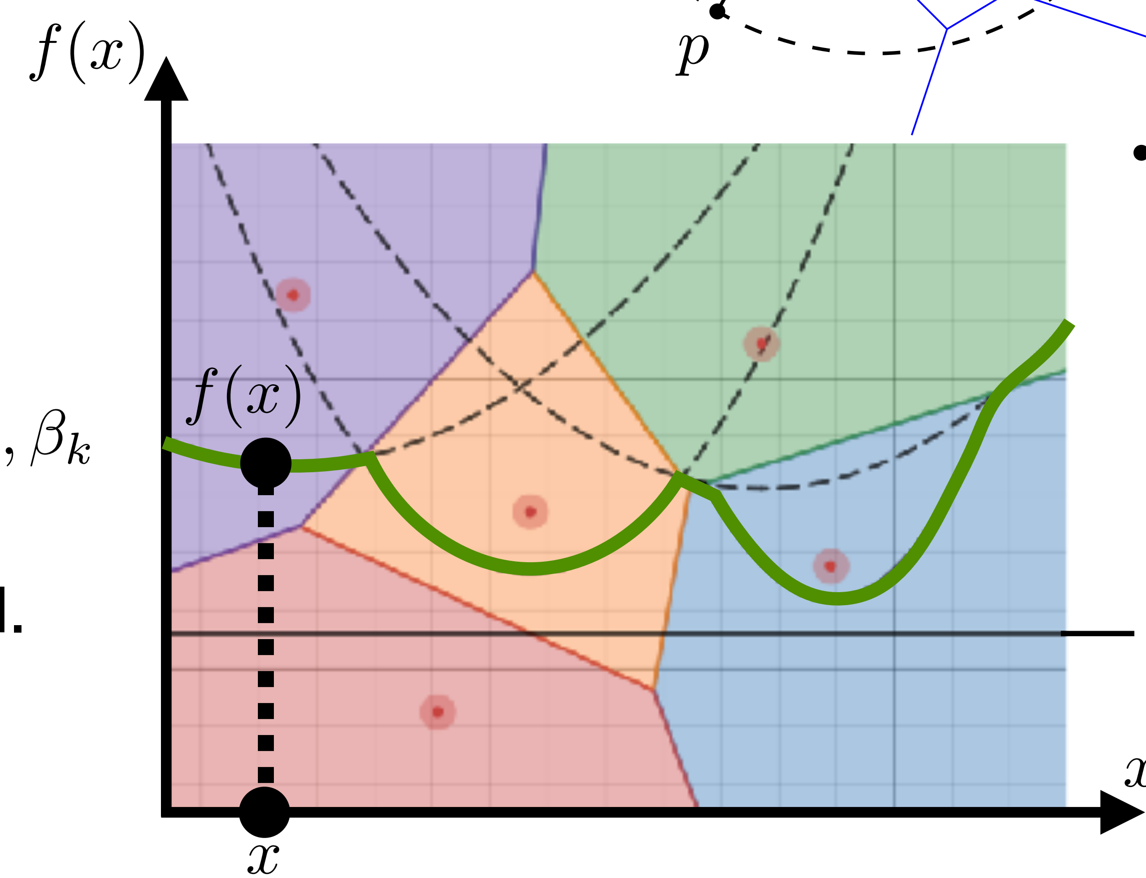

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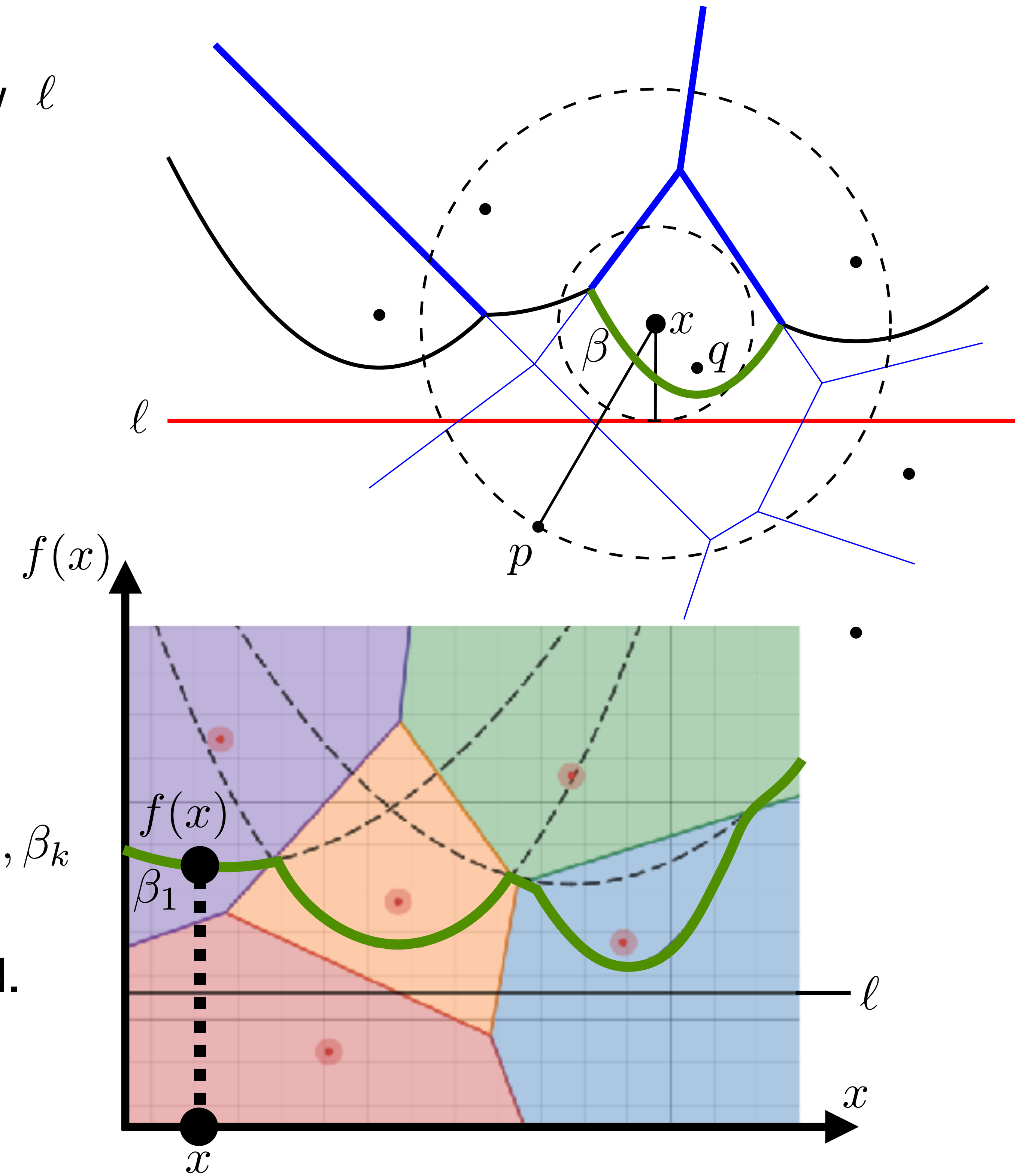

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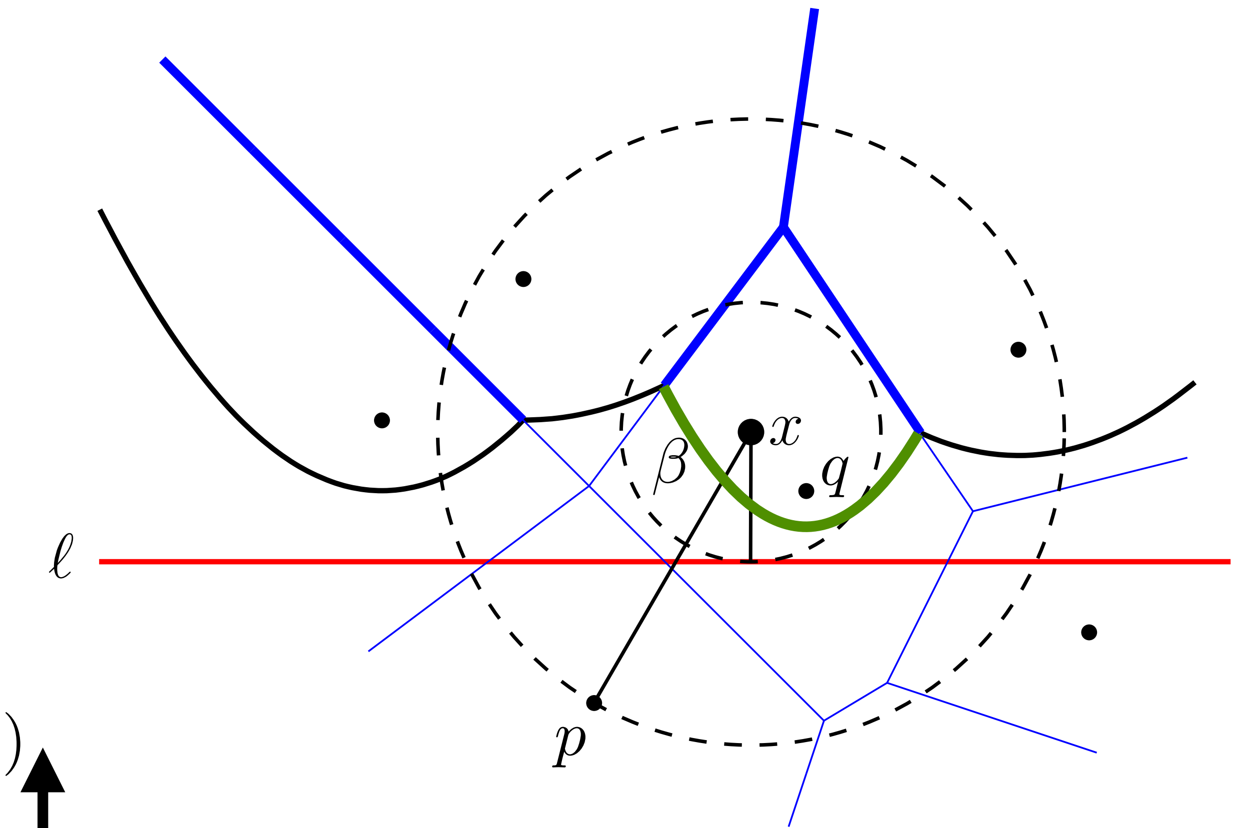

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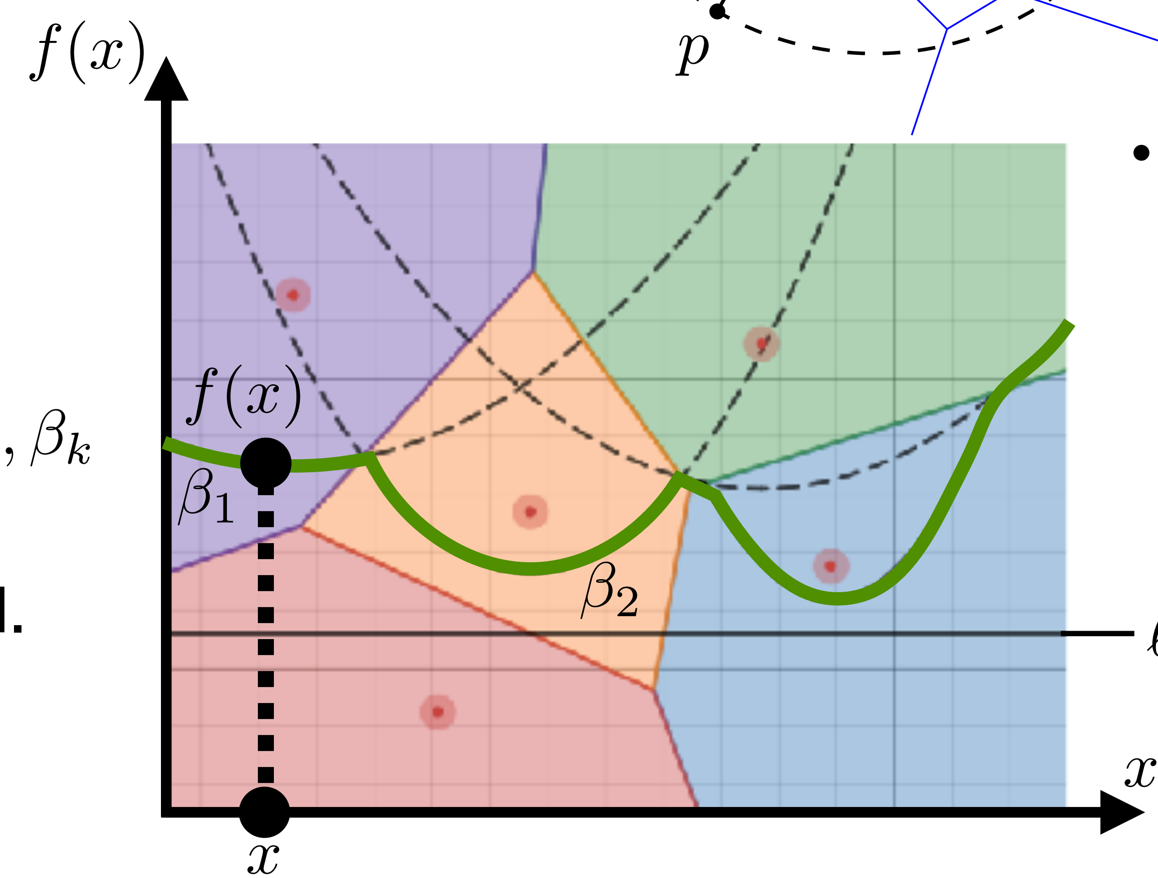

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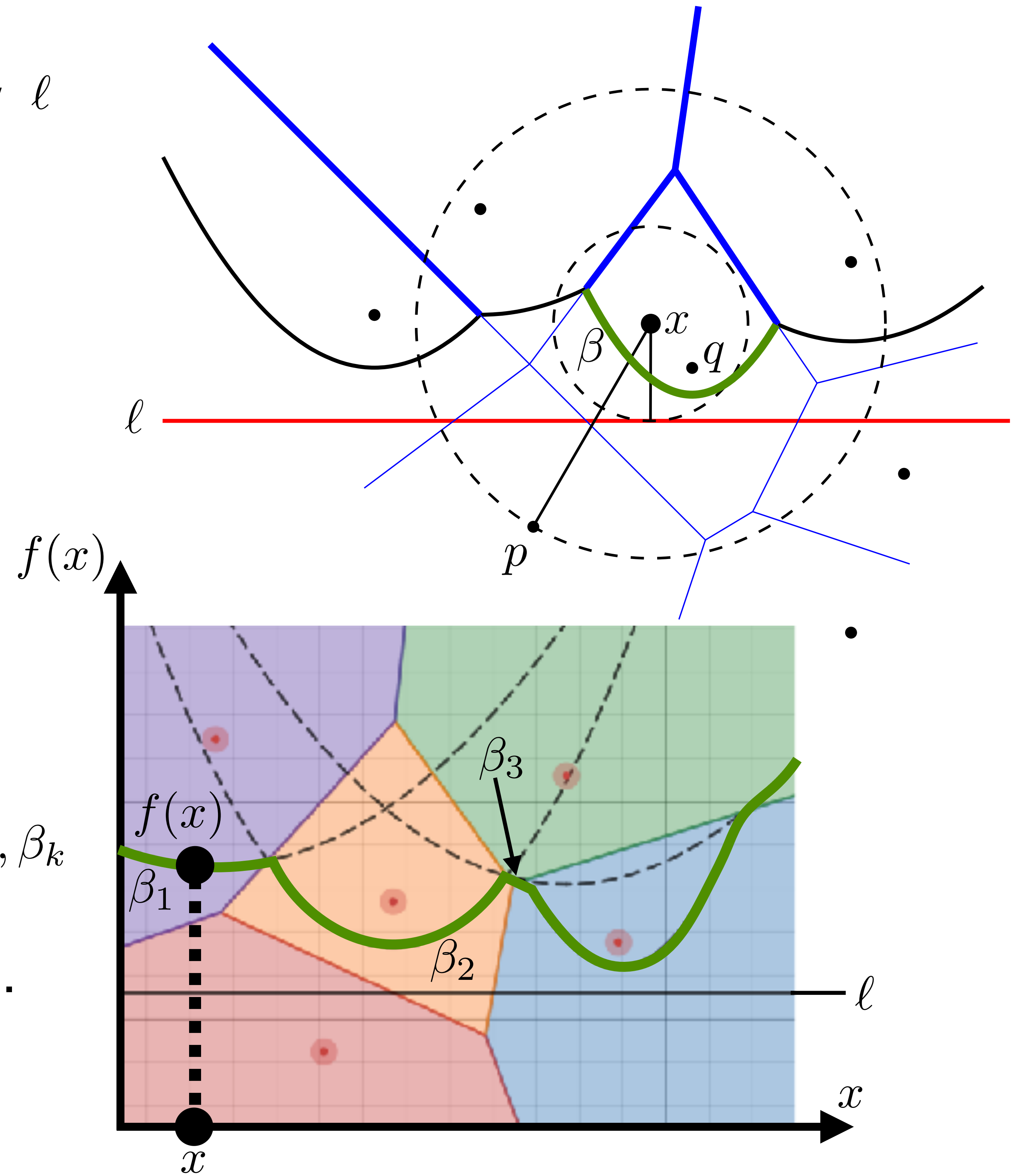

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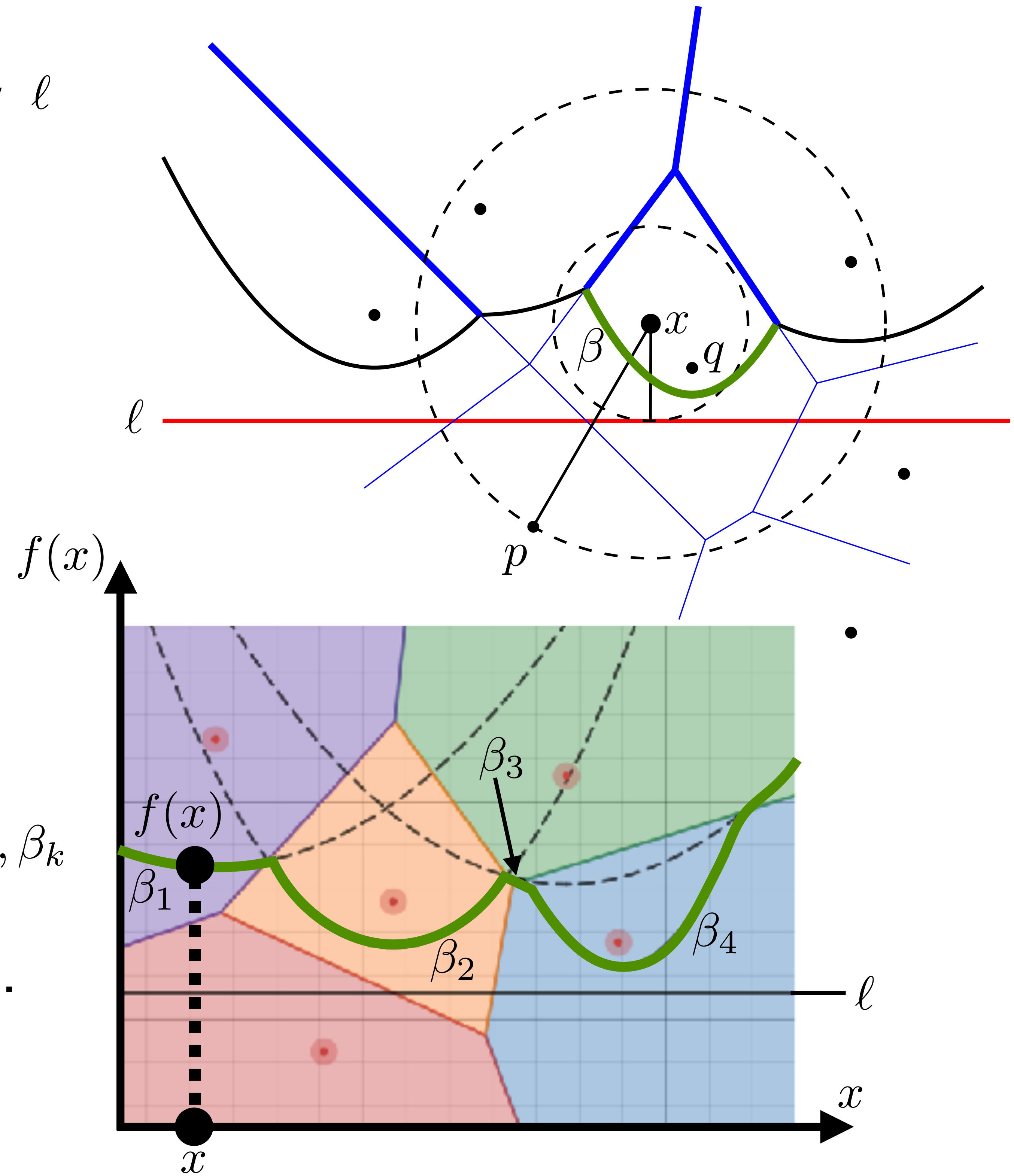

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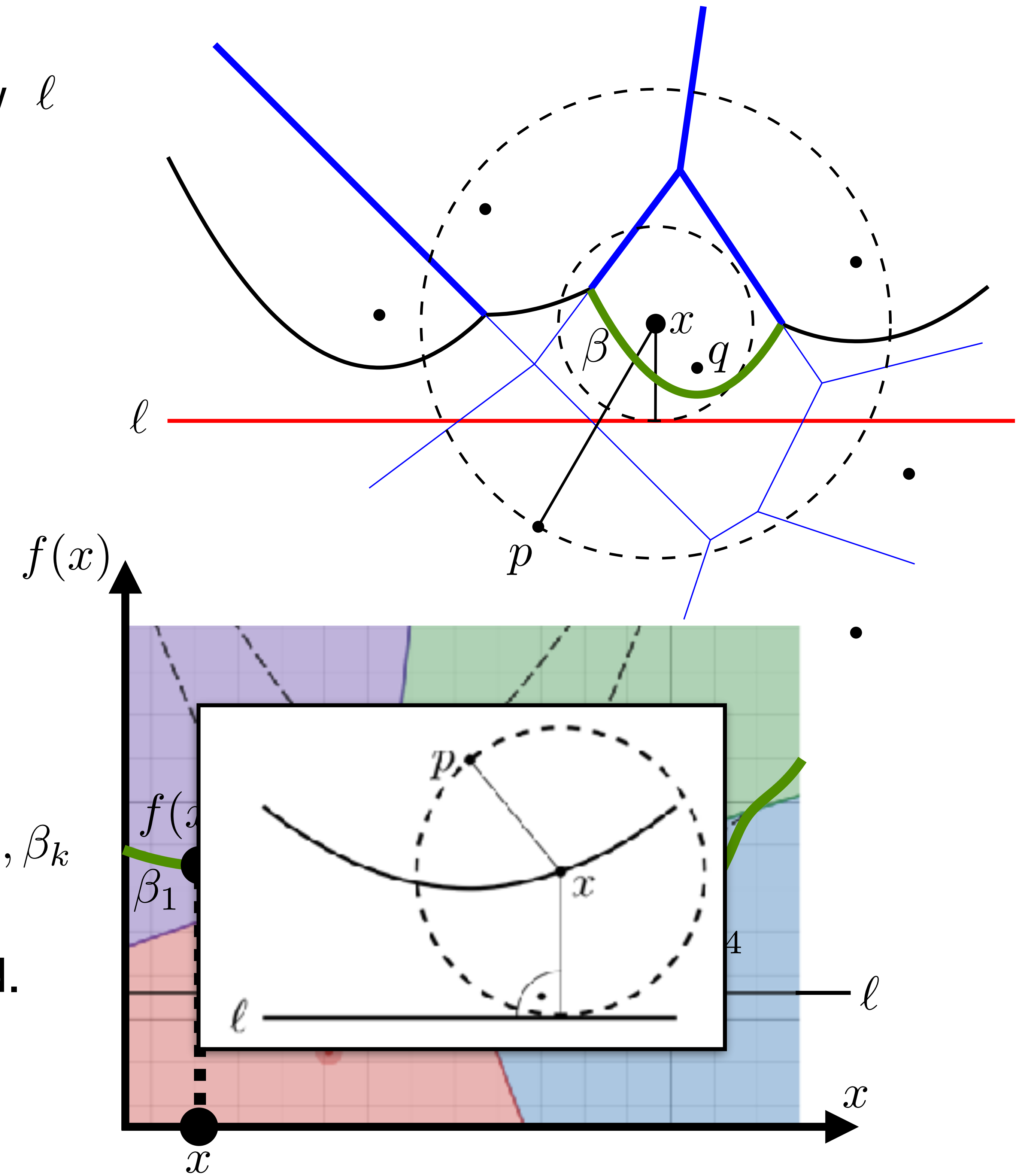

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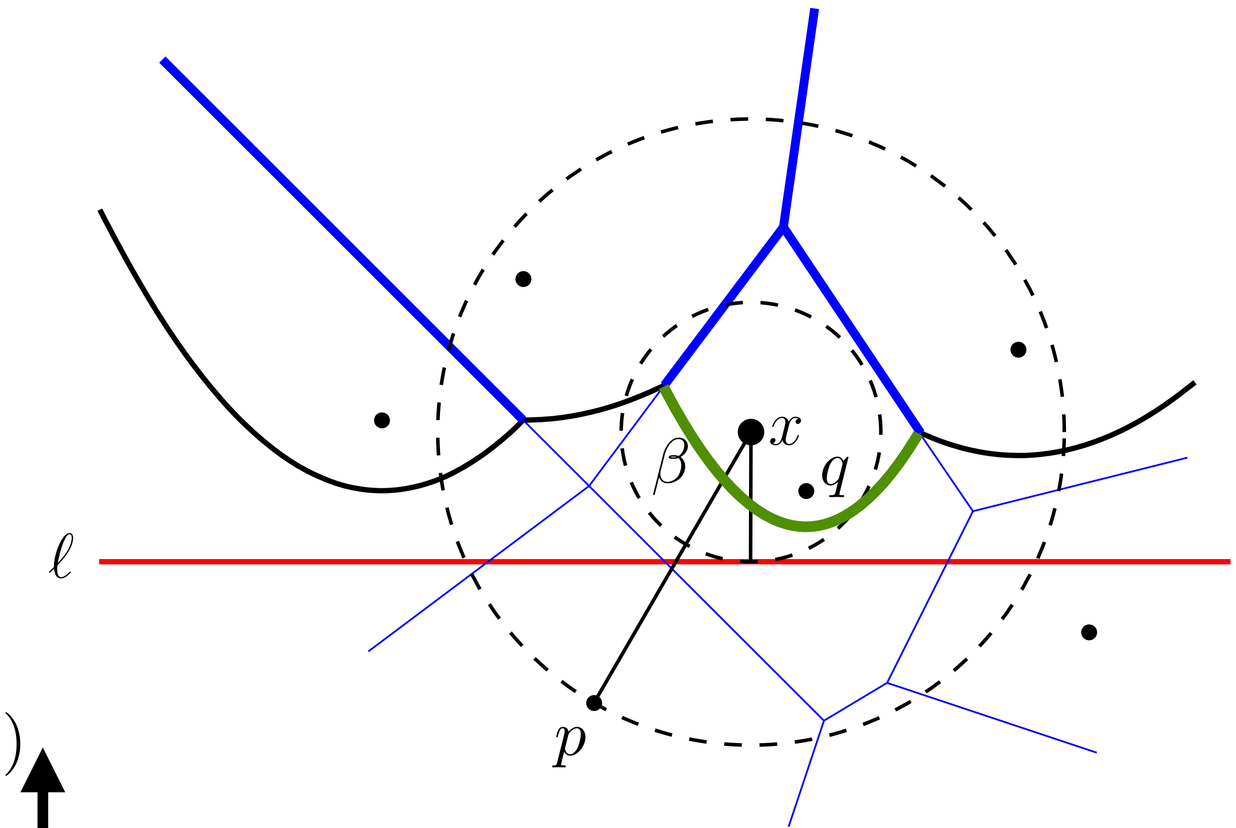

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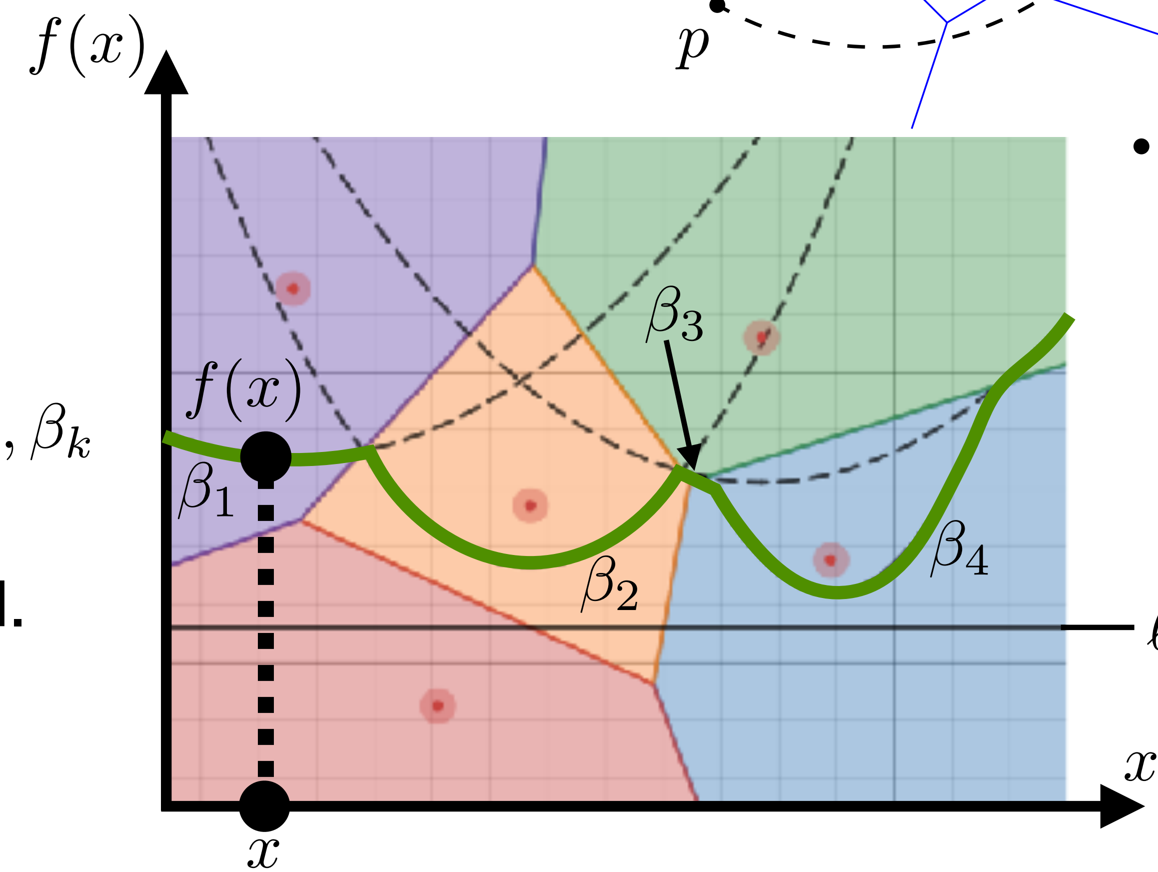
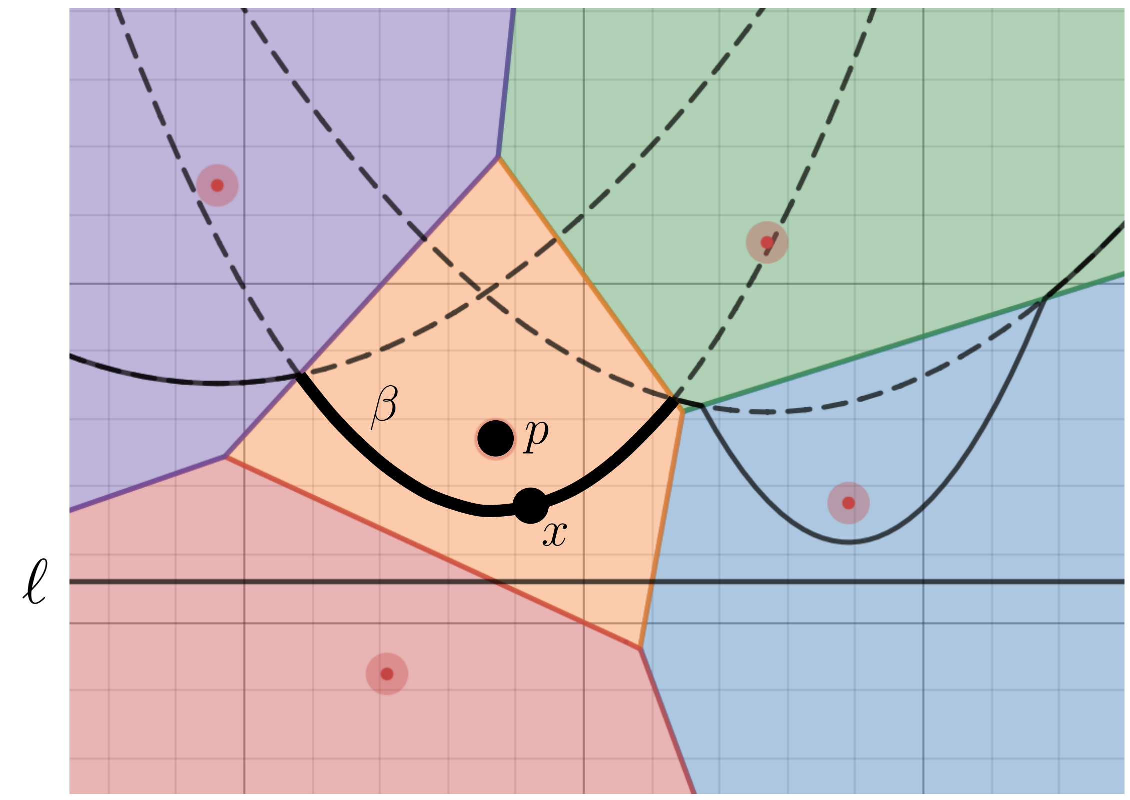


Lemma 4.15

$p \in \mathcal{P}$ defines arc β on beach line

$\Rightarrow$ p is nearest site $\forall x \in \beta$.



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- Assume: $\exists q \in P : d(x, q) < d(x, p)$
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Corollary 4.16

Intersection points of adjacent arcs lie on Voronoi edges.

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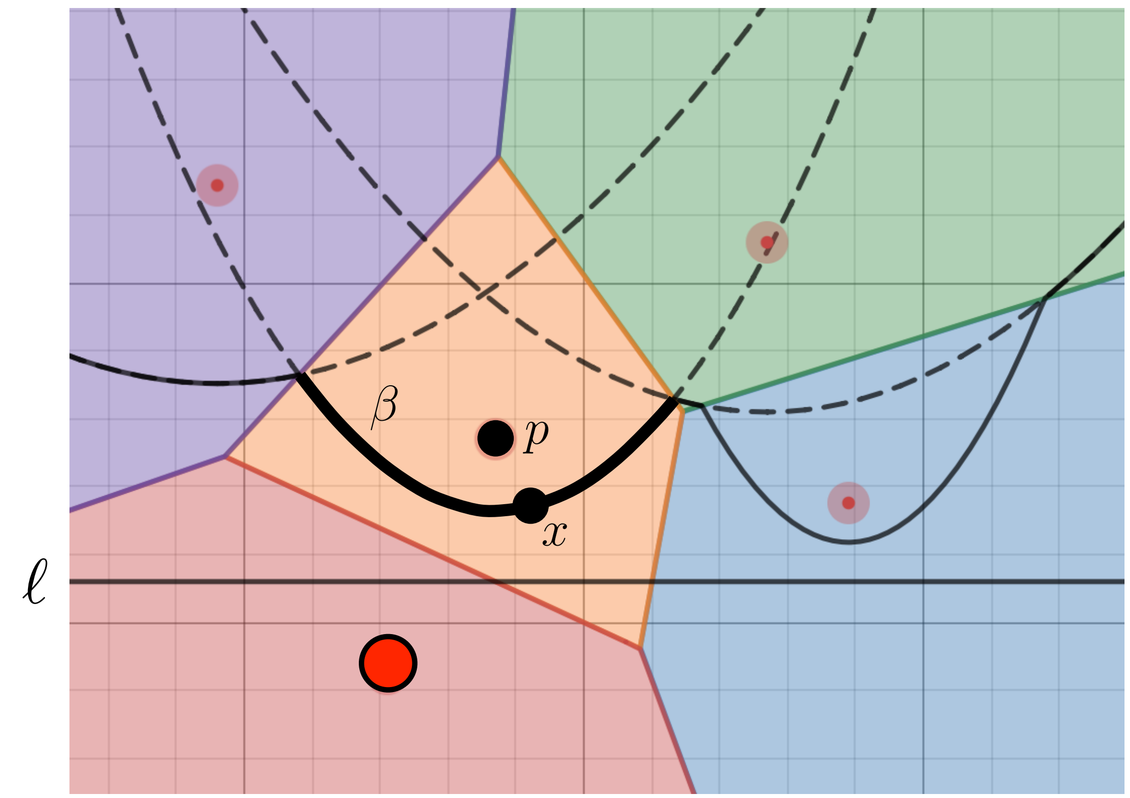

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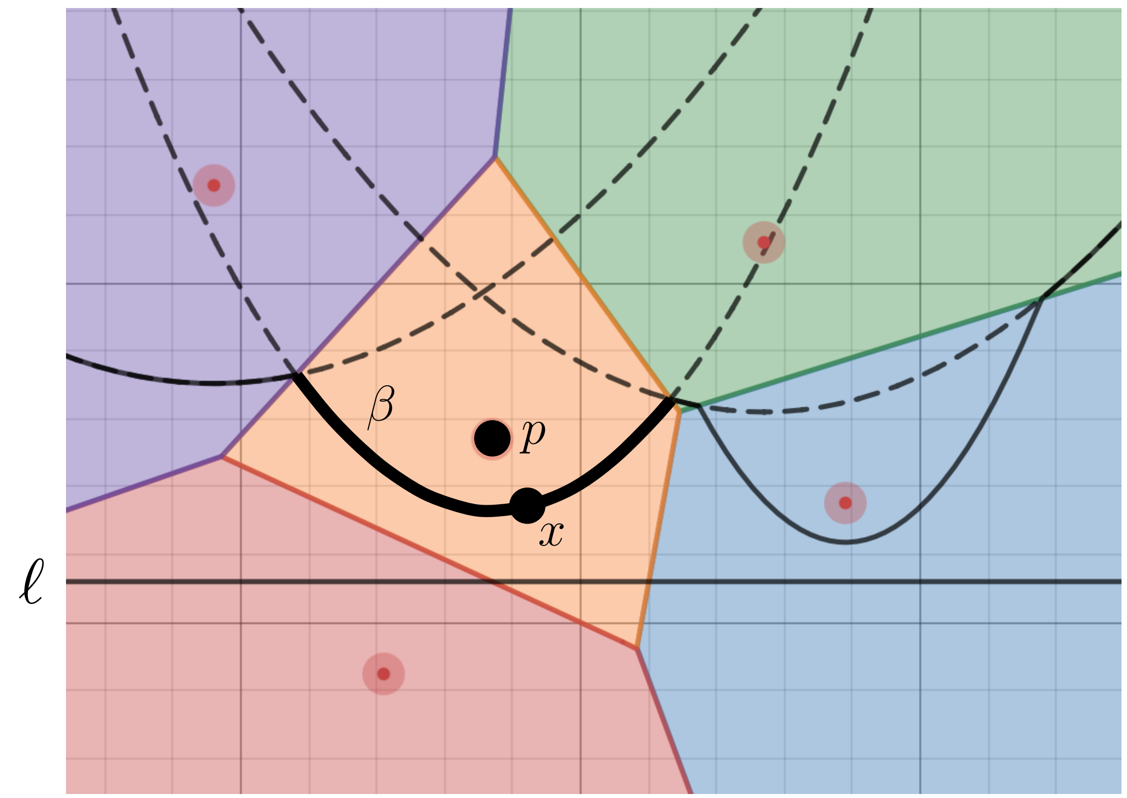

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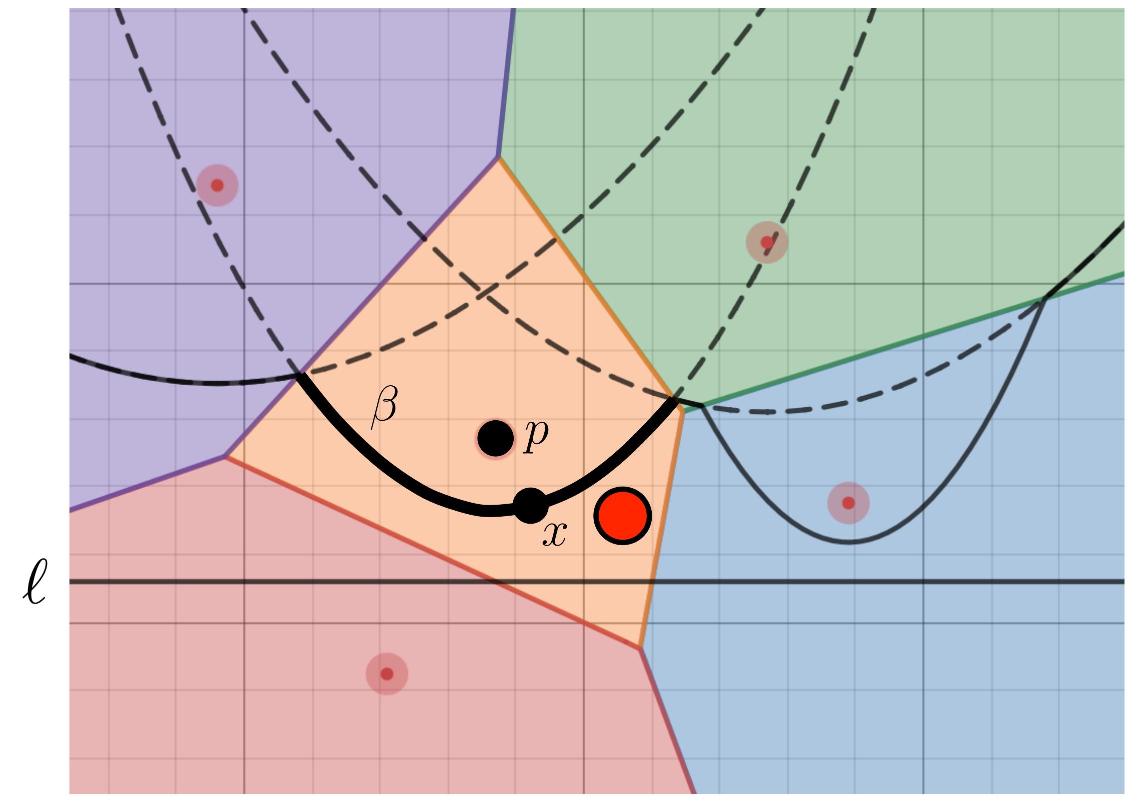

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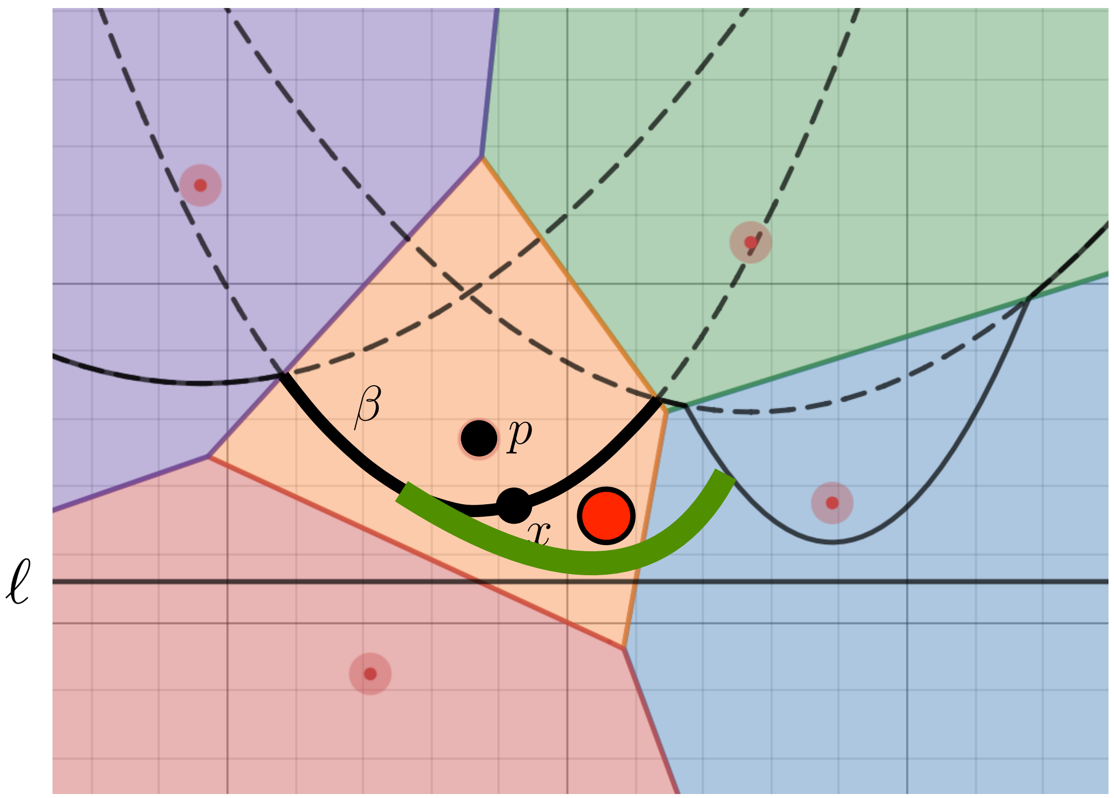

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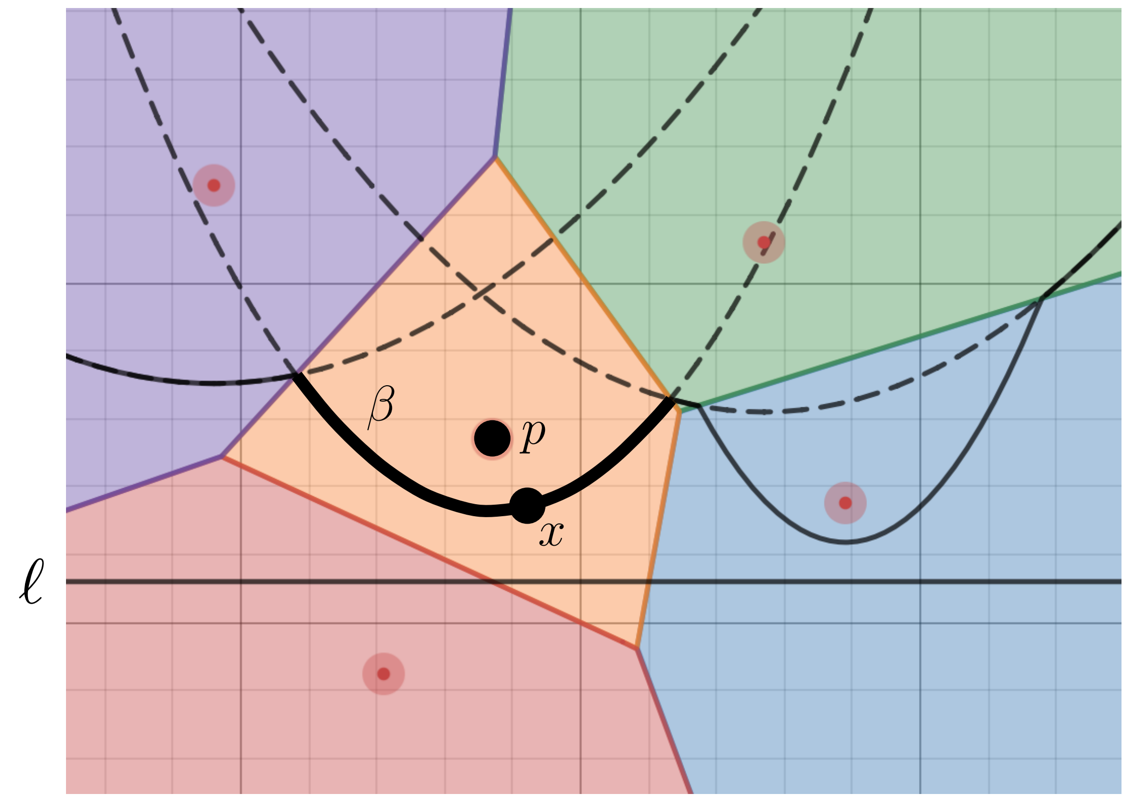

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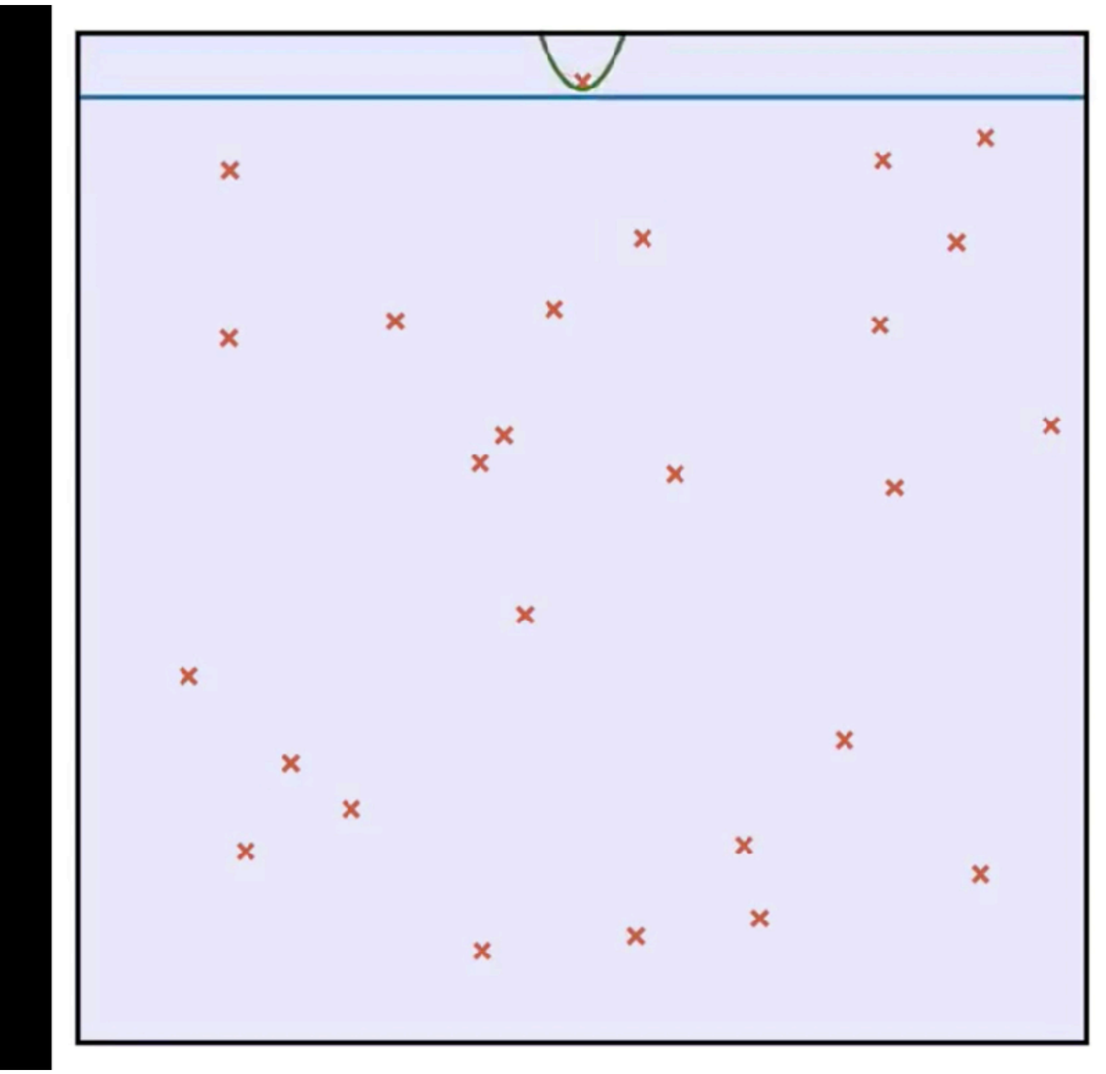
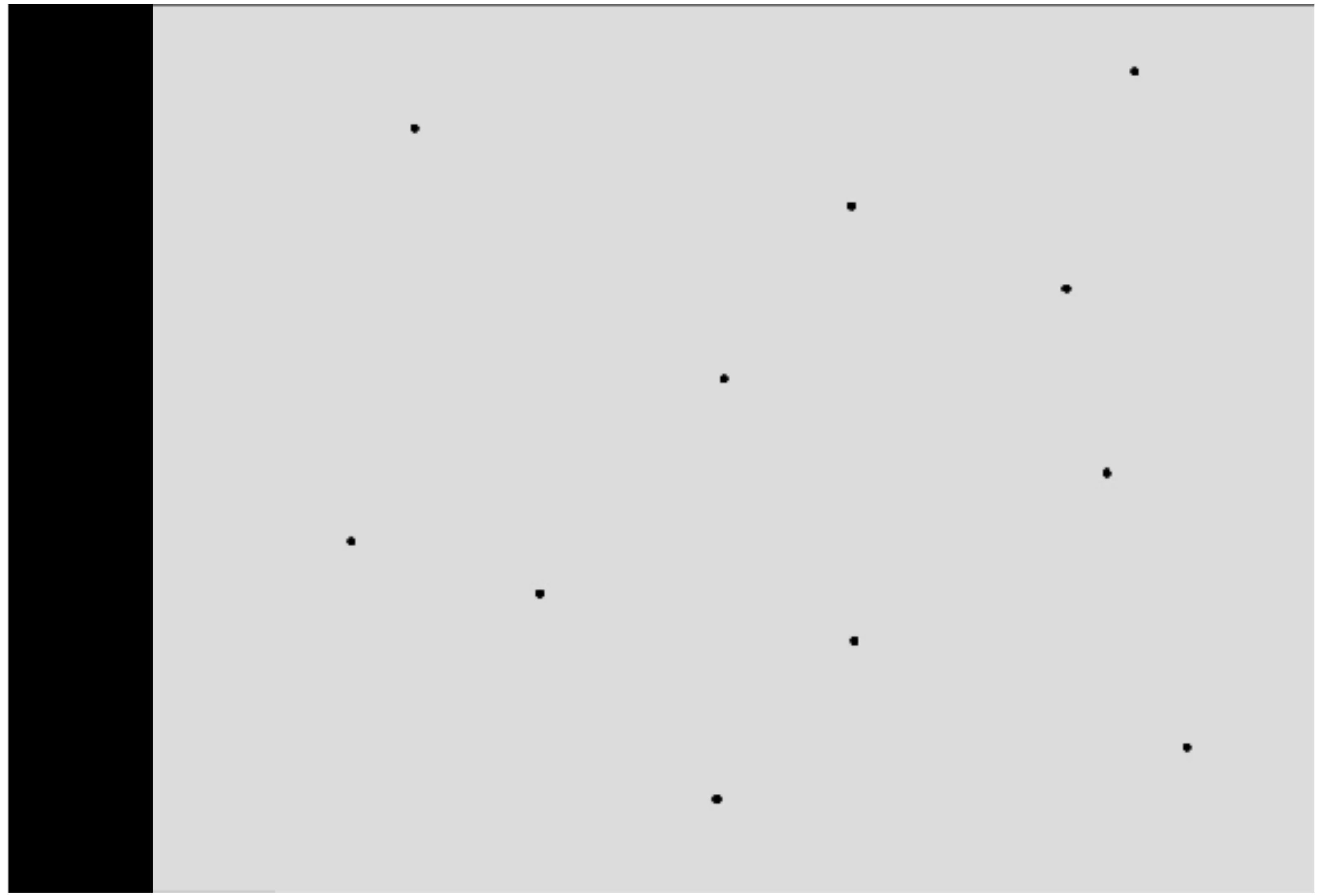
- Plane sweep
- Compute $Vor(p)$ in guaranteed regions $\rightarrow$ draw Voronoi edges.





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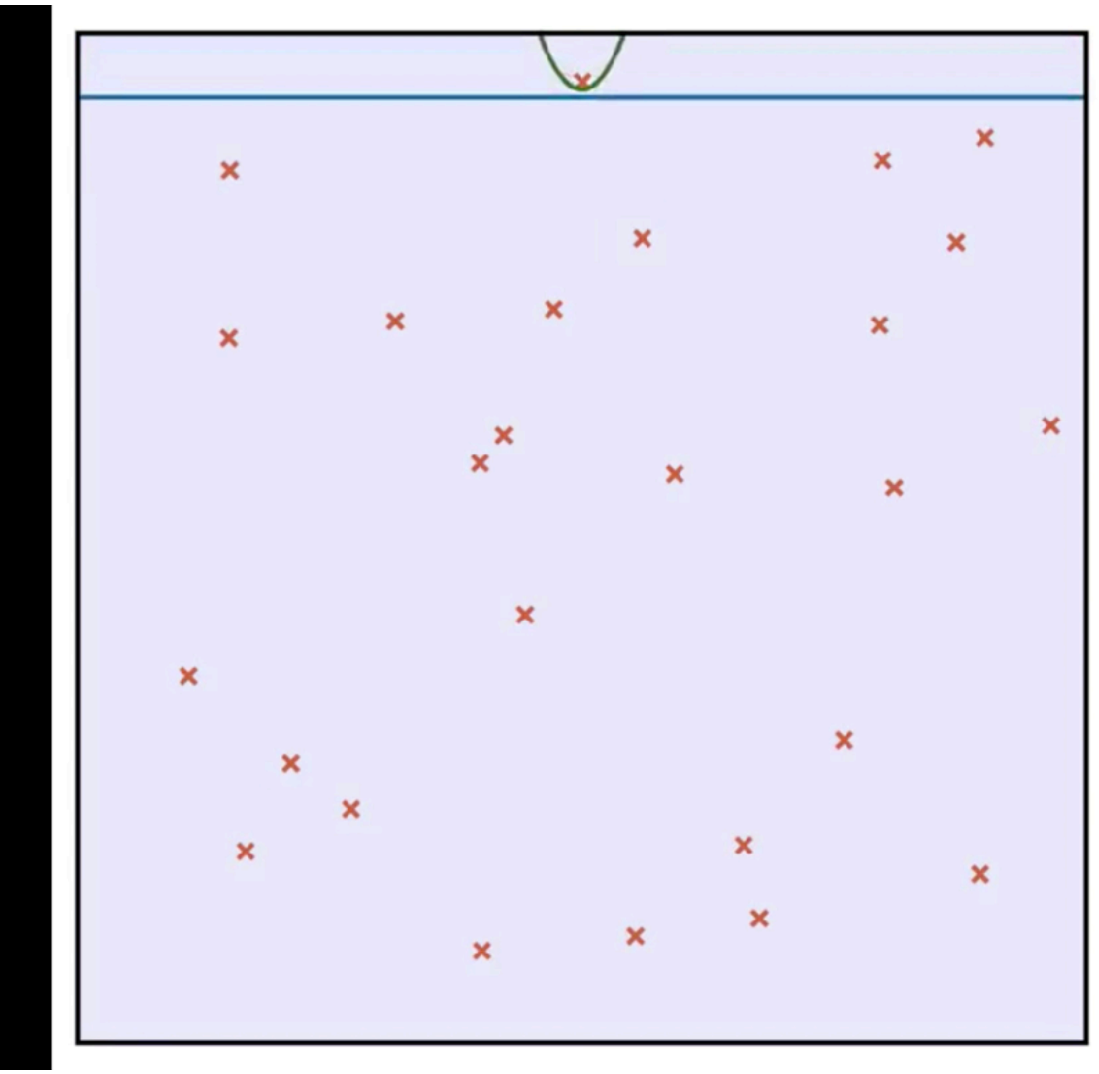

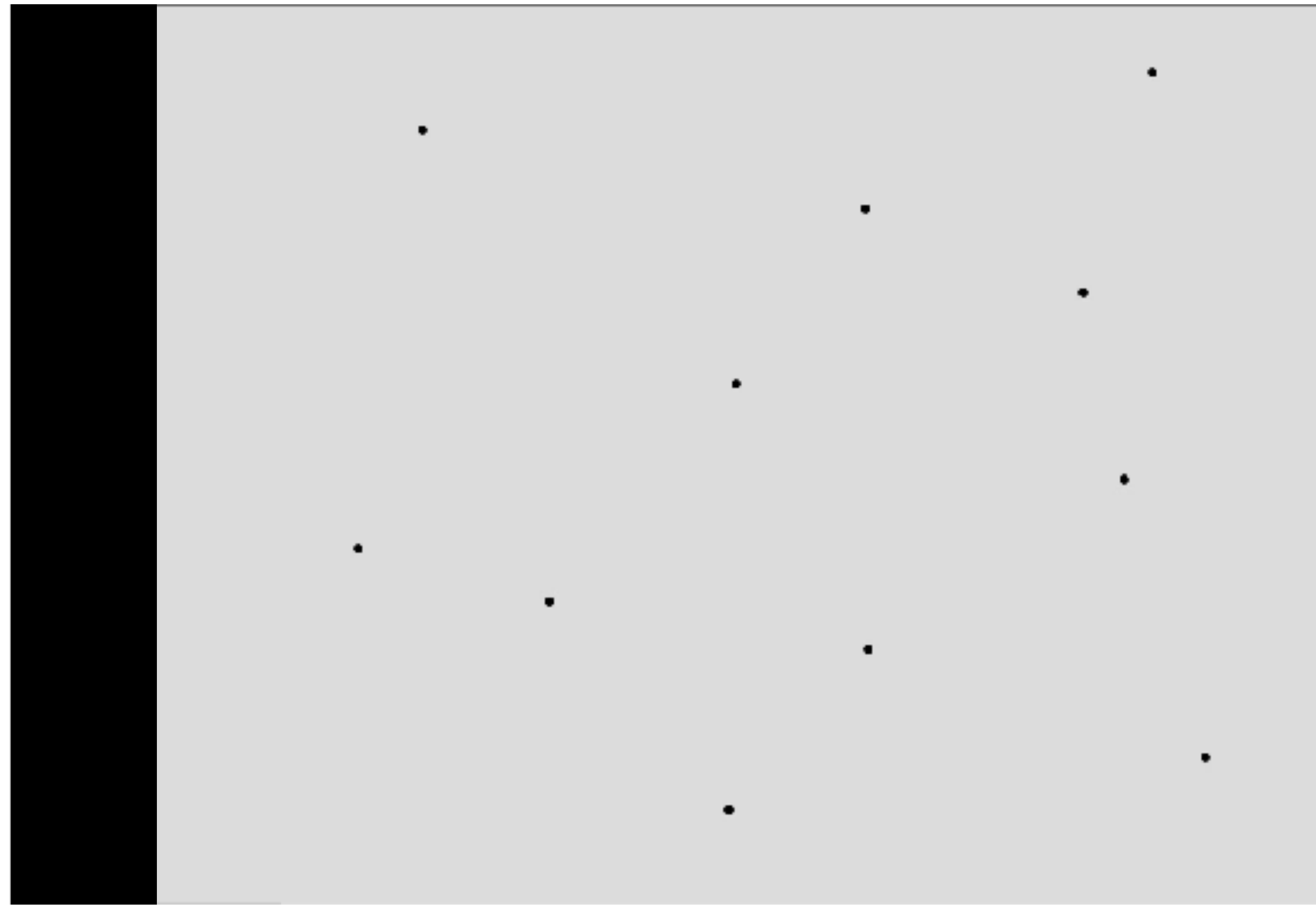

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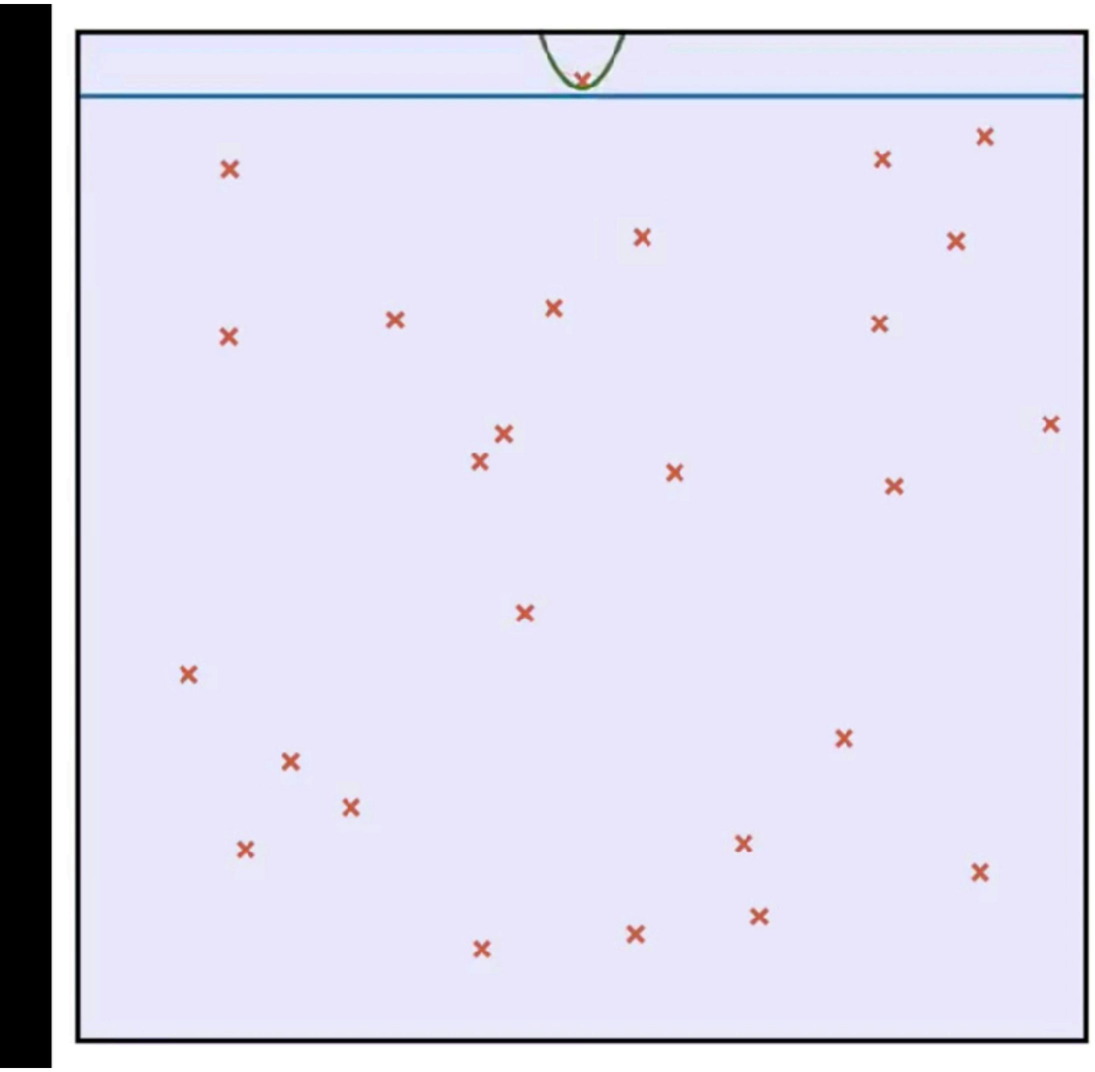

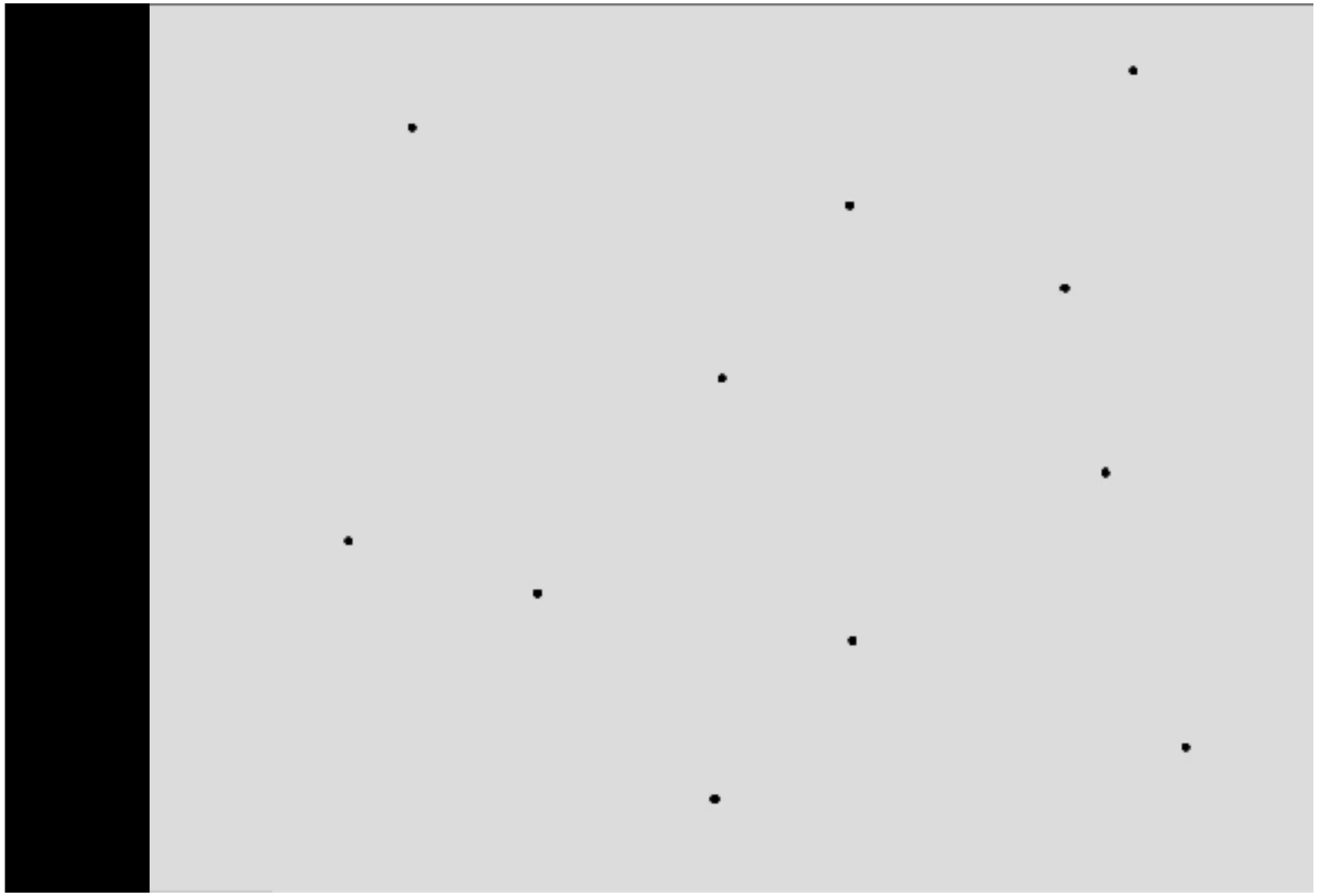

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Point events:

- Sweep line ℓ reaches $p \in \mathcal{P}$

Issue

- Complexity of beach line?

Lemma 4.17

New parabolic arcs can only occur at point events.

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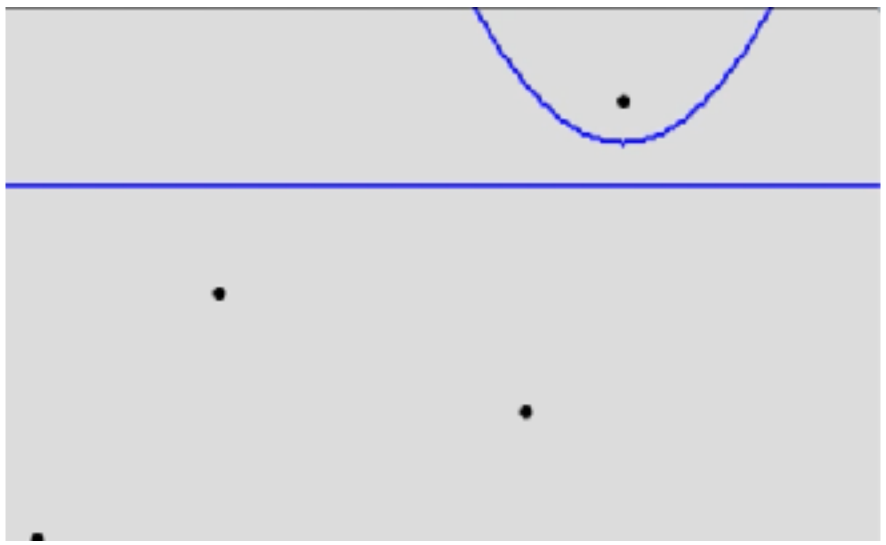

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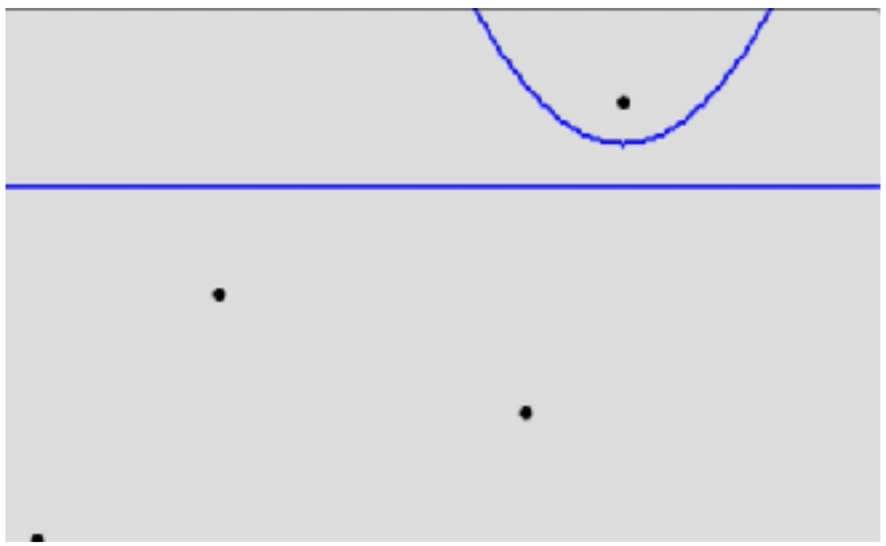

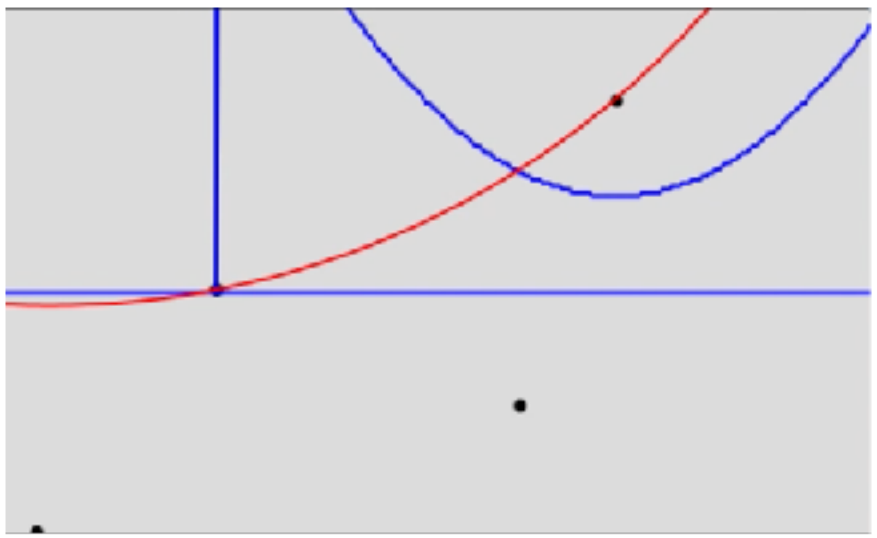

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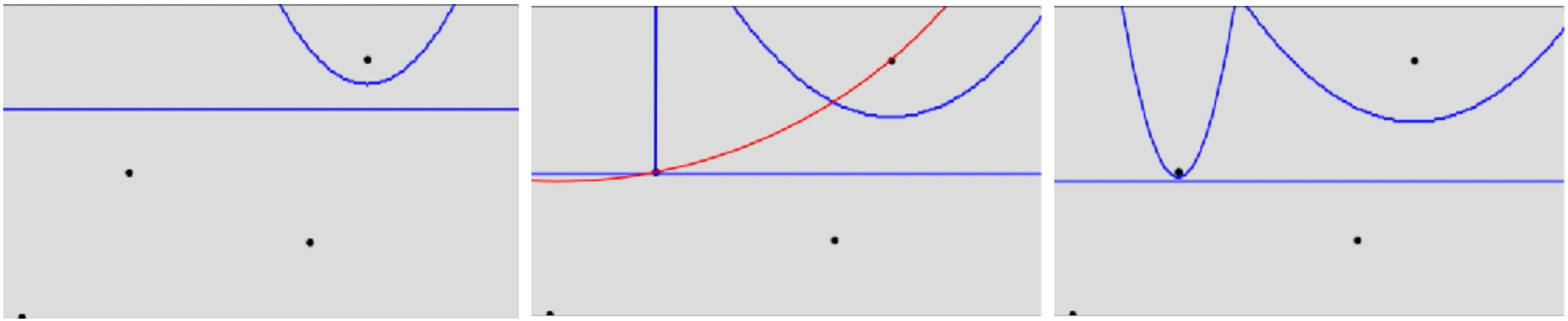

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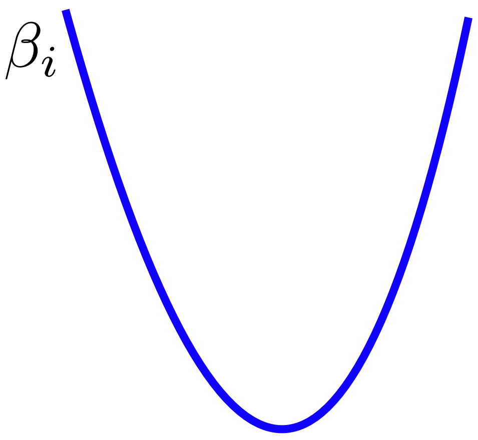

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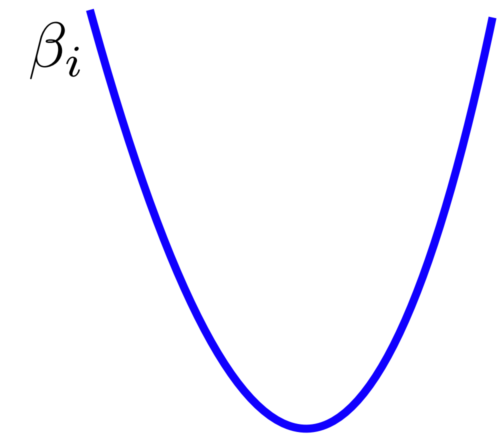

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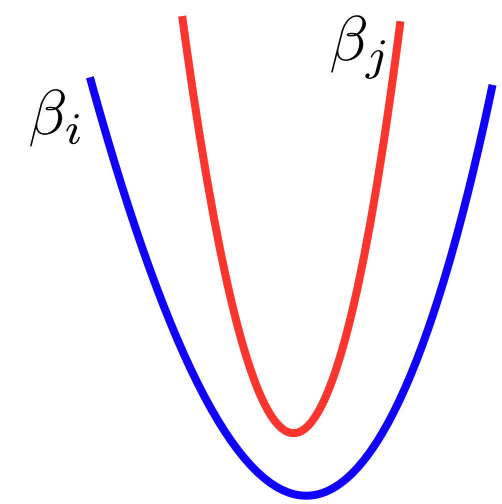

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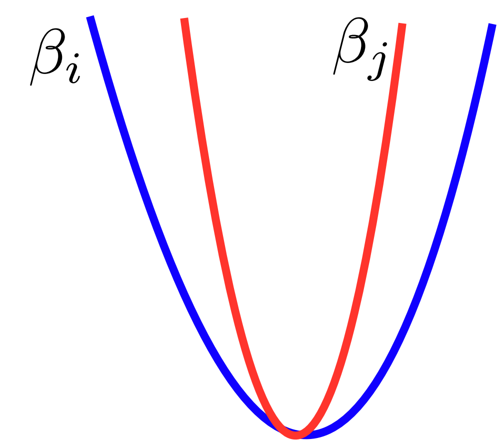

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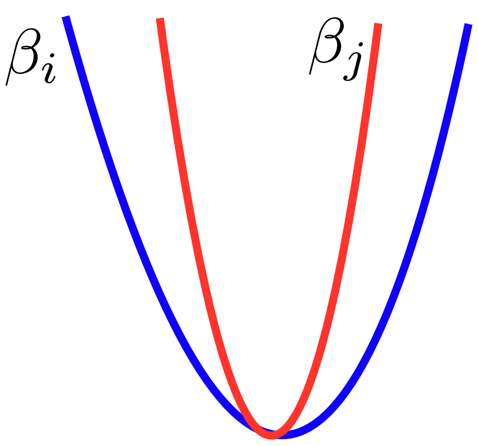

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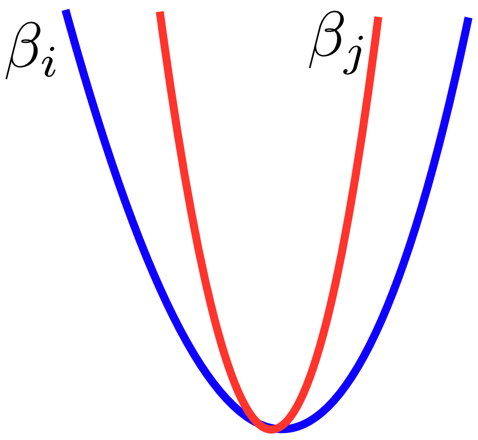
- $\ell.y$:= y-coordinate of ℓ at „piercing event".

  $\Rightarrow$ At moment $\ell.y$: One joint point



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New parabolic arcs can only occur at point events.

Proof:

- In general, parabolas correspond to sites, so:
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- $\ell.y$:= y-coordinate of ℓ at „piercing event“.

  $\Rightarrow$ At moment $\ell.y$: One joint point

- $p_i.y \neq p_j.y$, otherwise

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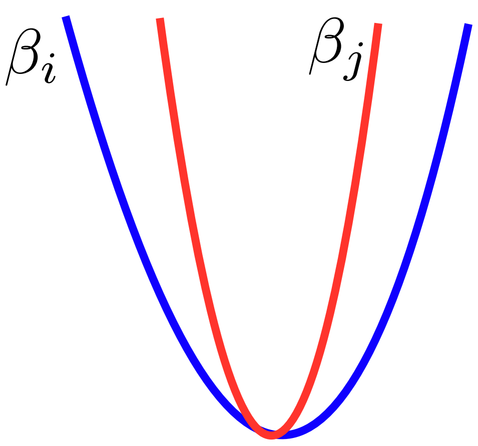

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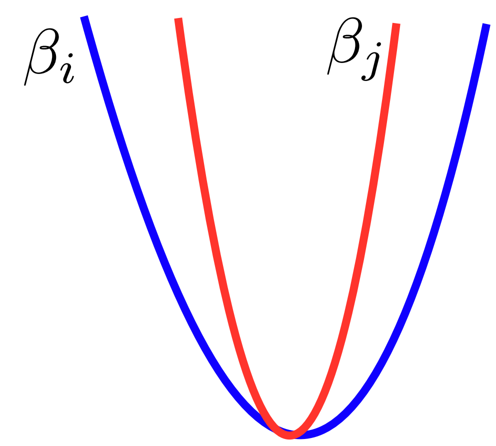

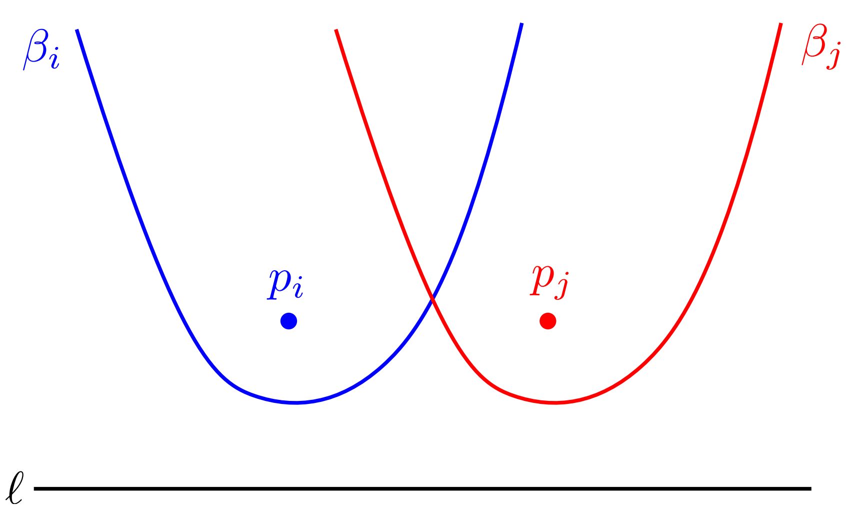

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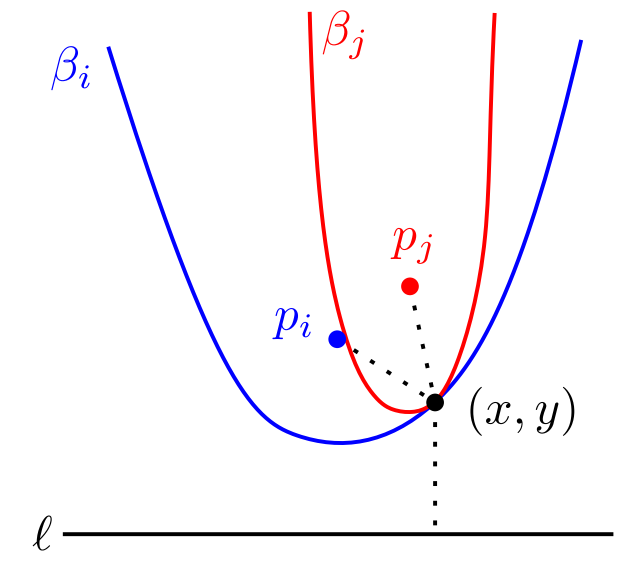

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- Equation for β_j:

$$(\ell.y - y)^2 = (p_j.x - x)^2 + (p_j.x - y)^2$$

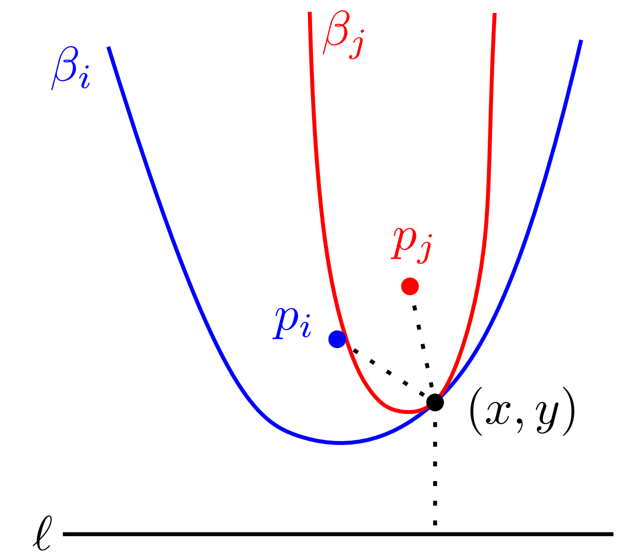
$$\Leftrightarrow \ell.y^2 - 2 \cdot \ell.y \cdot y + y^2$$

$$= p_j.x^2 - 2 \cdot x \cdot p_j.x + x^2 + p_j.y^2 - 2 \cdot y \cdot p_j.y + y^2$$

$$\Leftrightarrow y \cdot 2\,(p_j.y - \ell.y)$$

$$= x^2 - 2 \cdot x \cdot p_j.x + p_j.x^2 + p_j.y^2 - \ell.y^2$$

$$\Leftrightarrow y = \frac{x^2 - 2 \cdot x \cdot p_j.x + p_j.x^2 + p_j.y^2 - \ell.y^2}{2\,(p_j.y - \ell.y)}$$



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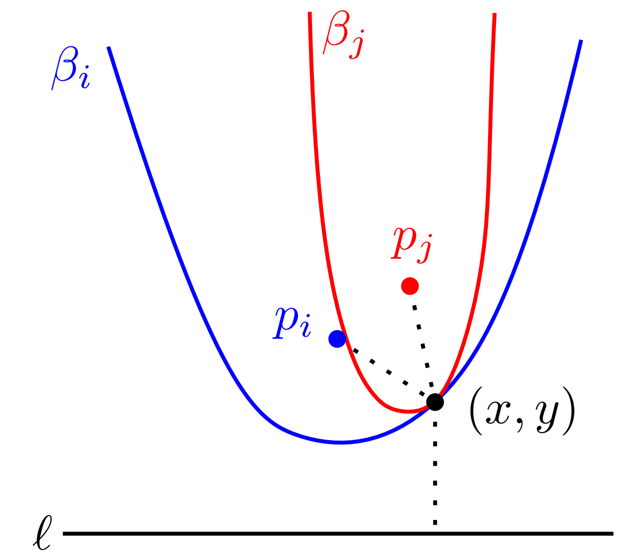

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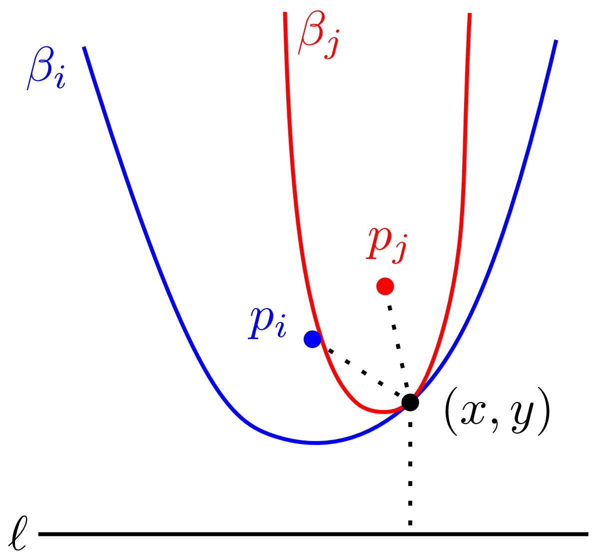

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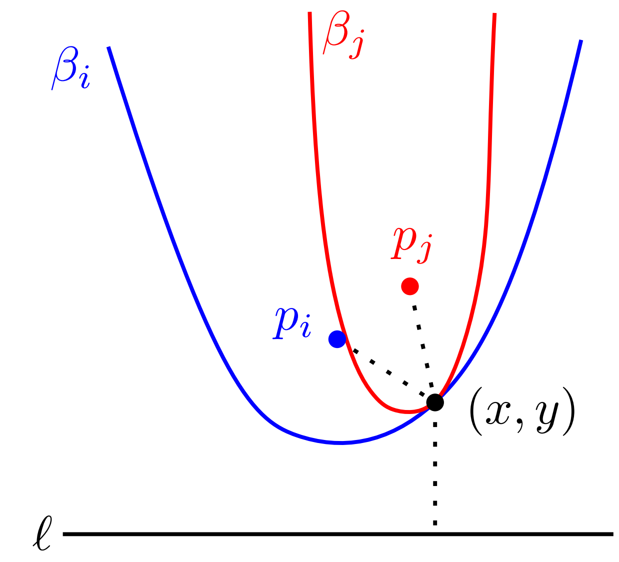

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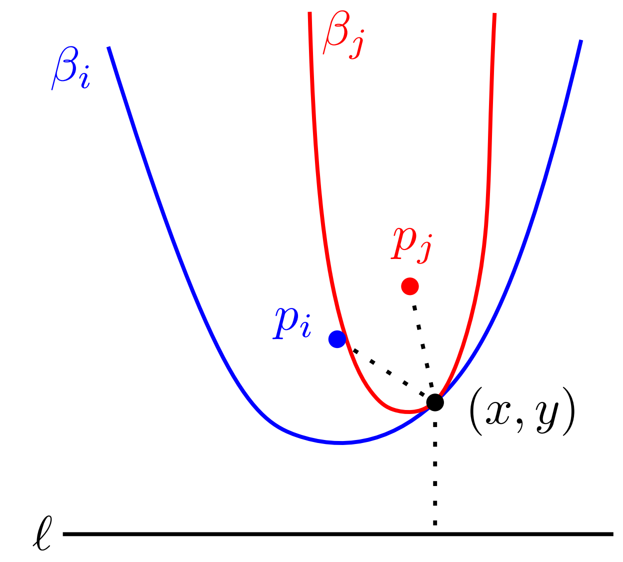

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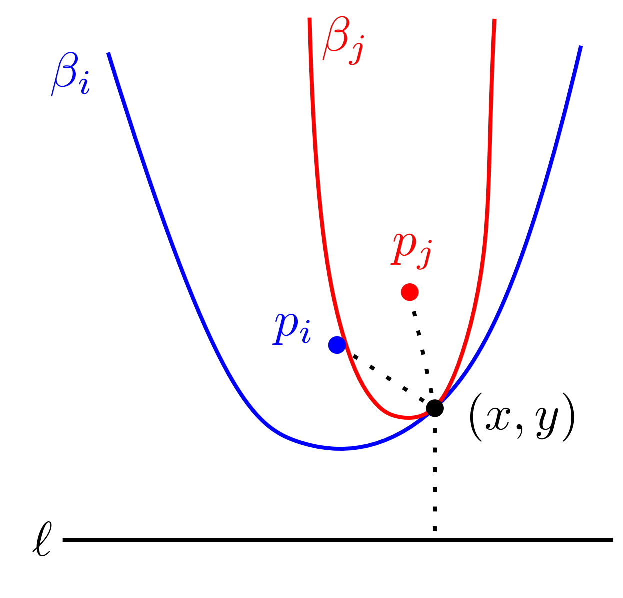

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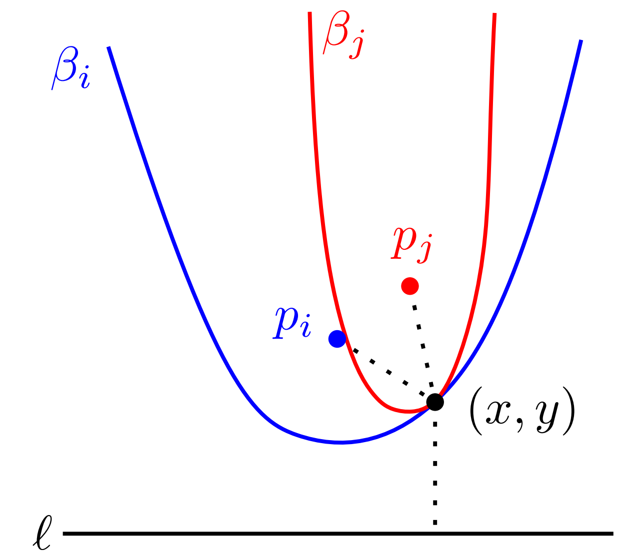

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$\Rightarrow$ Two intersection points ⚡

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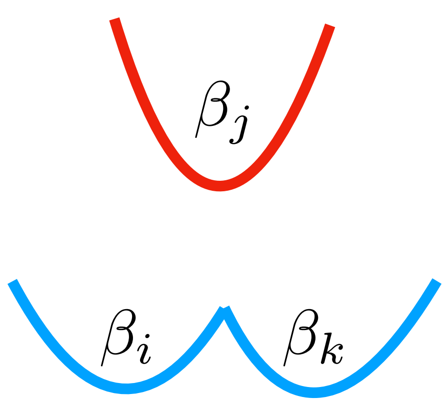

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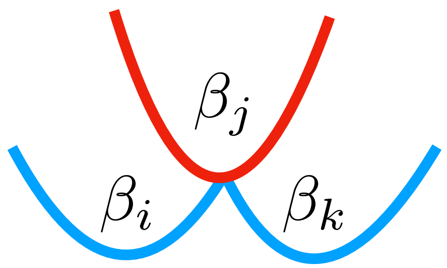

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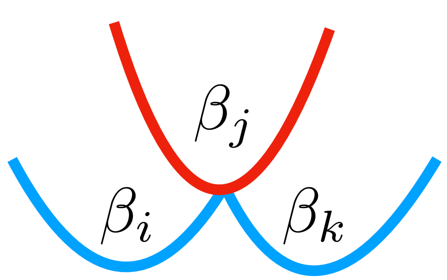

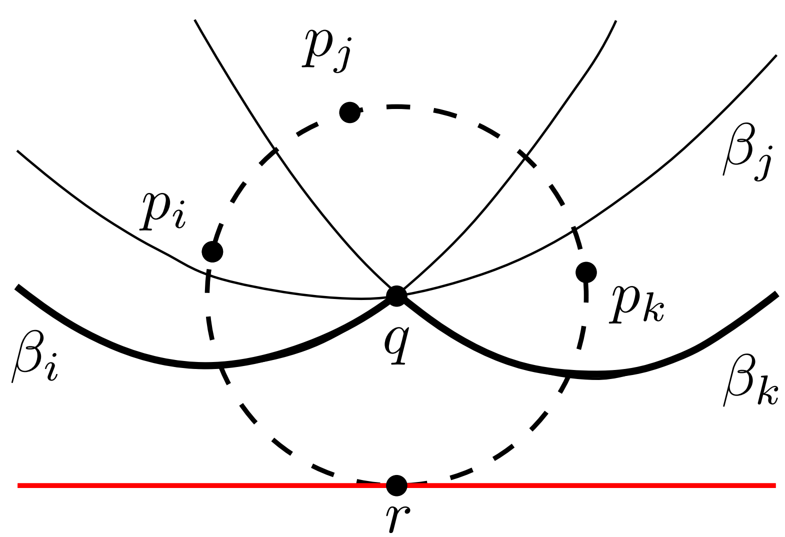

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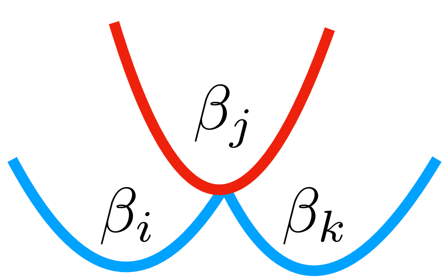

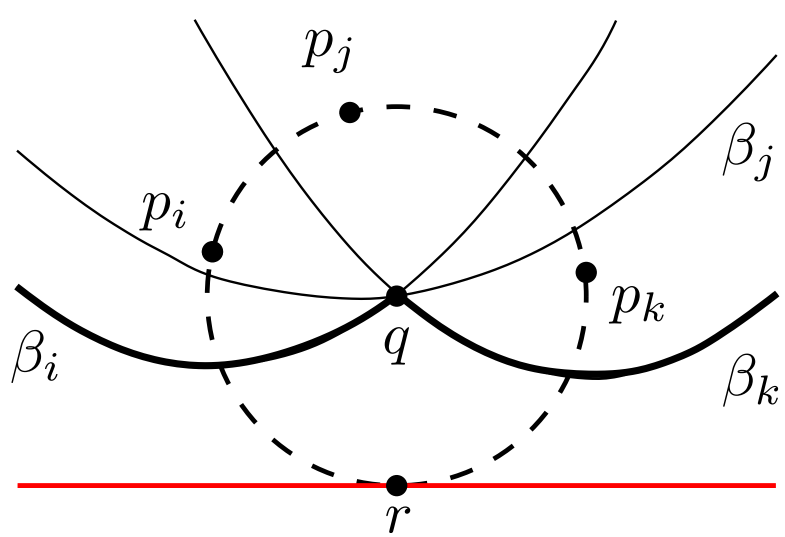

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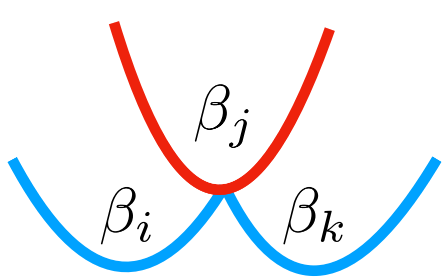

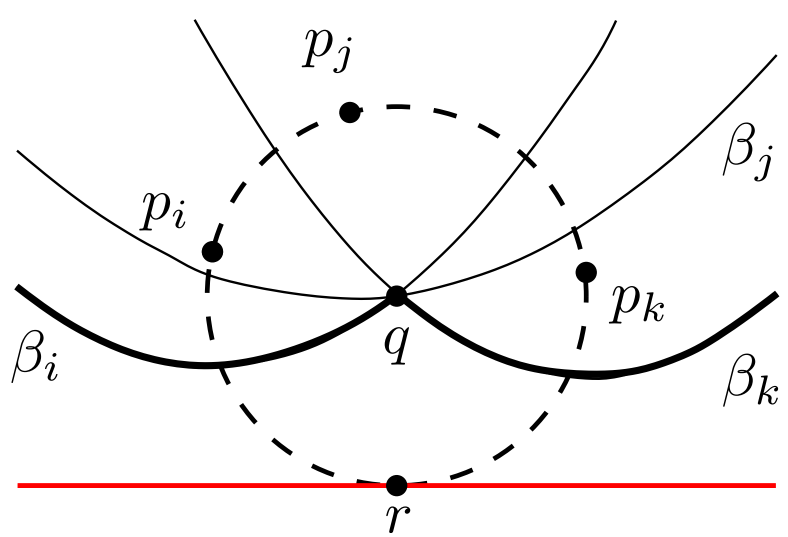

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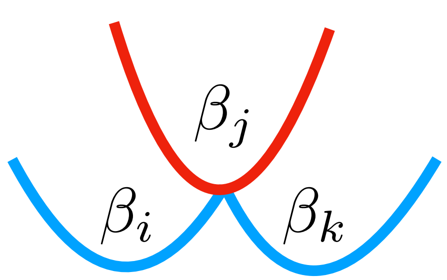

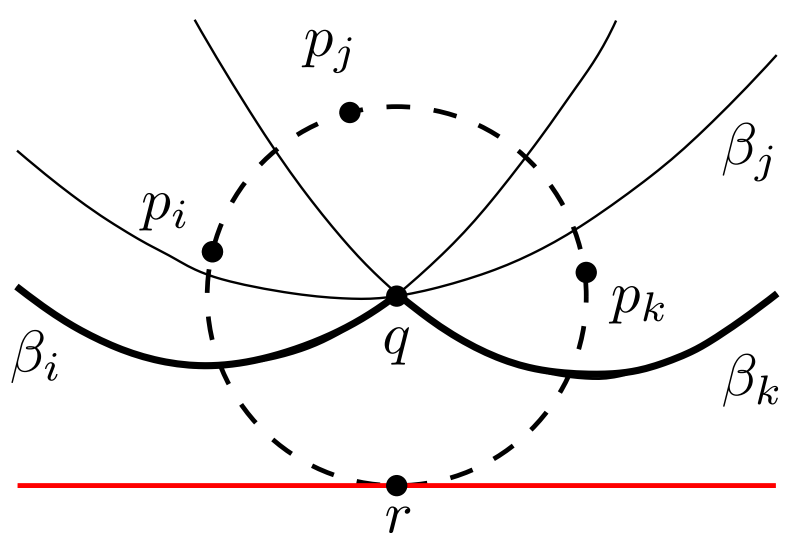

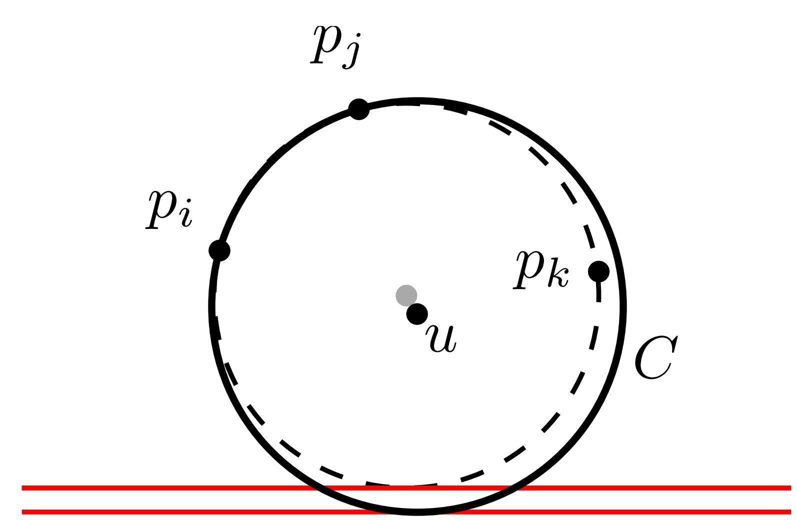

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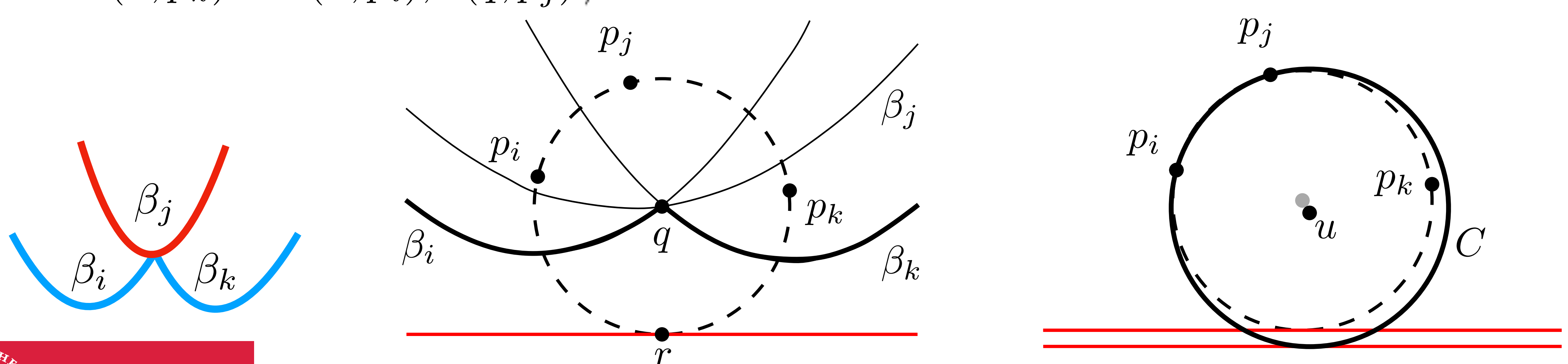

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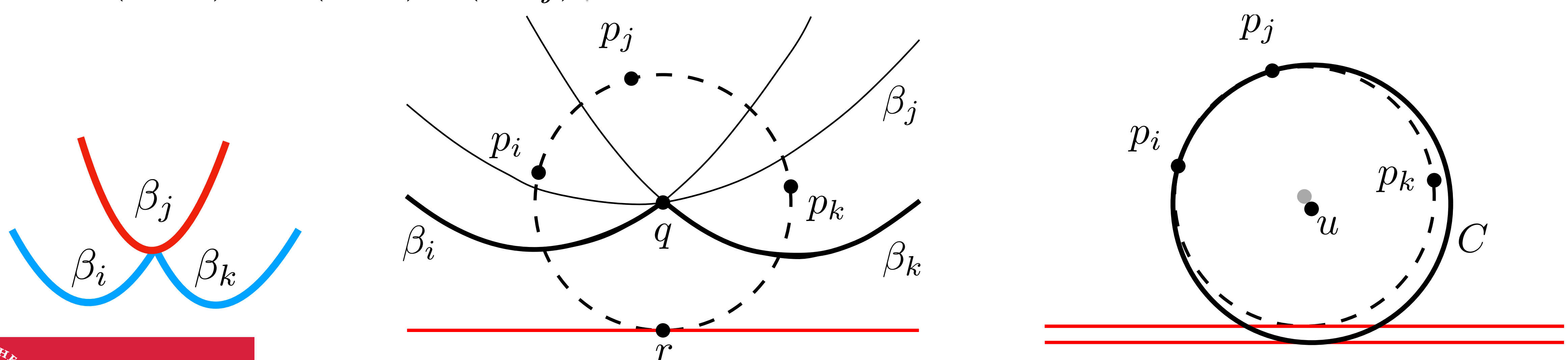
$\square$



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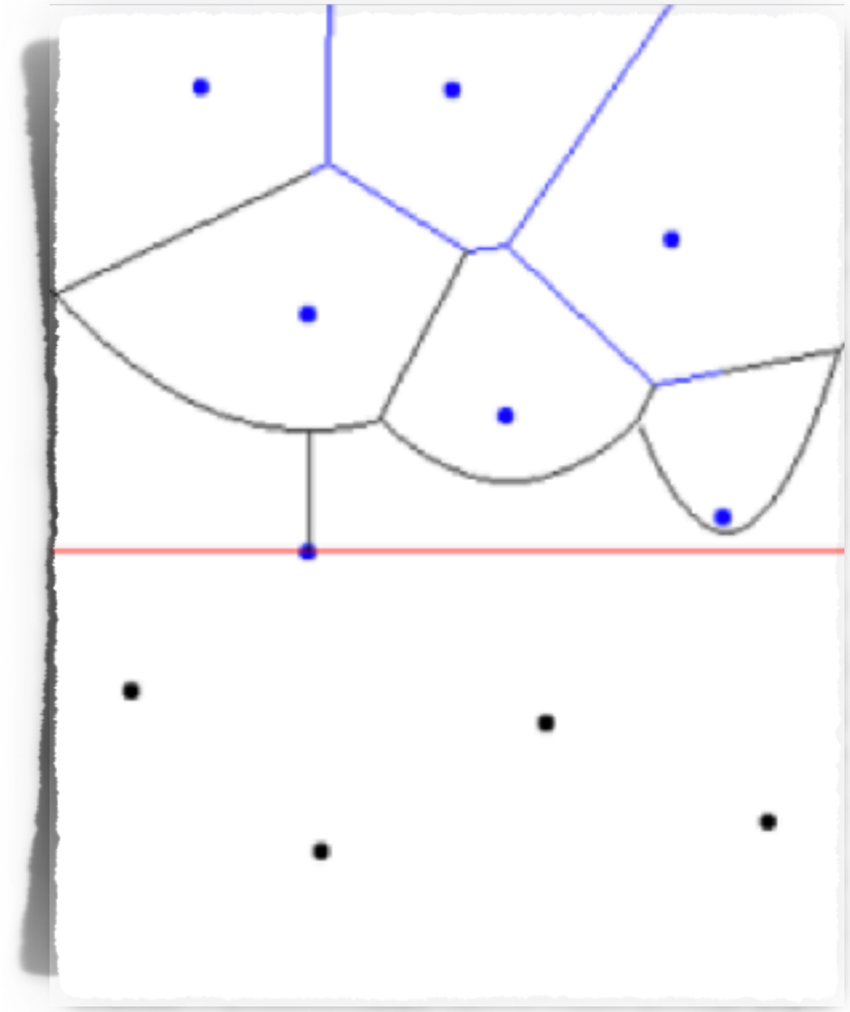

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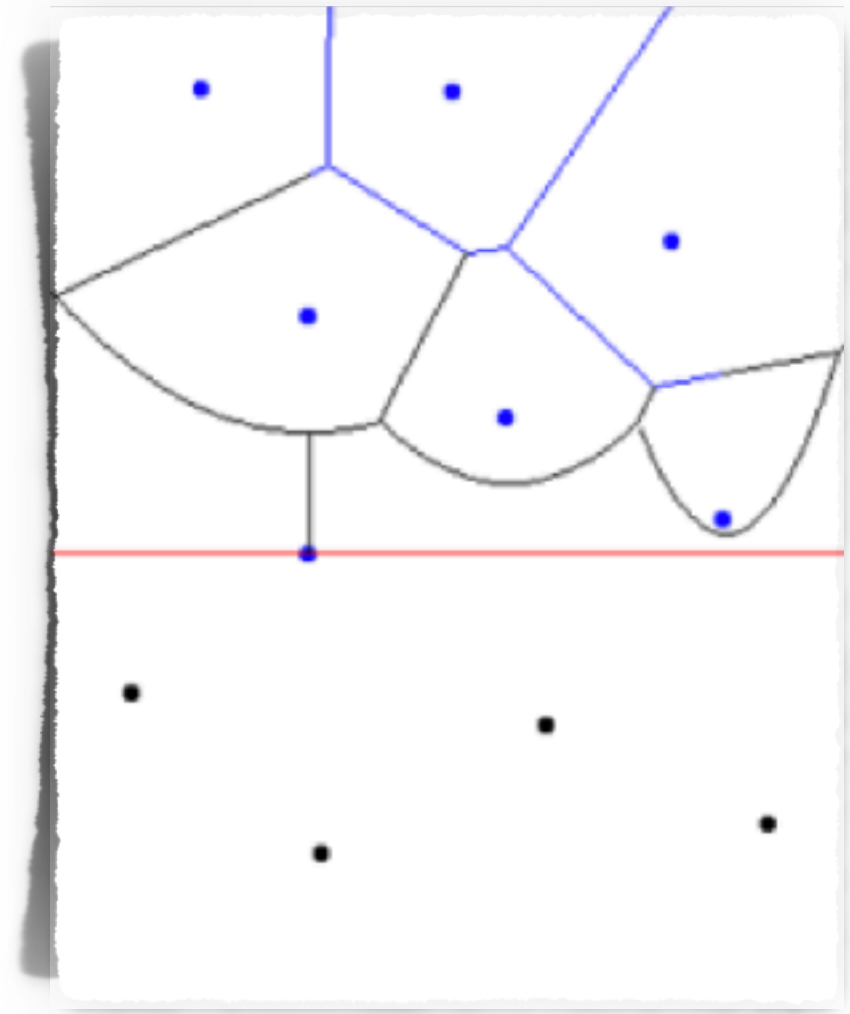

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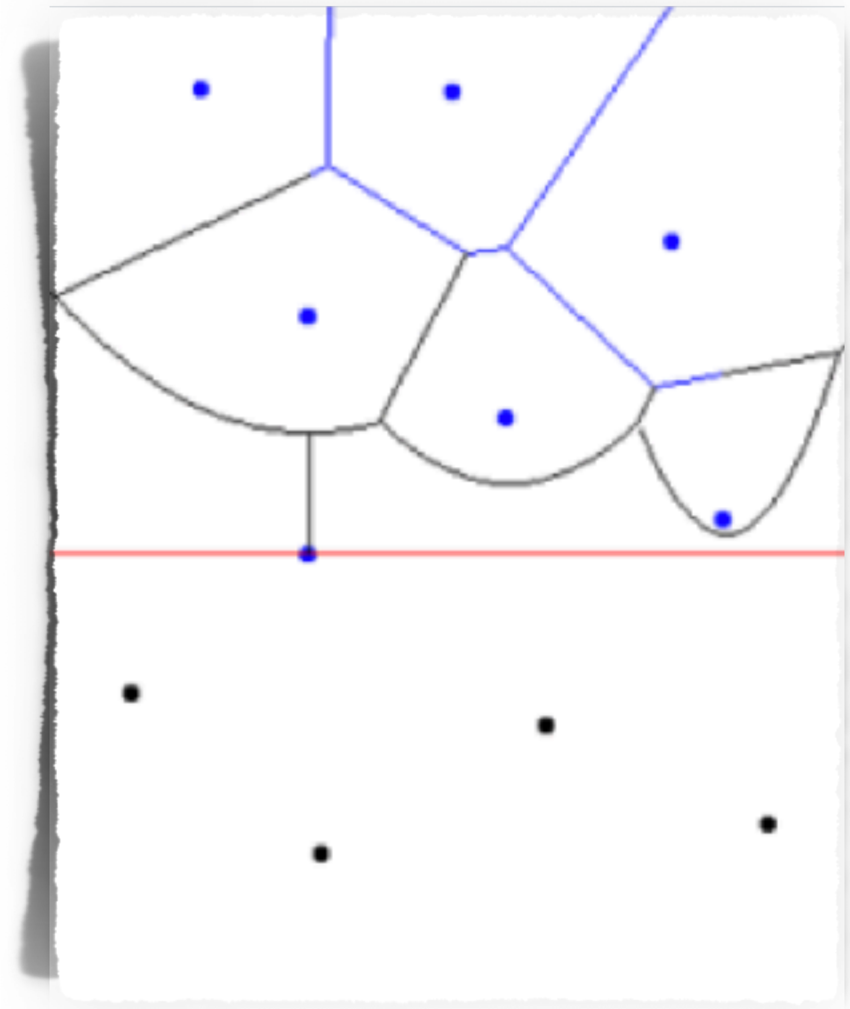

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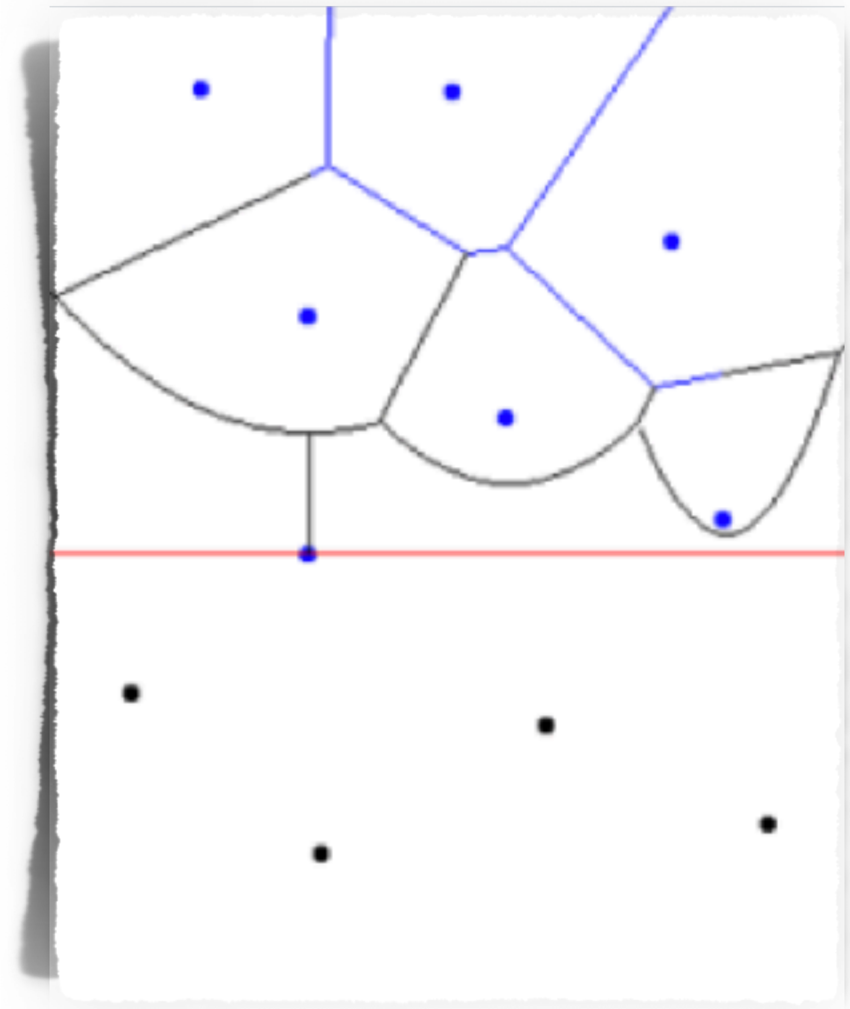

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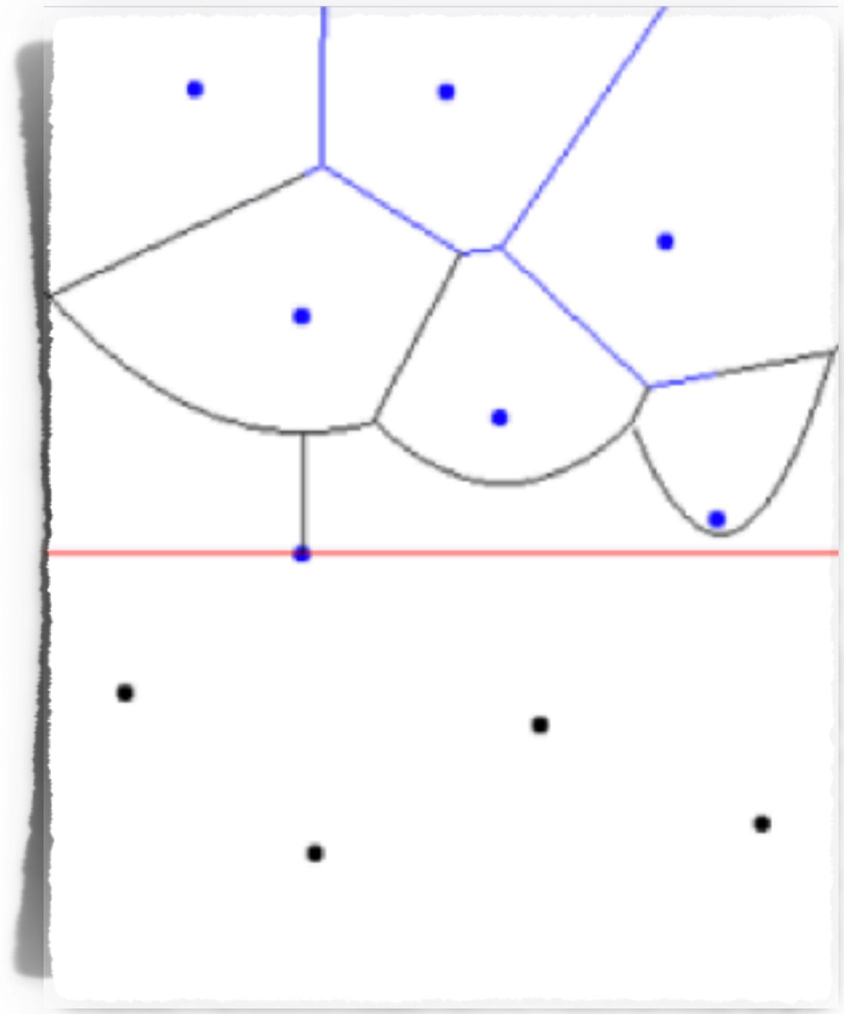

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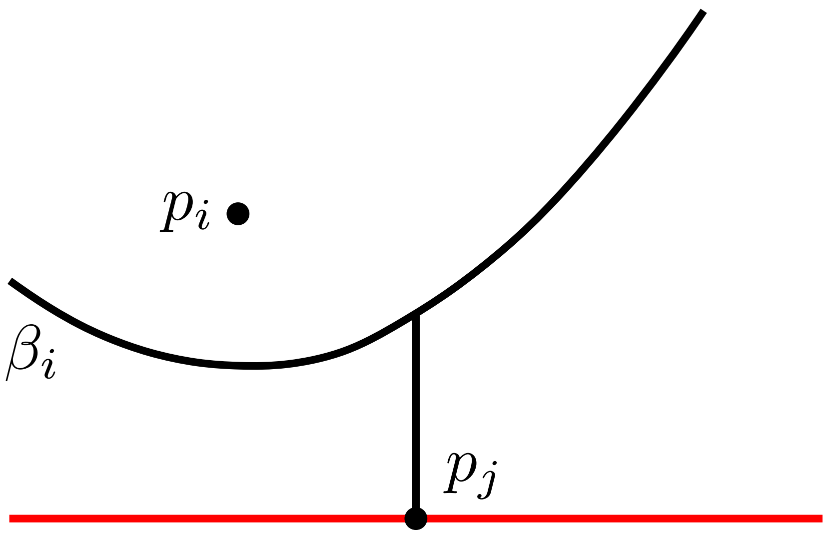

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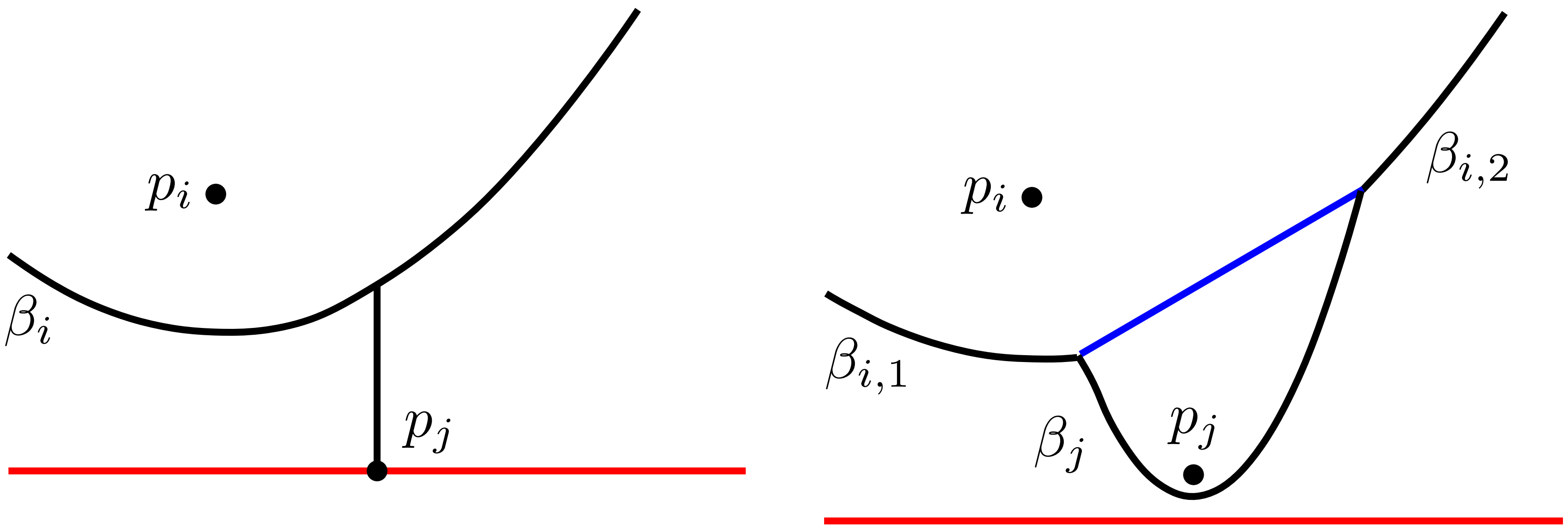

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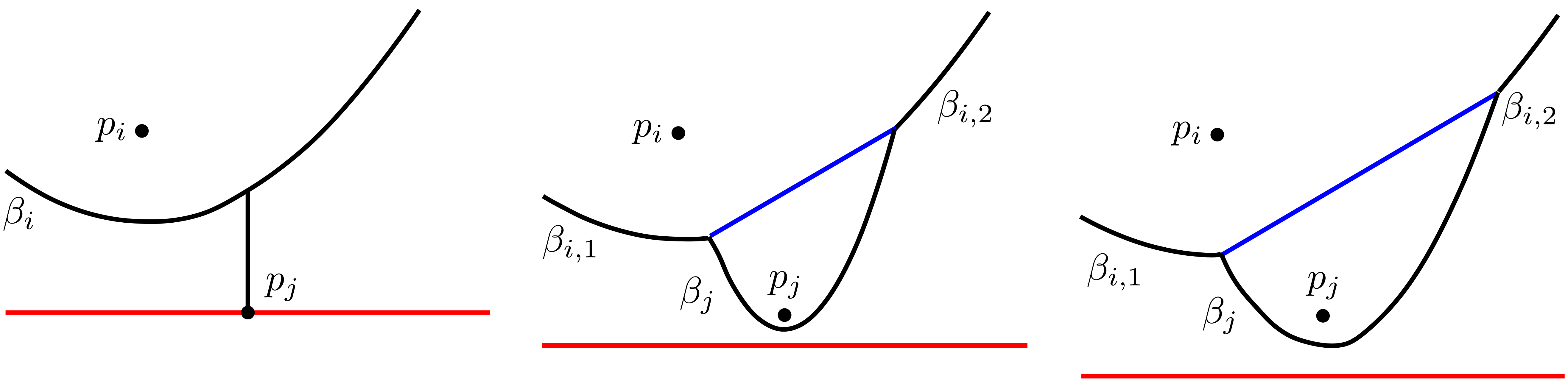

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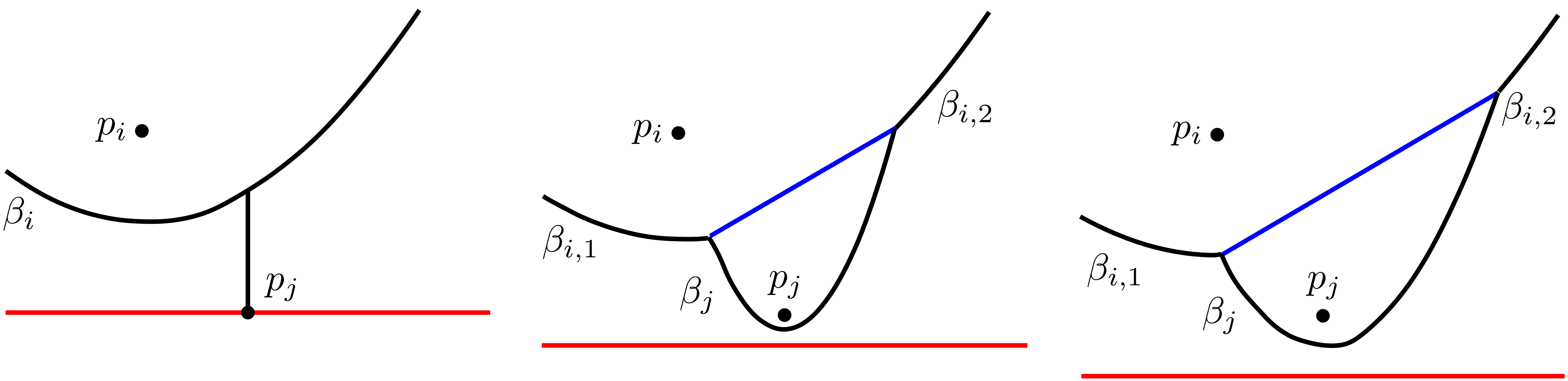

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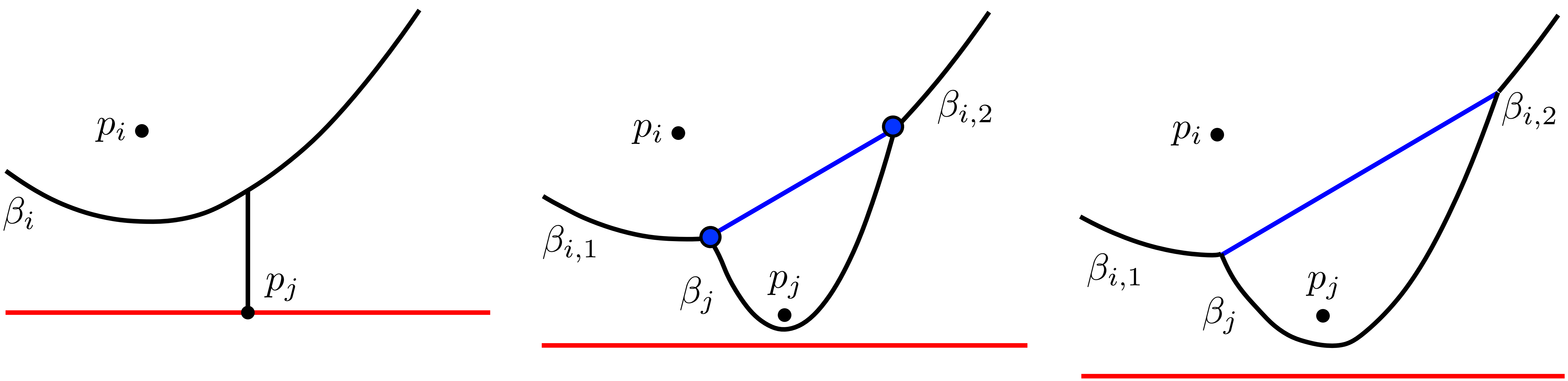

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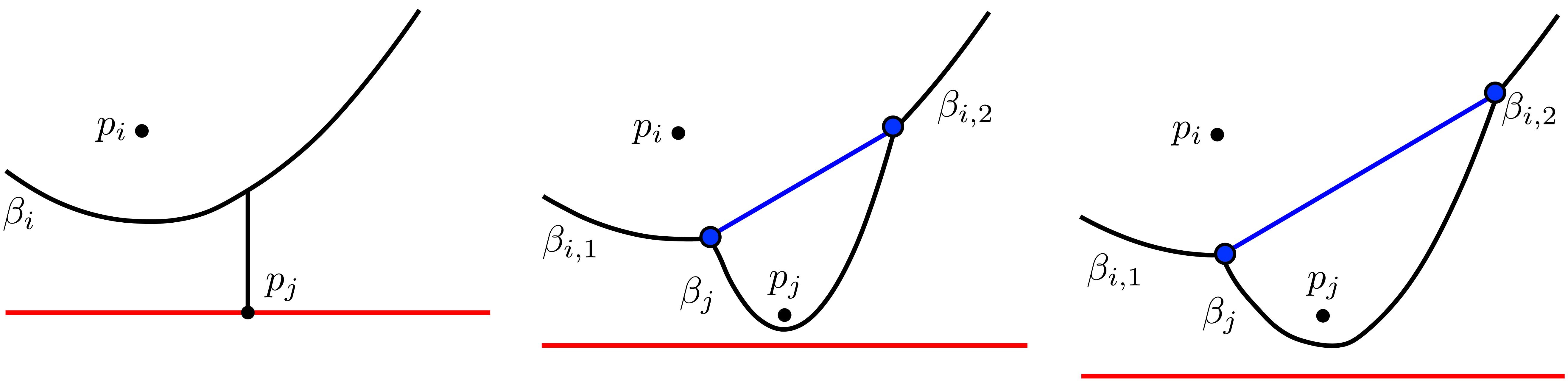

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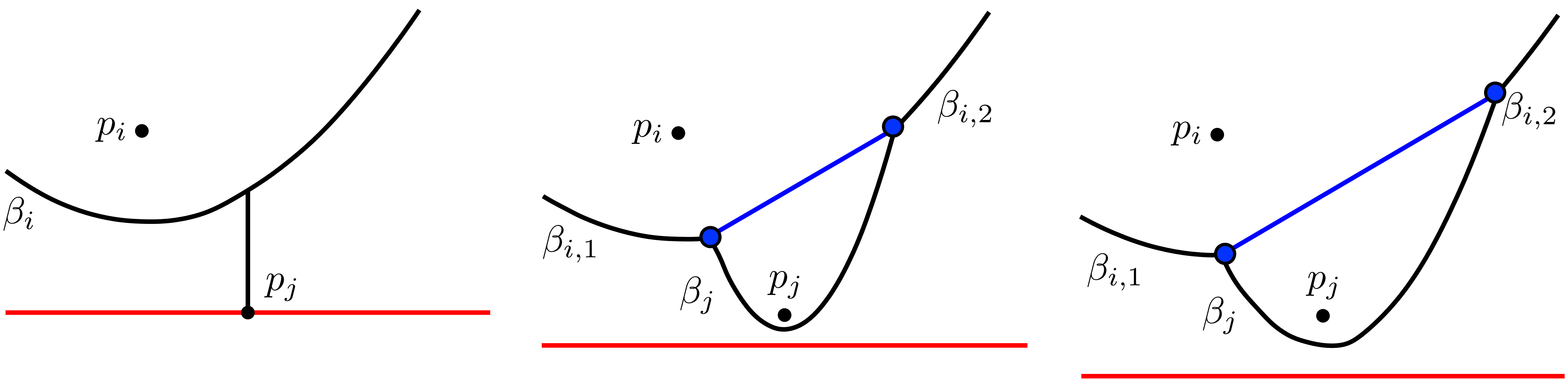

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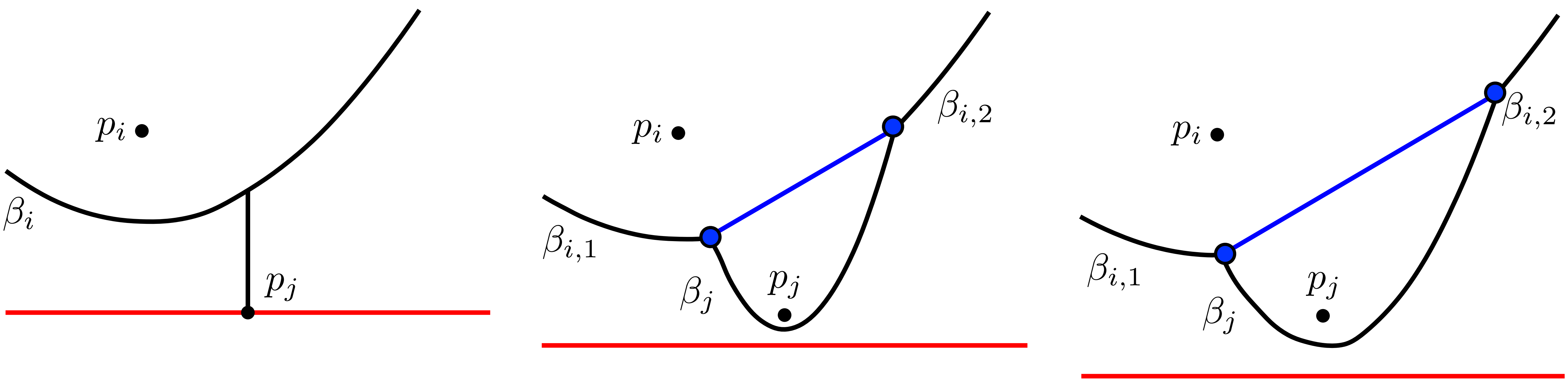

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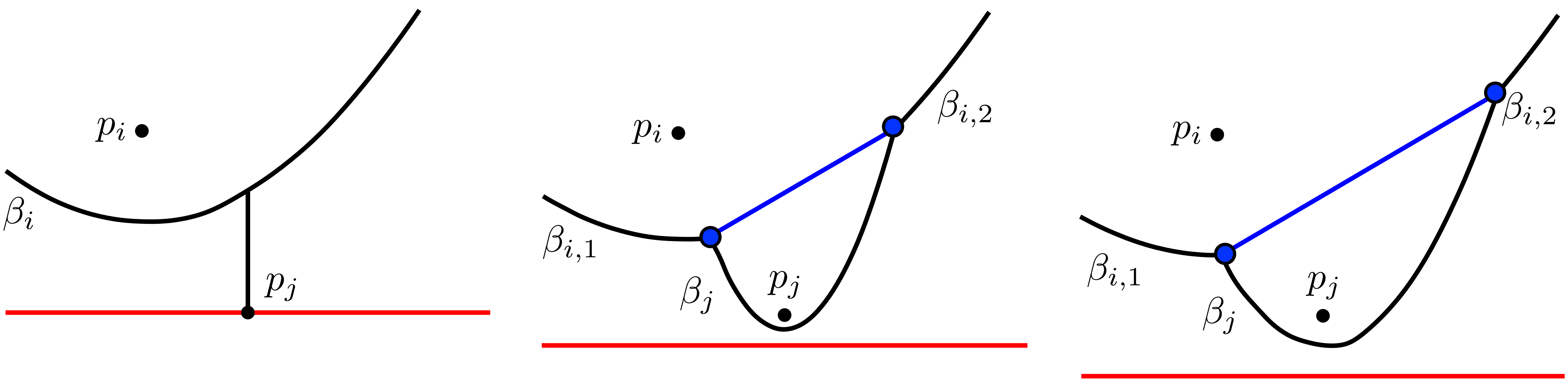

- Edge i.g. not connected to already computed part of $Vor(\mathcal{P})$

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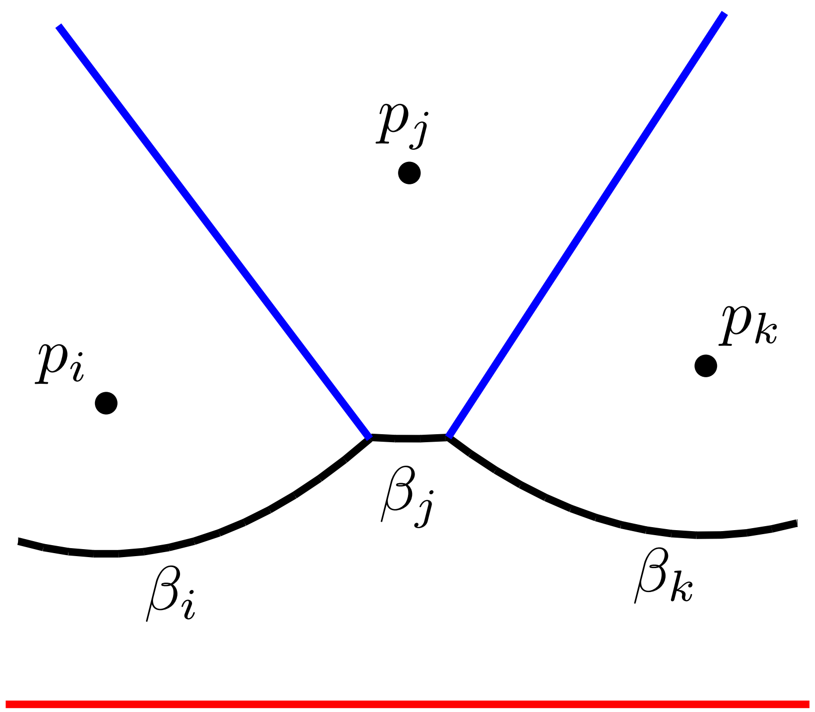

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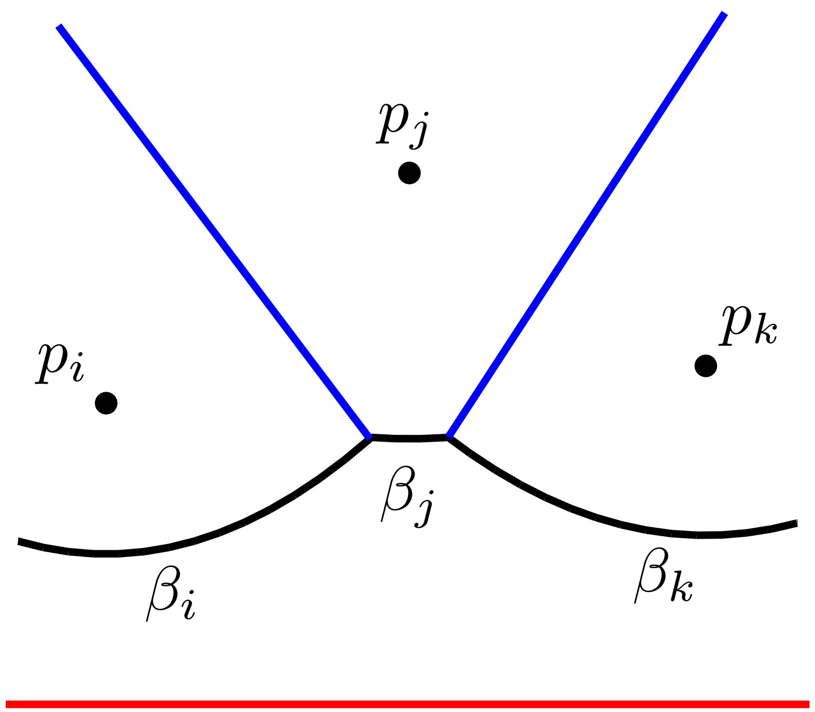

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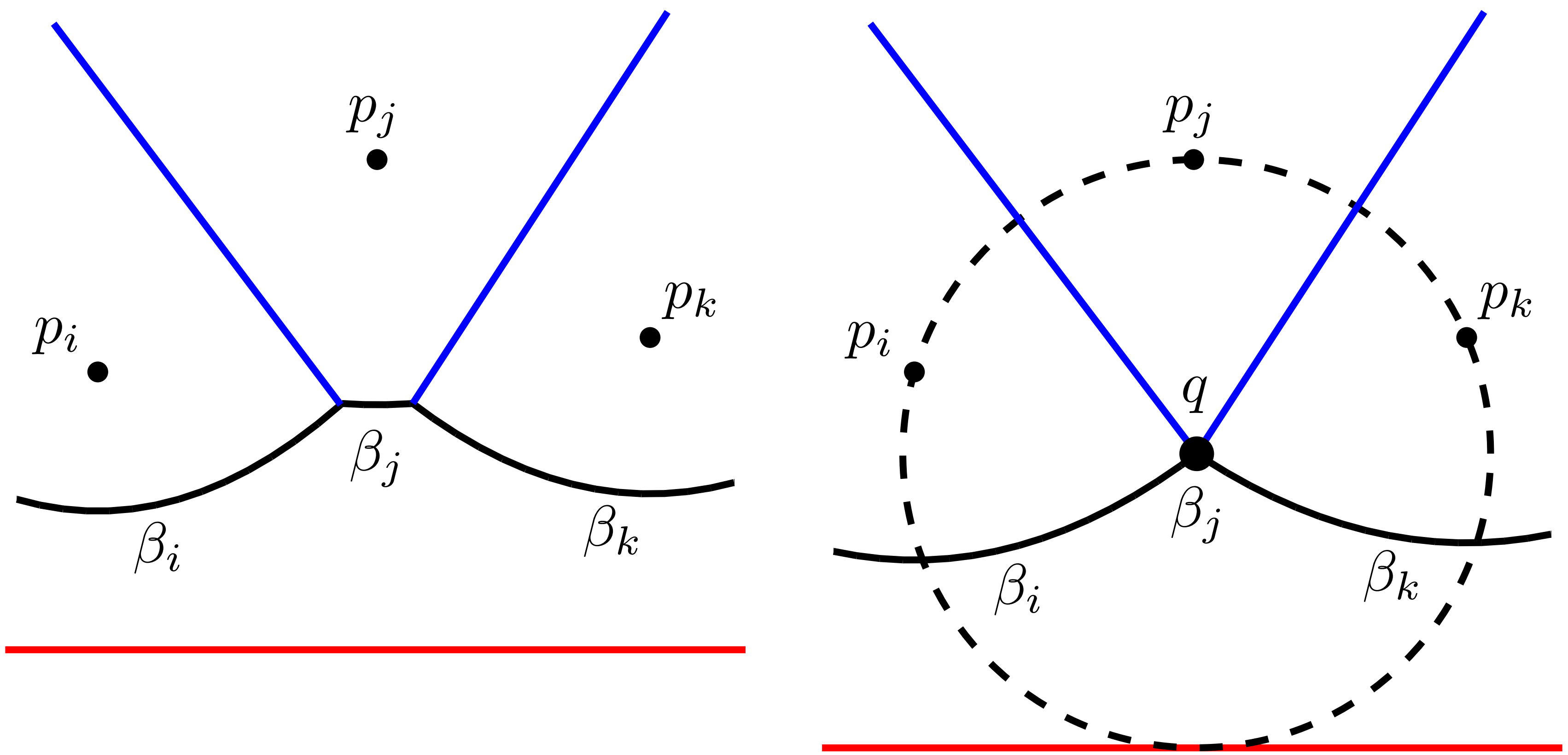

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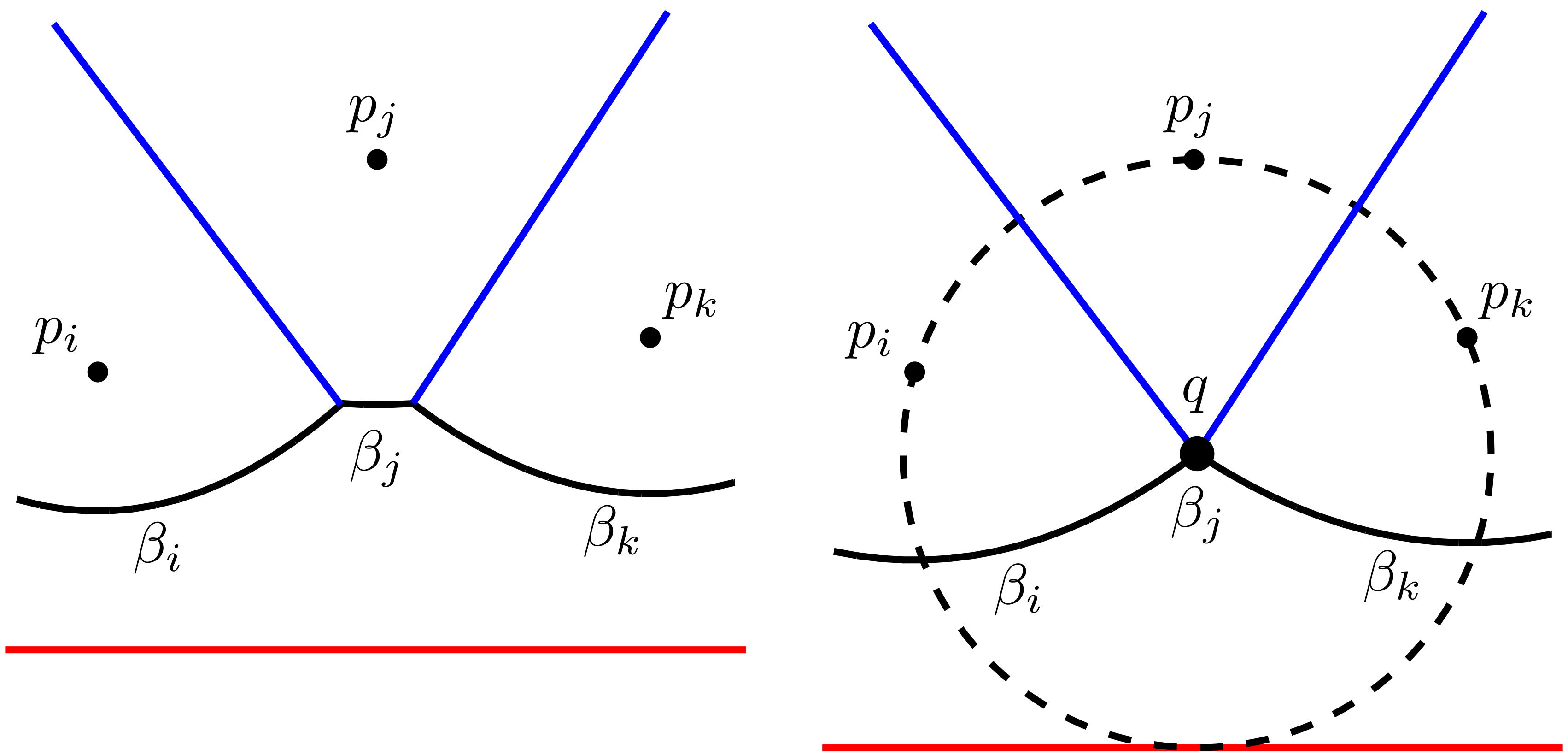

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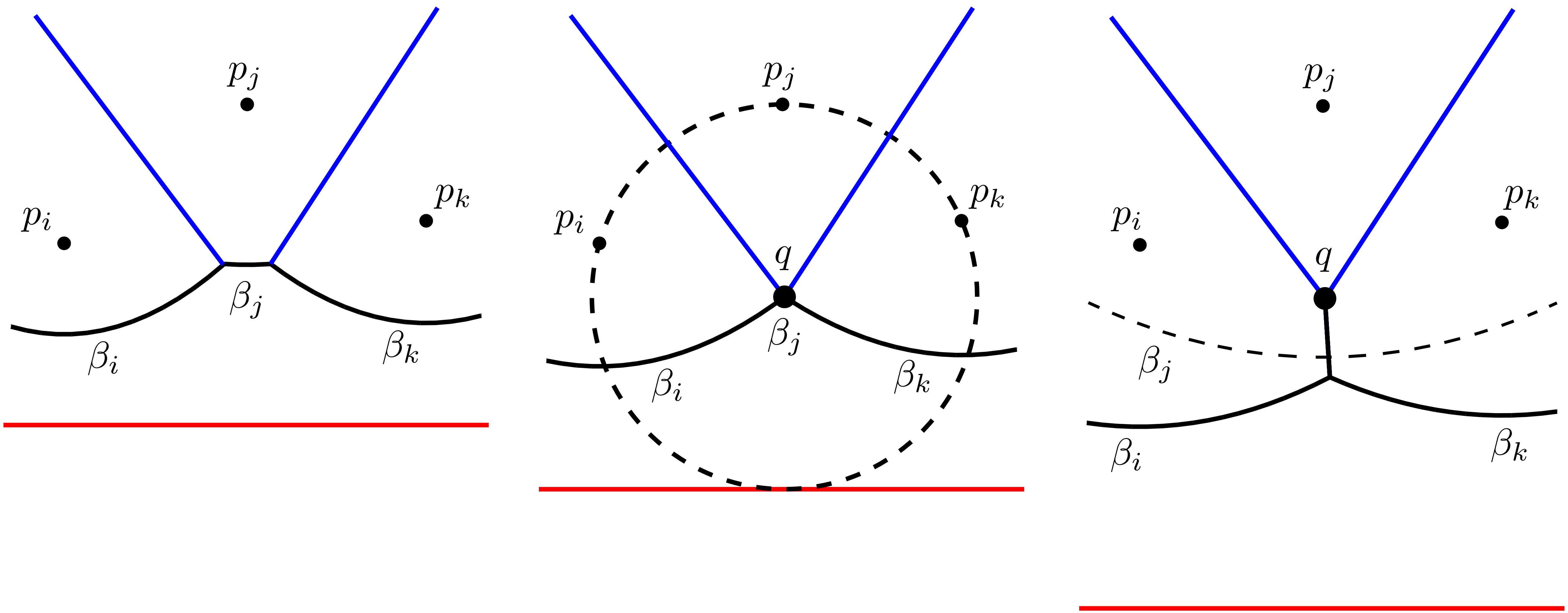

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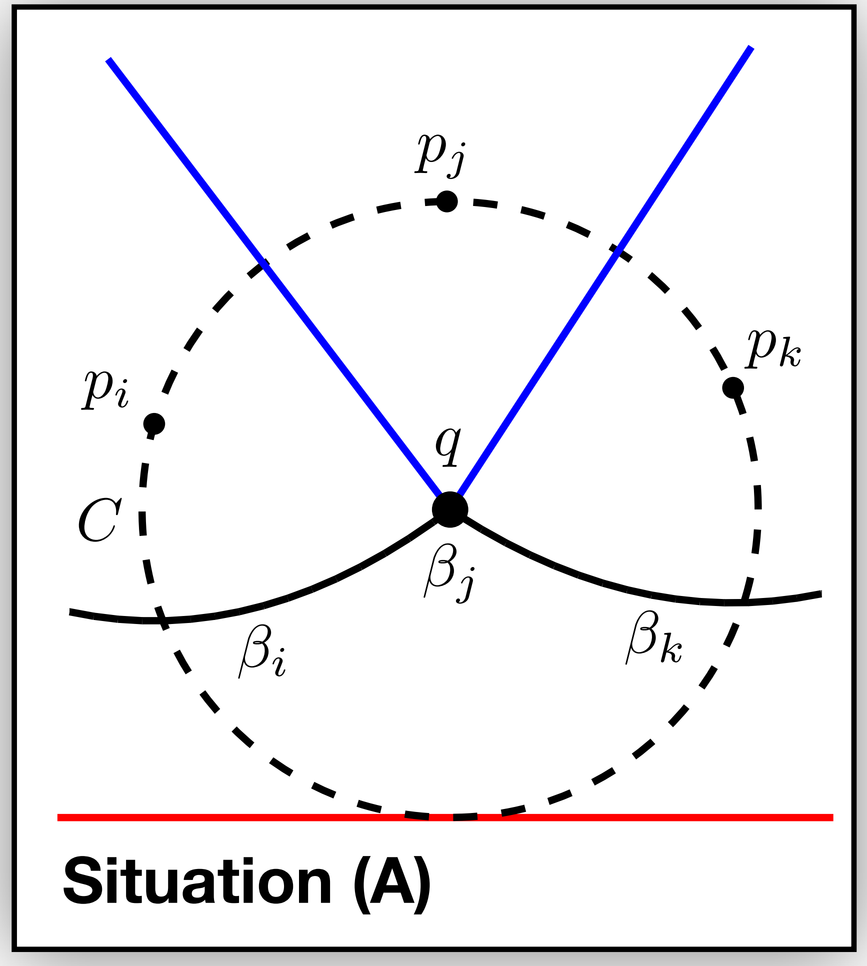

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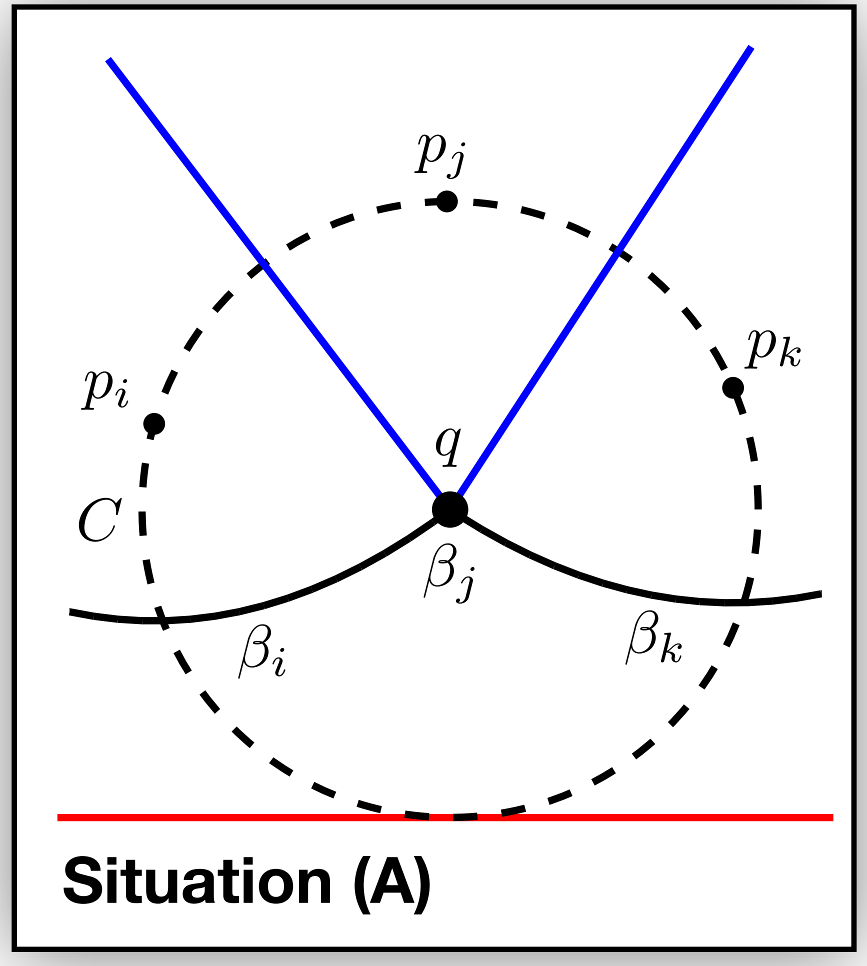

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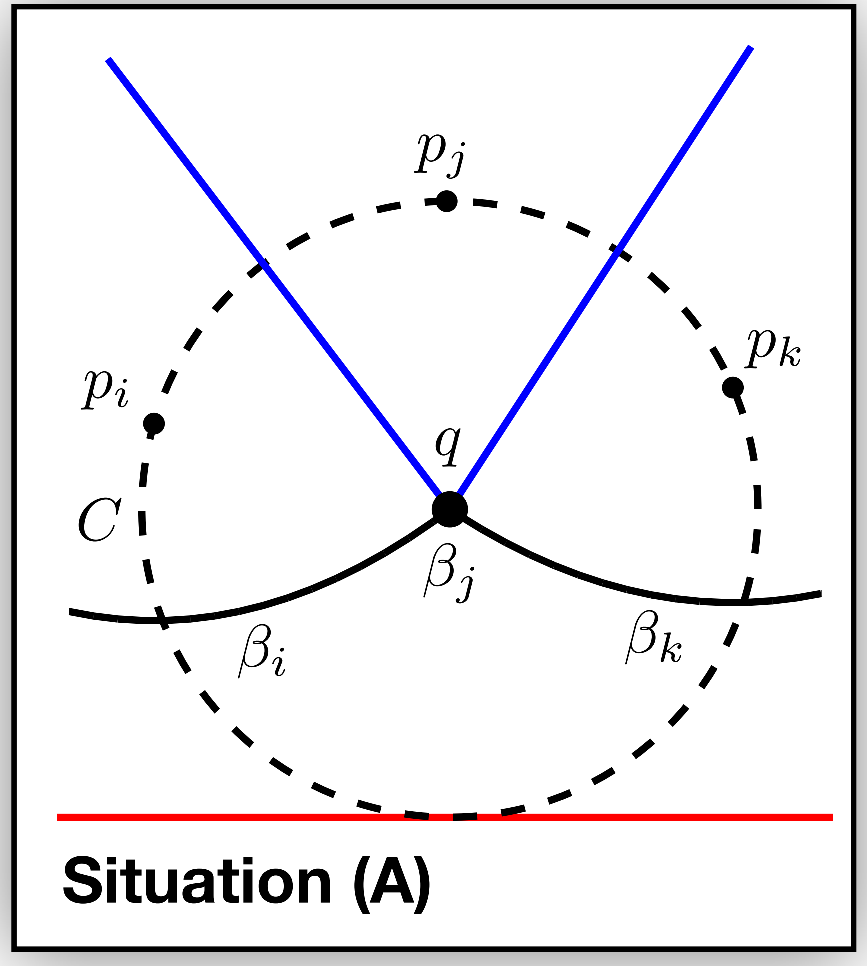

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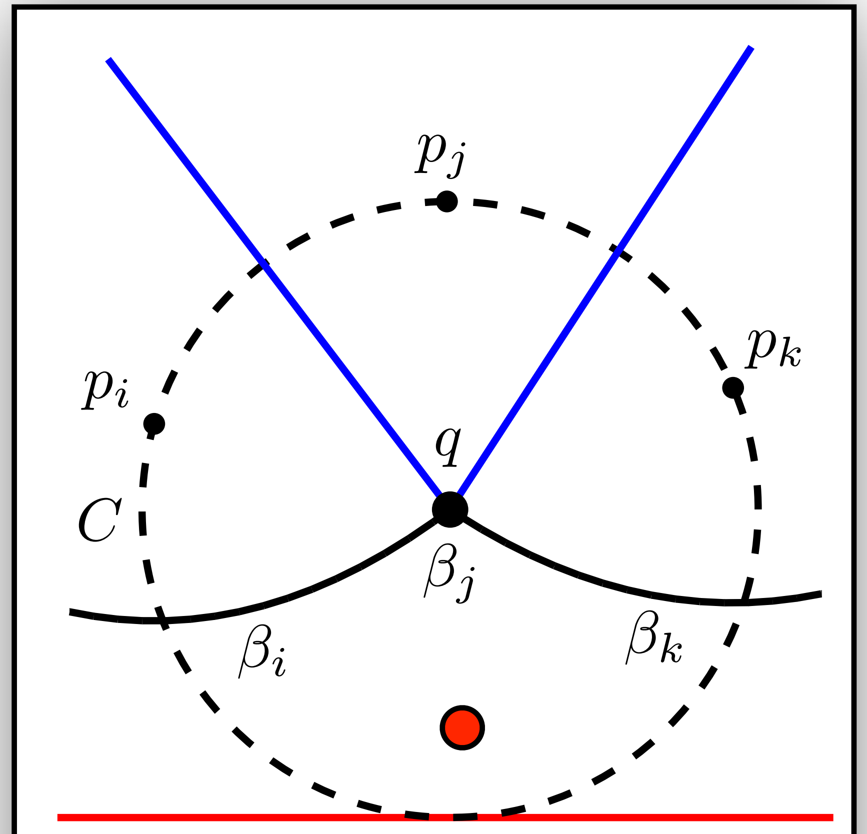

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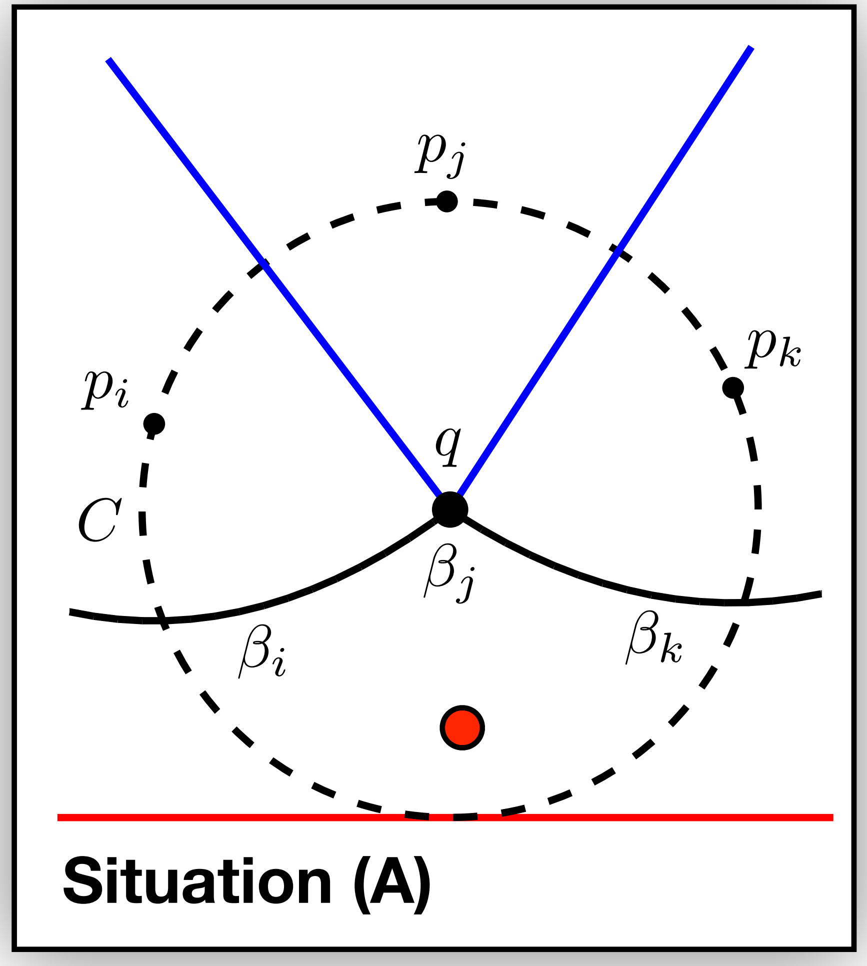

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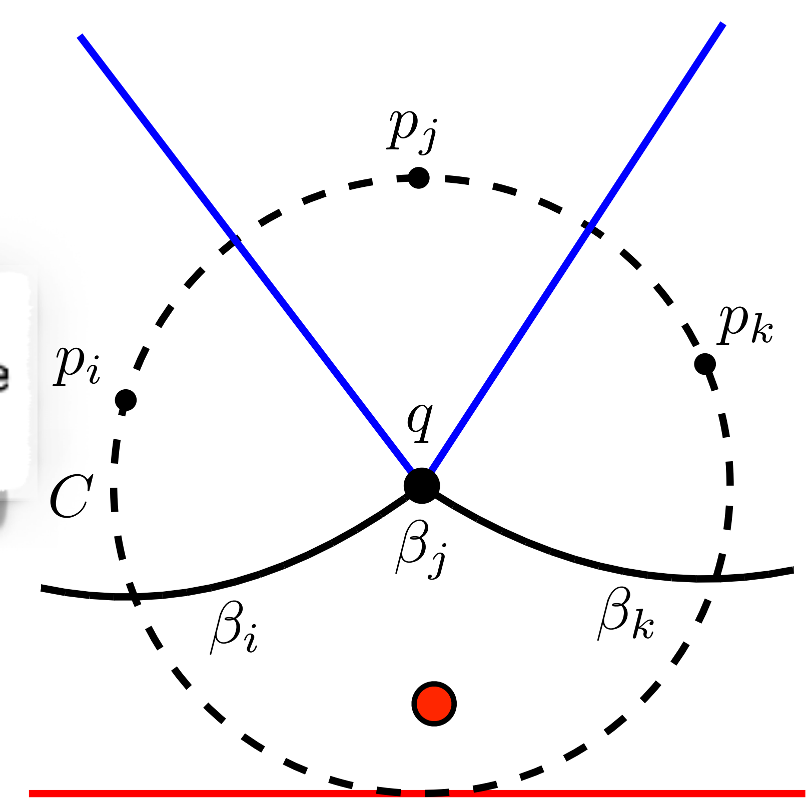

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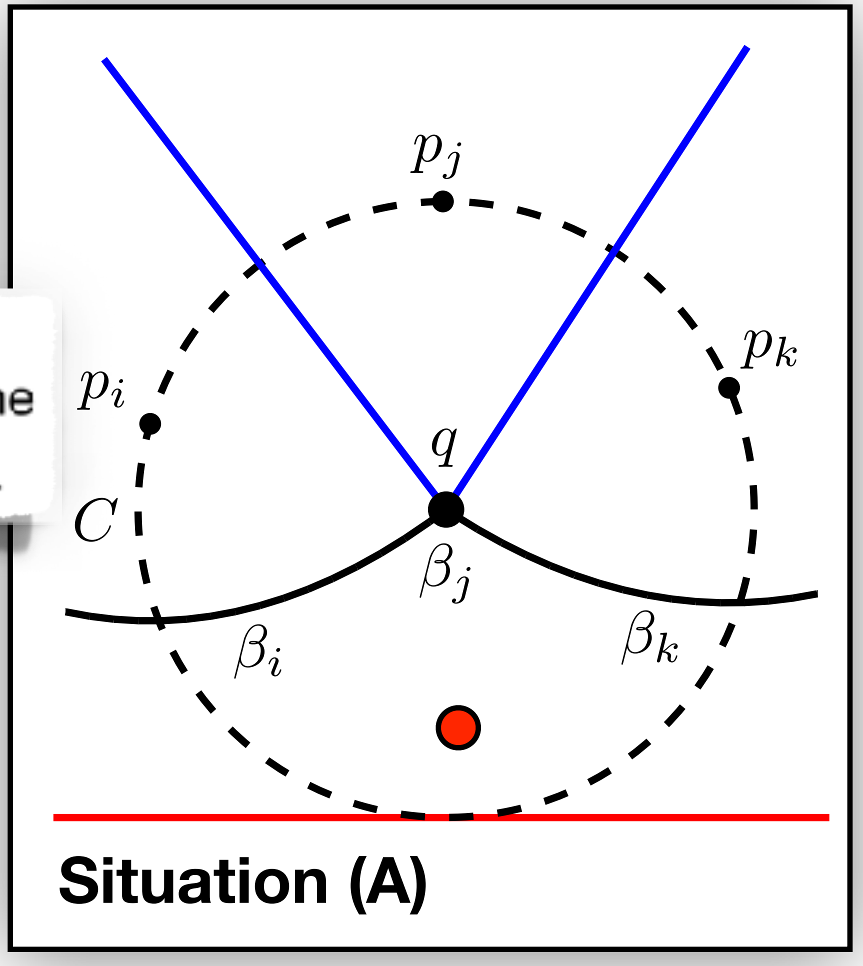

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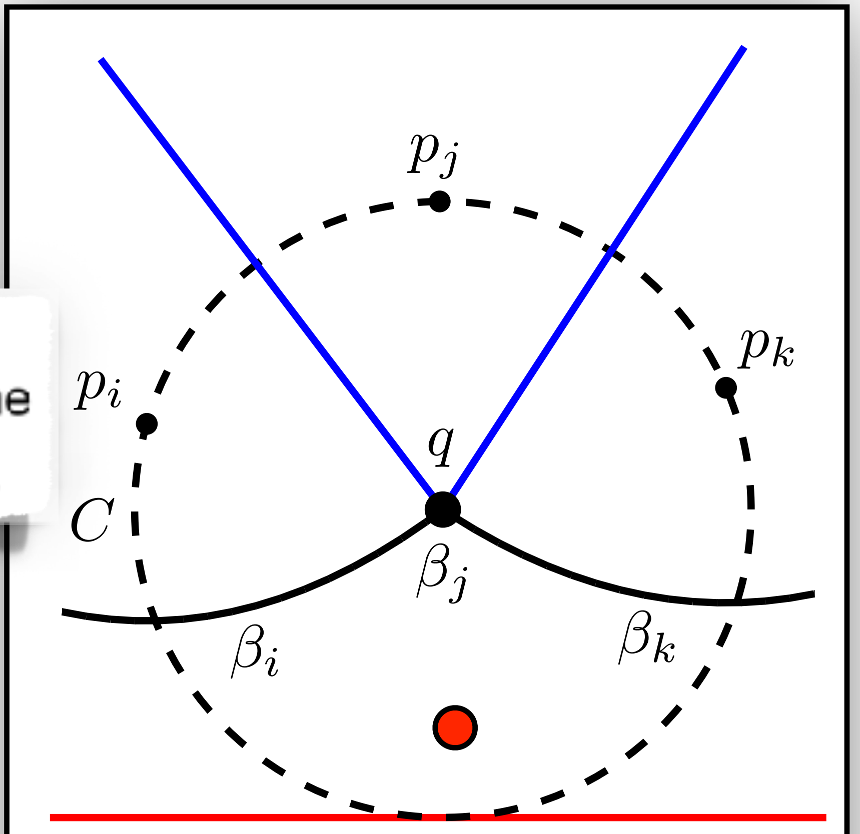

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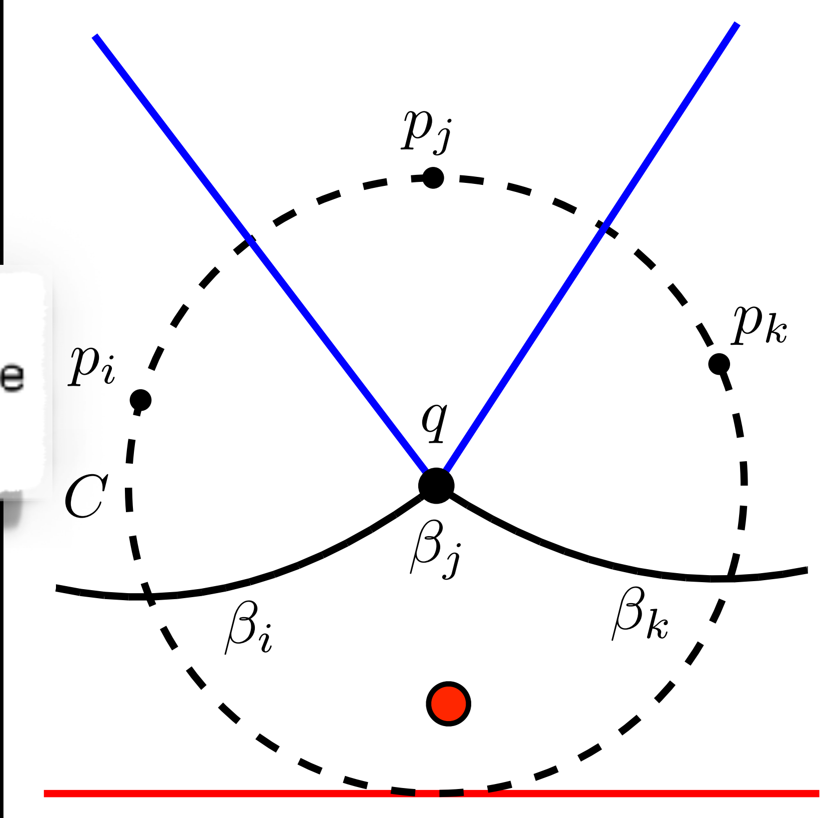

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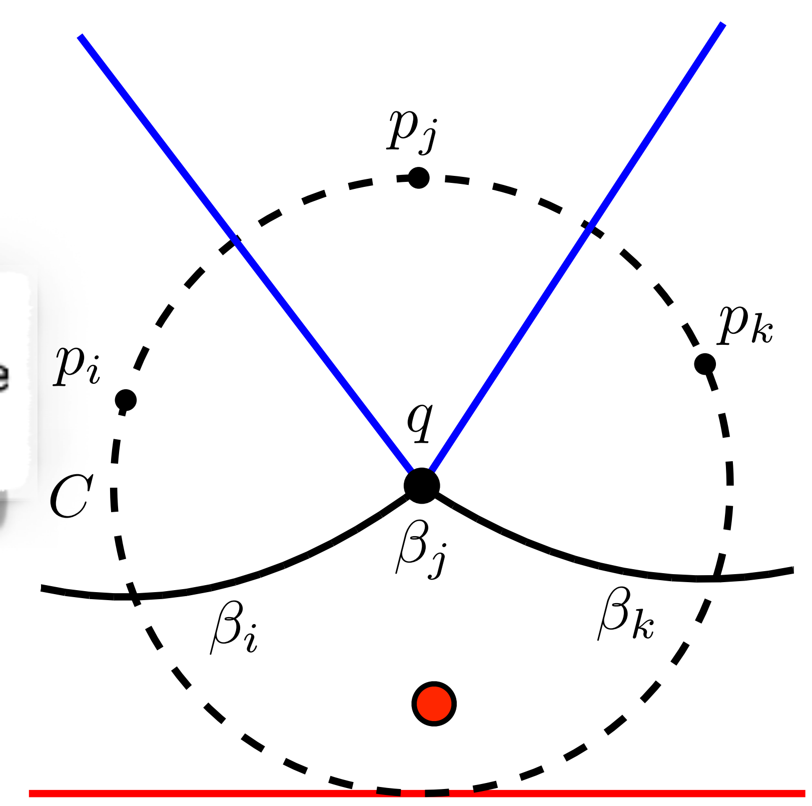

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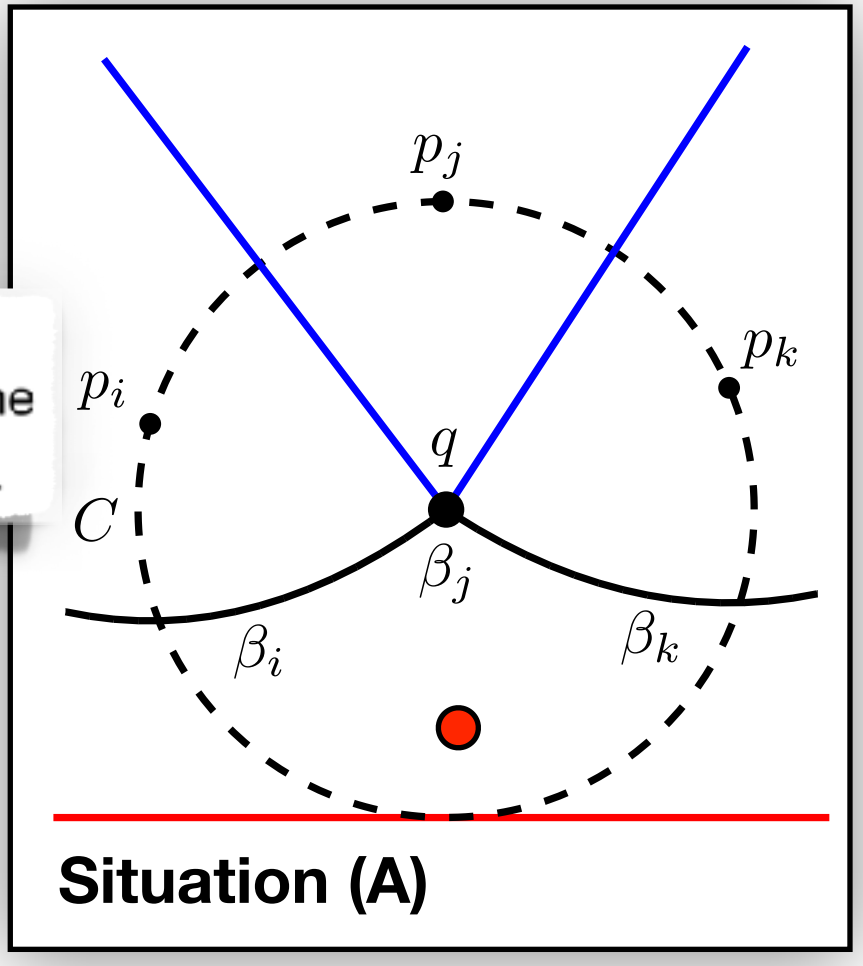

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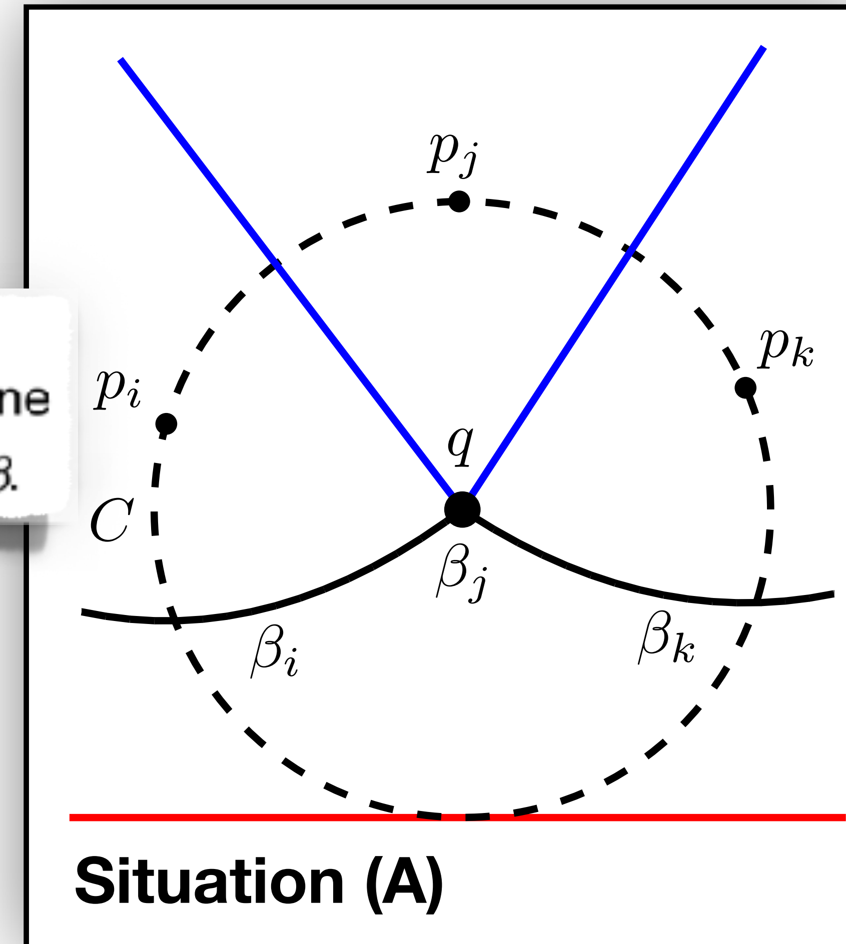

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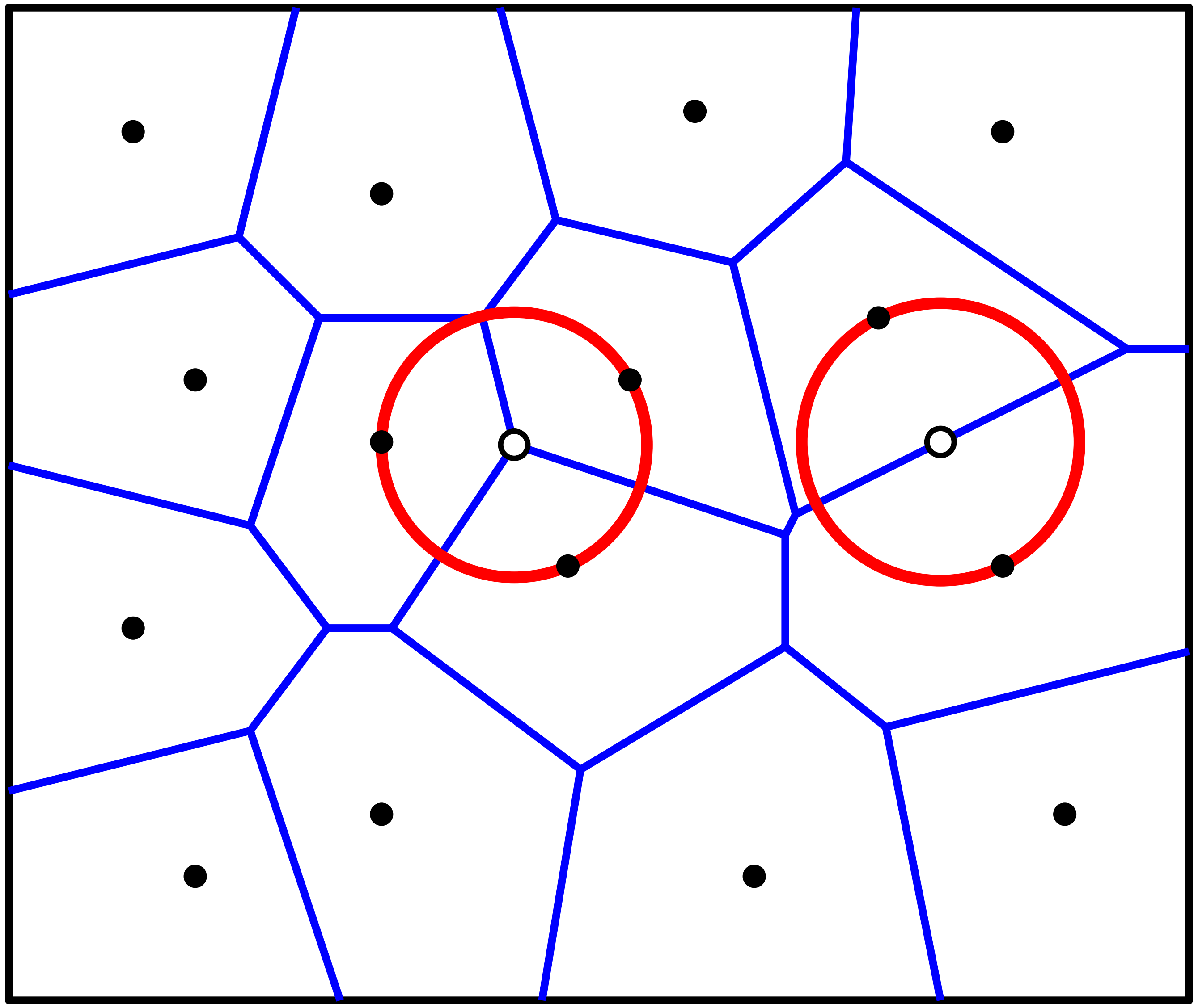

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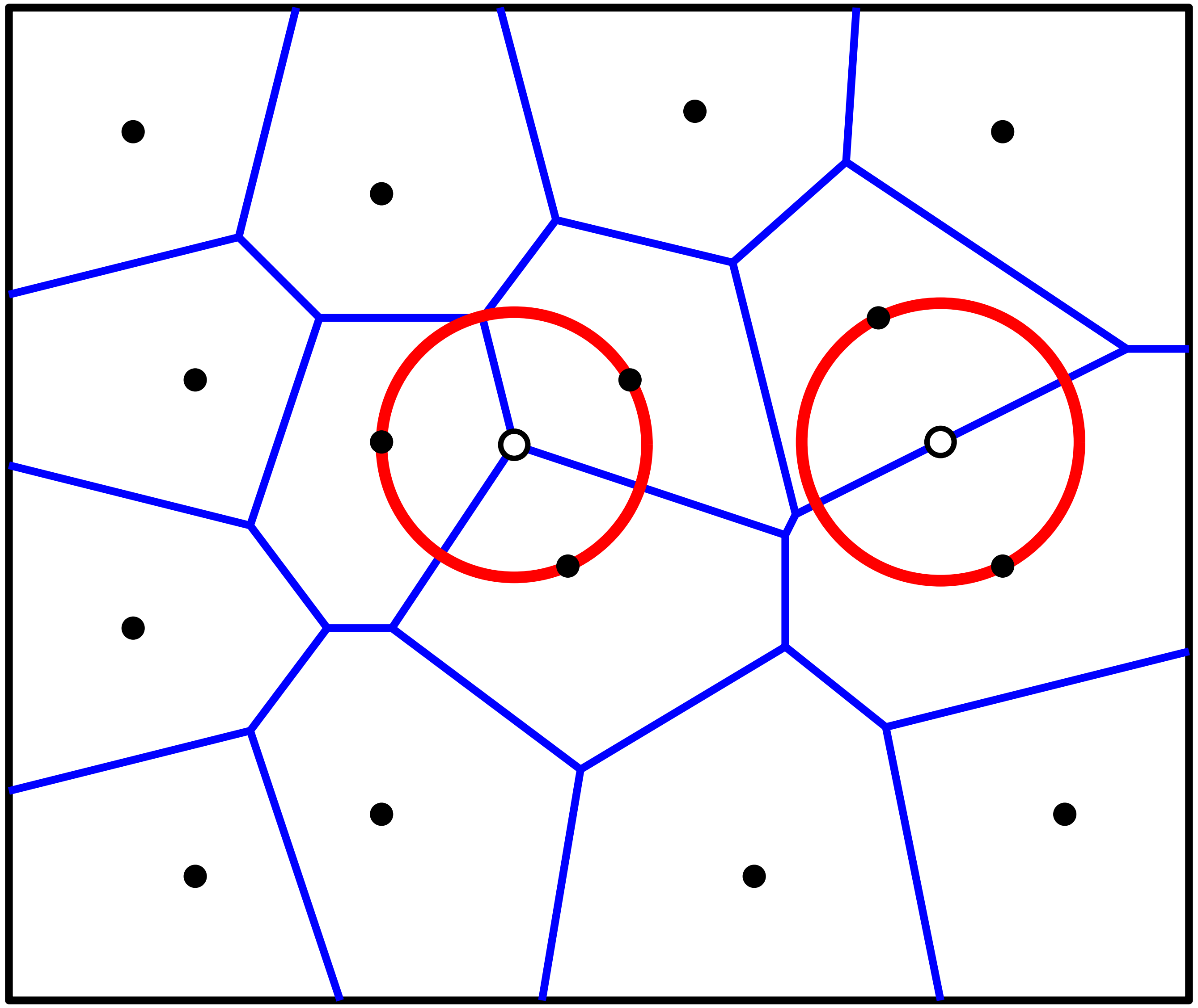

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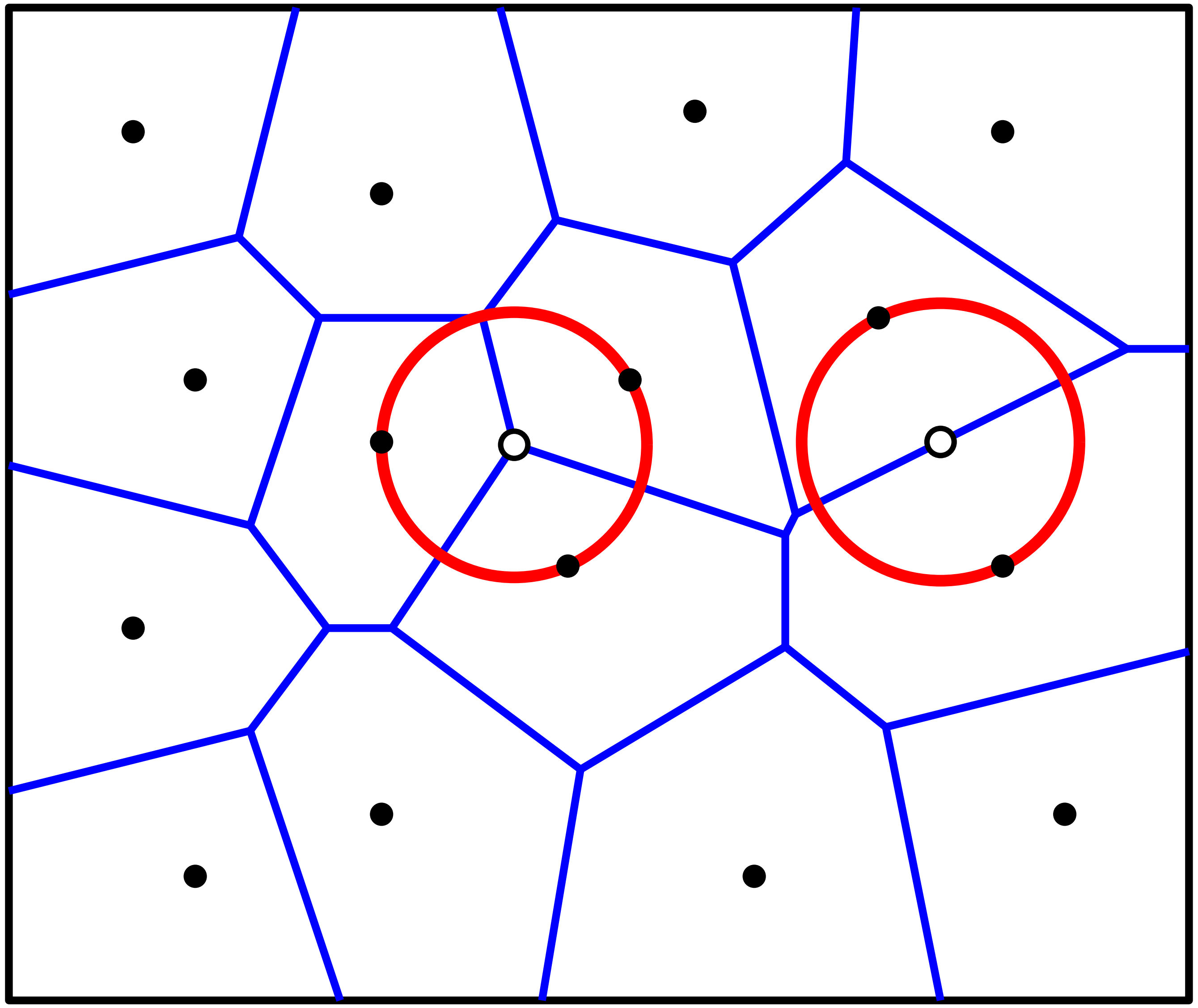

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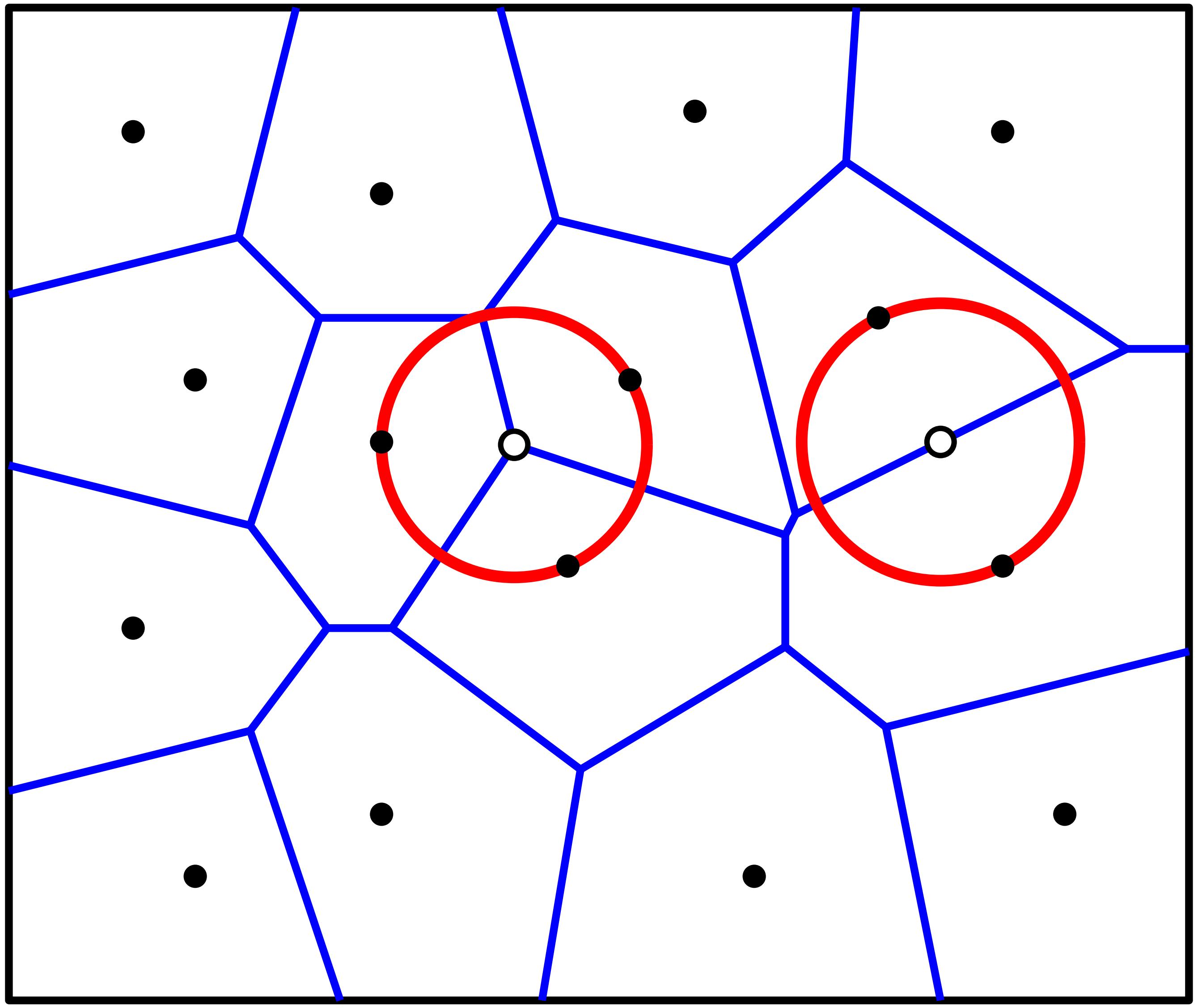

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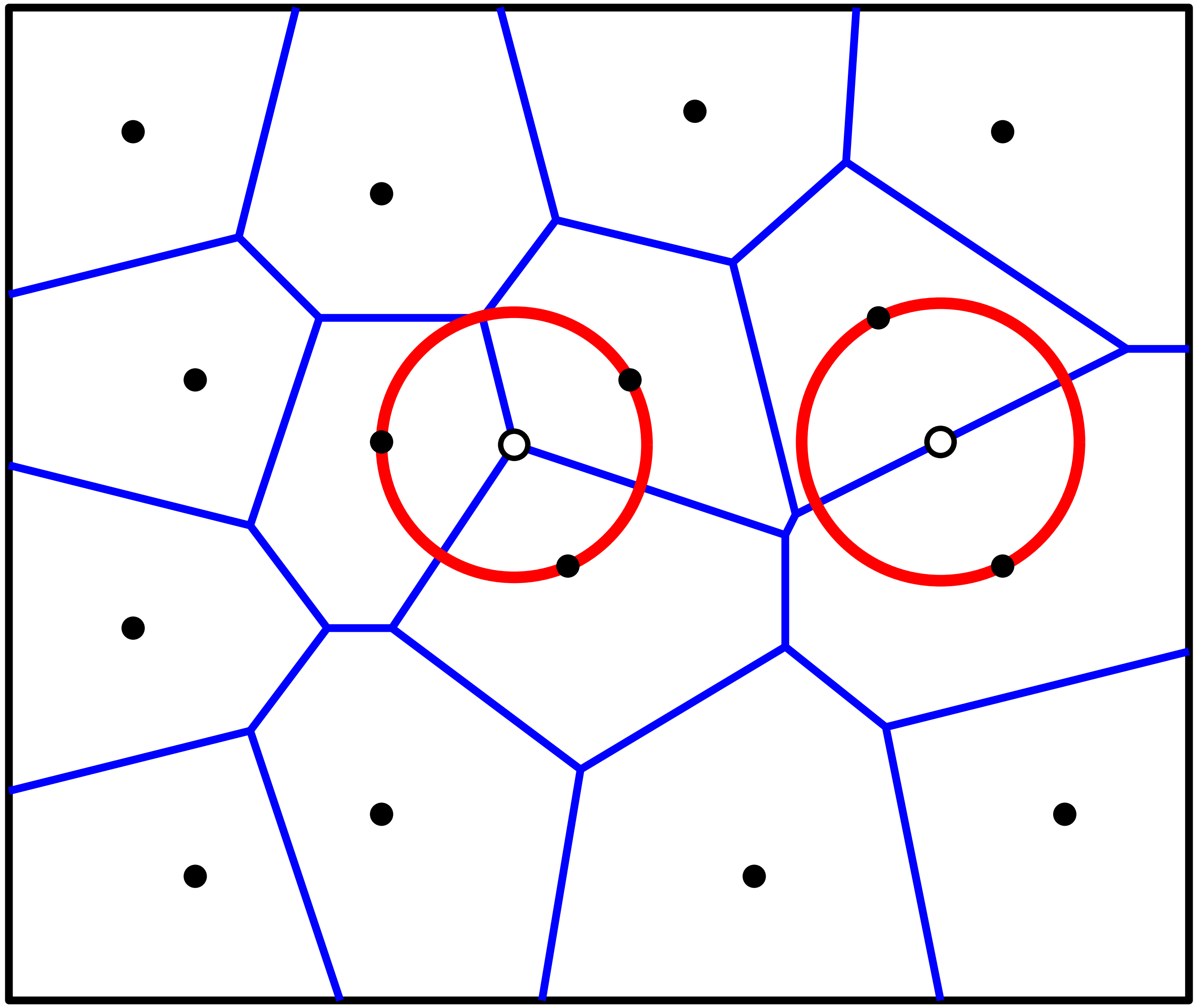

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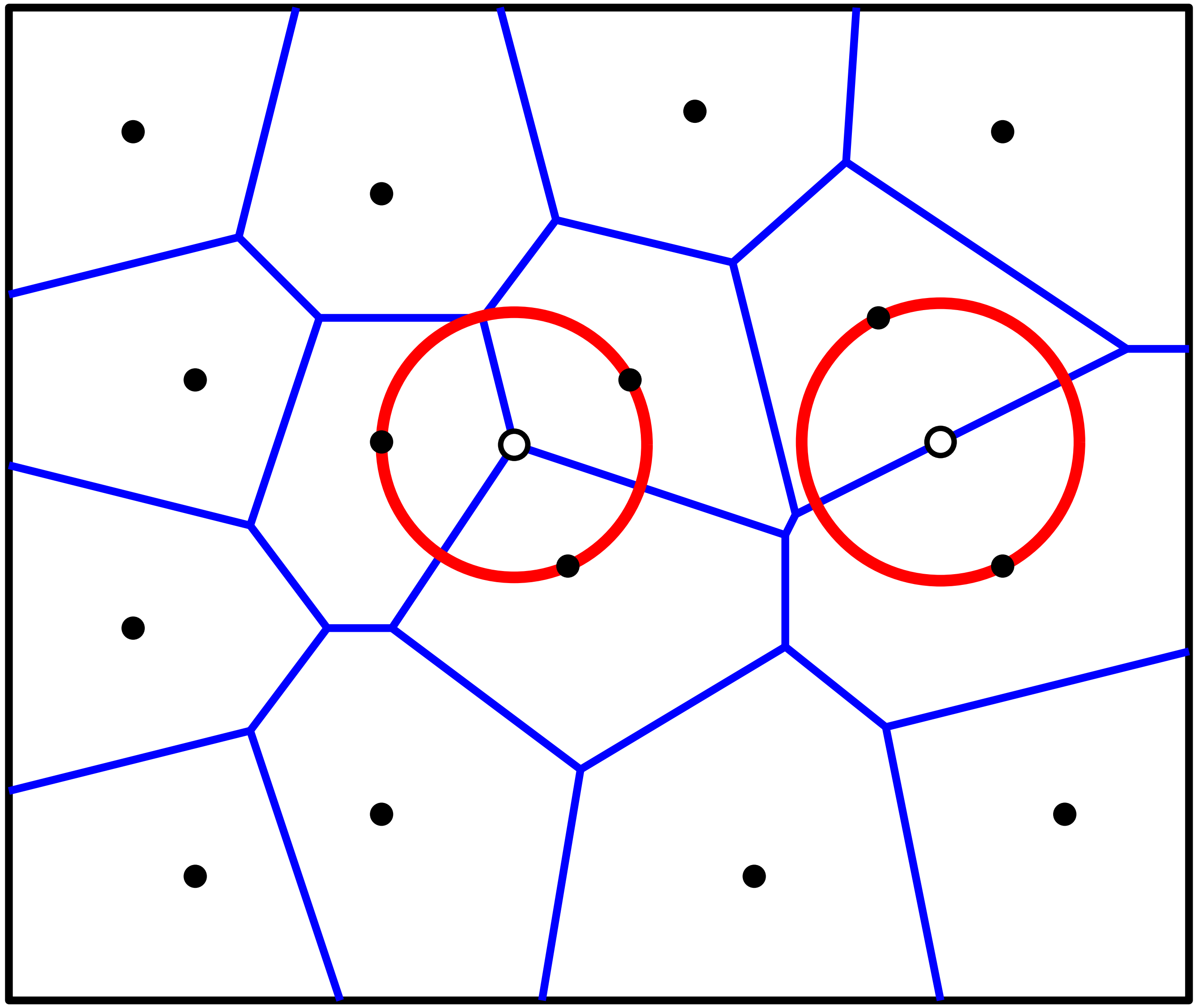

Proof:

Straightforward, because nearest neighbor to center lies on circle.

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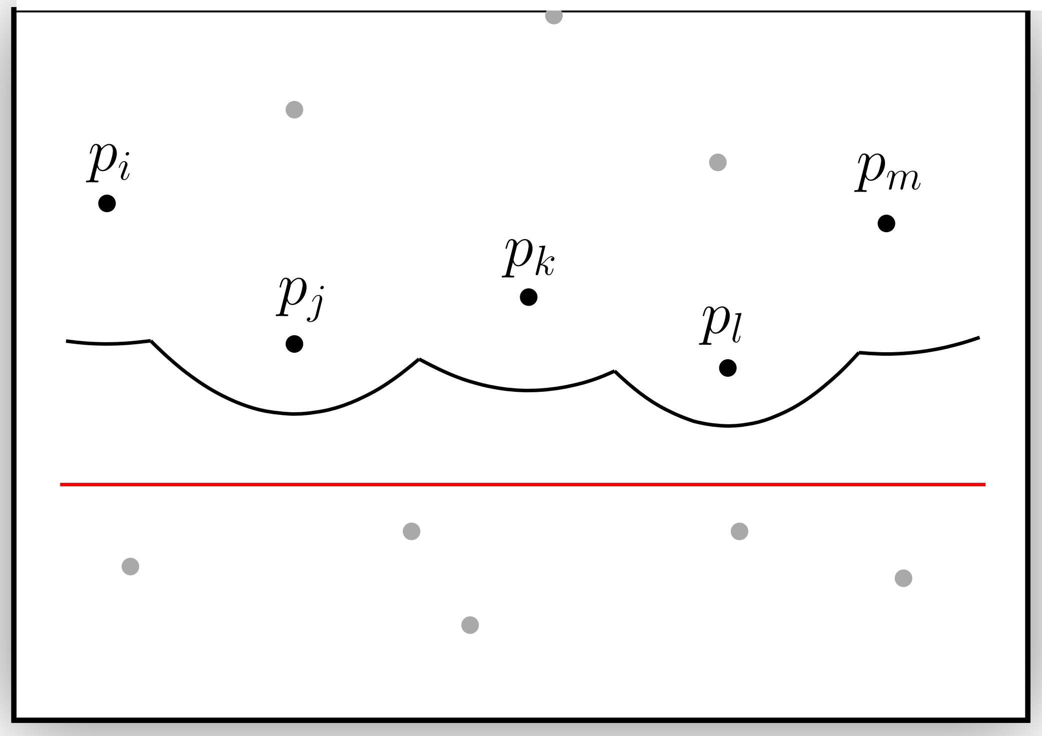

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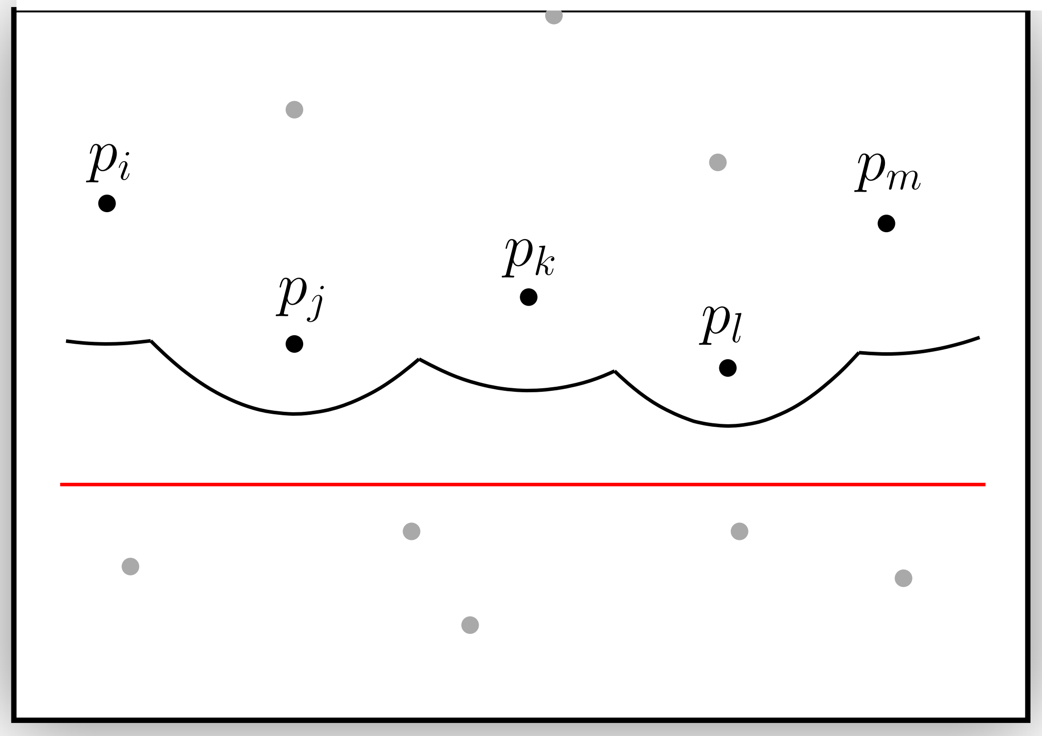

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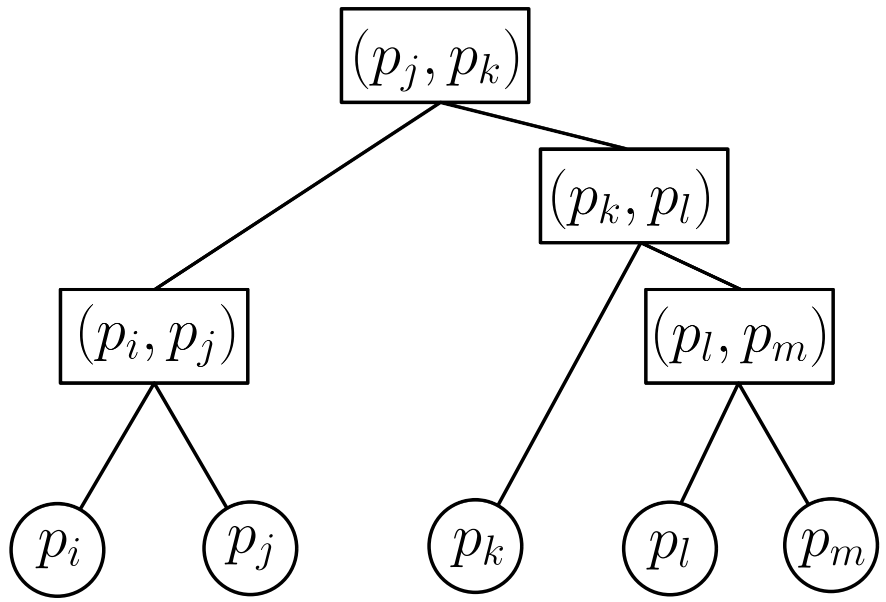

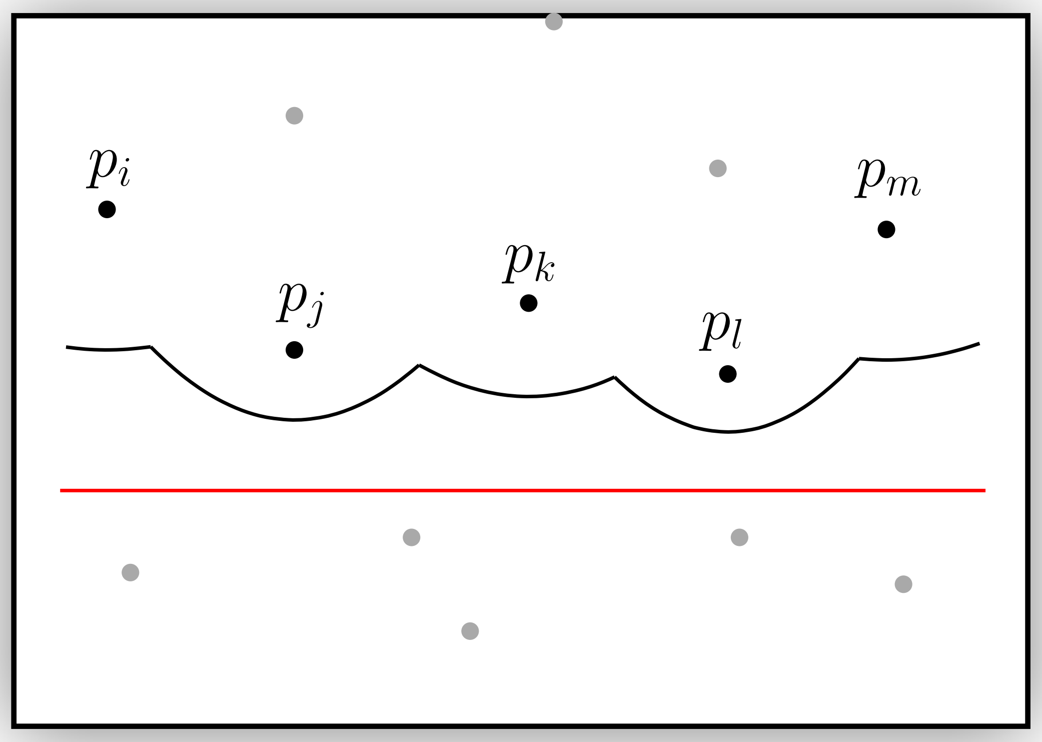

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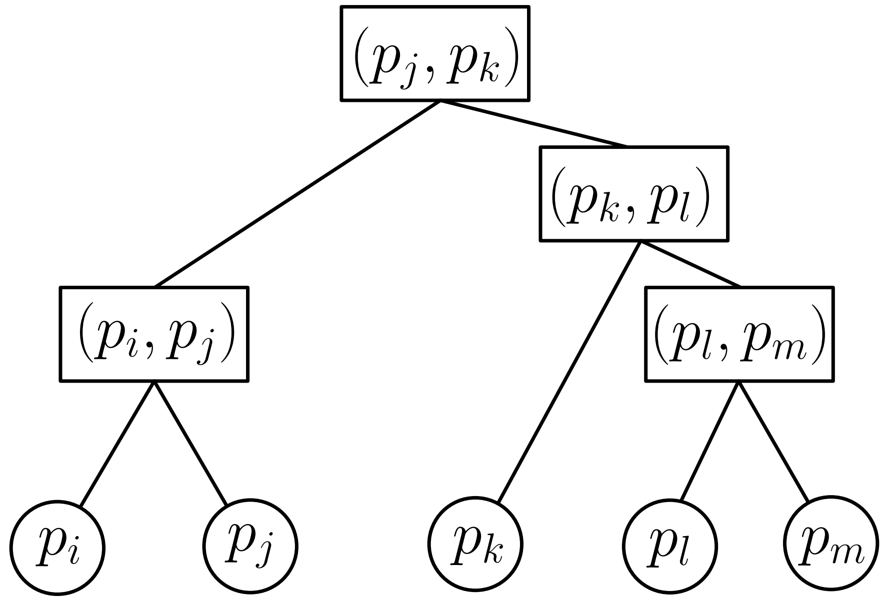

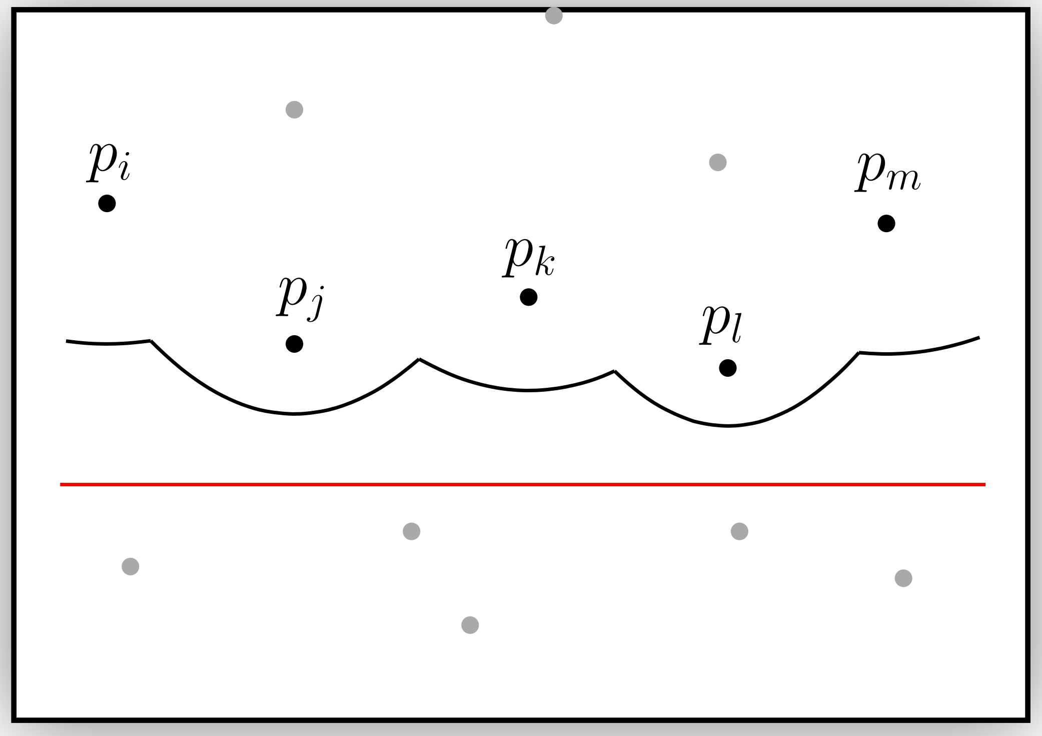

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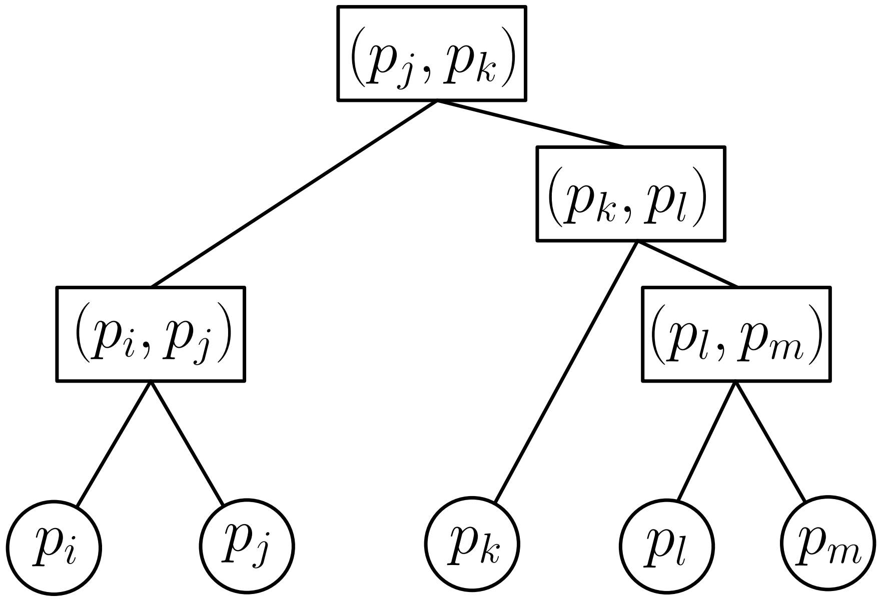

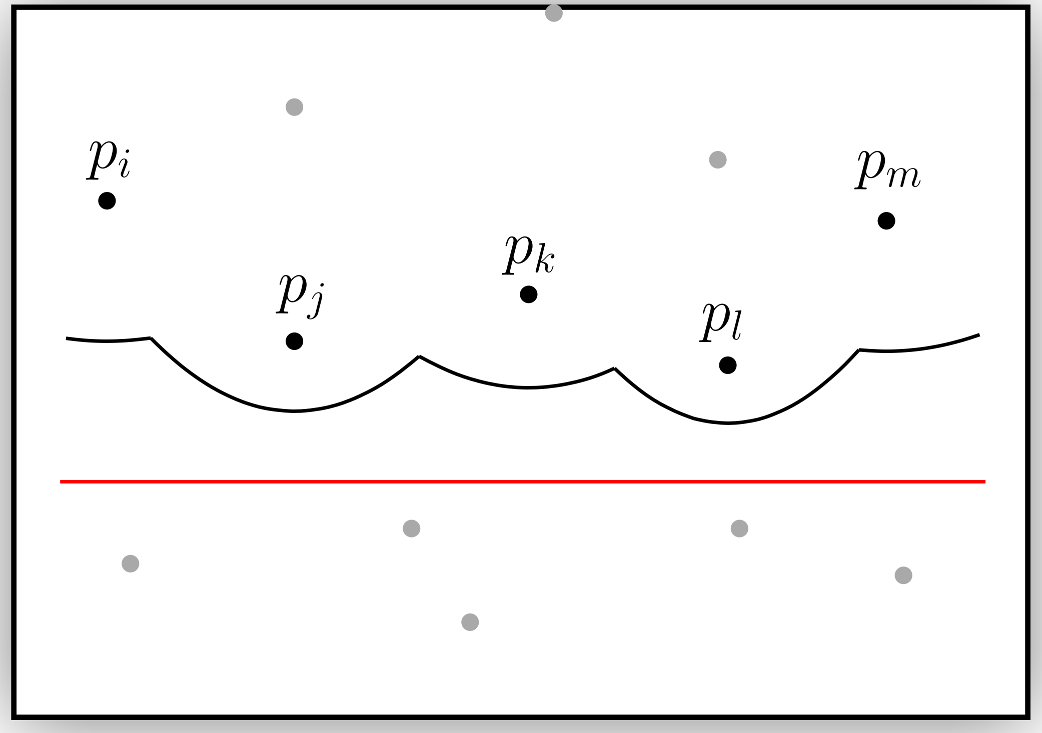

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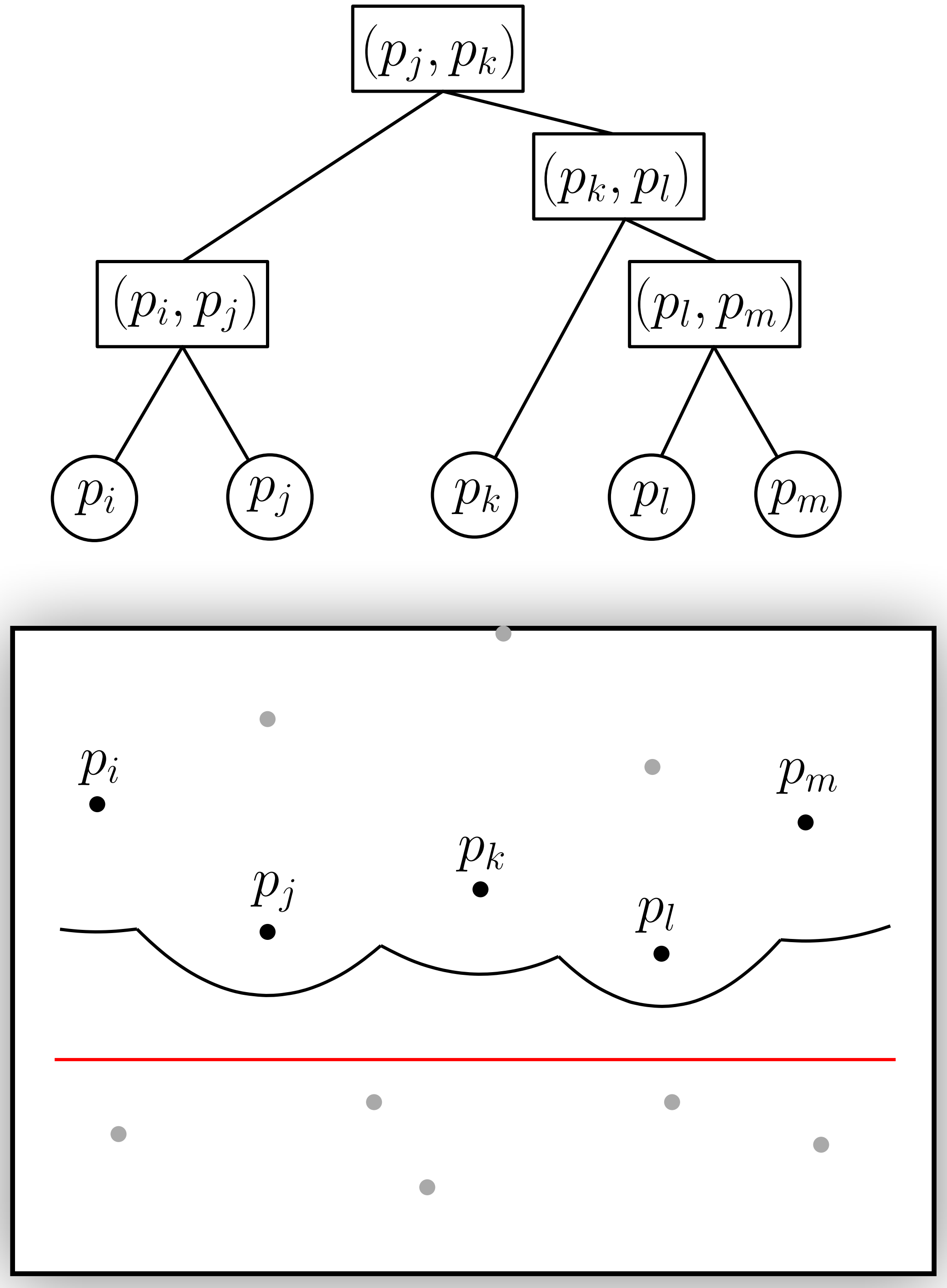

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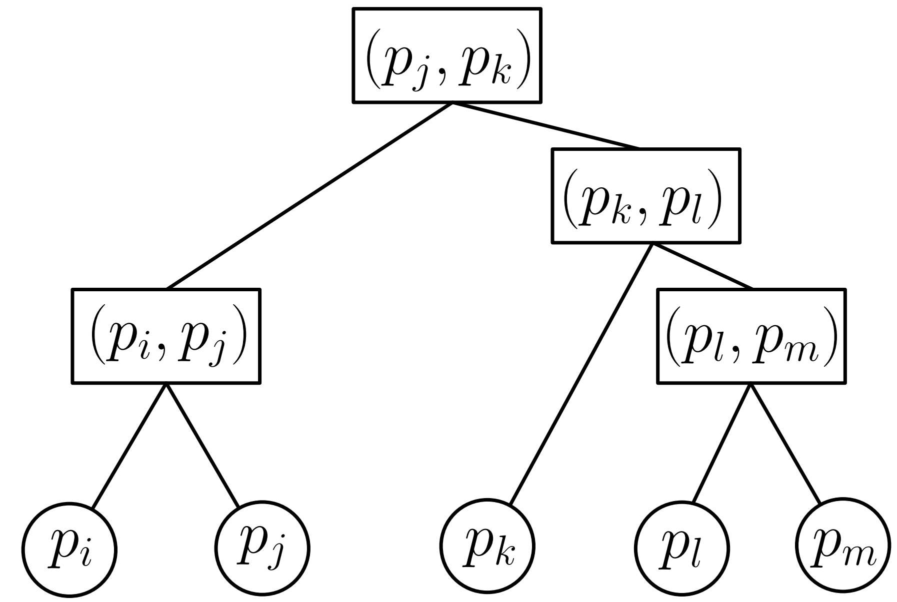

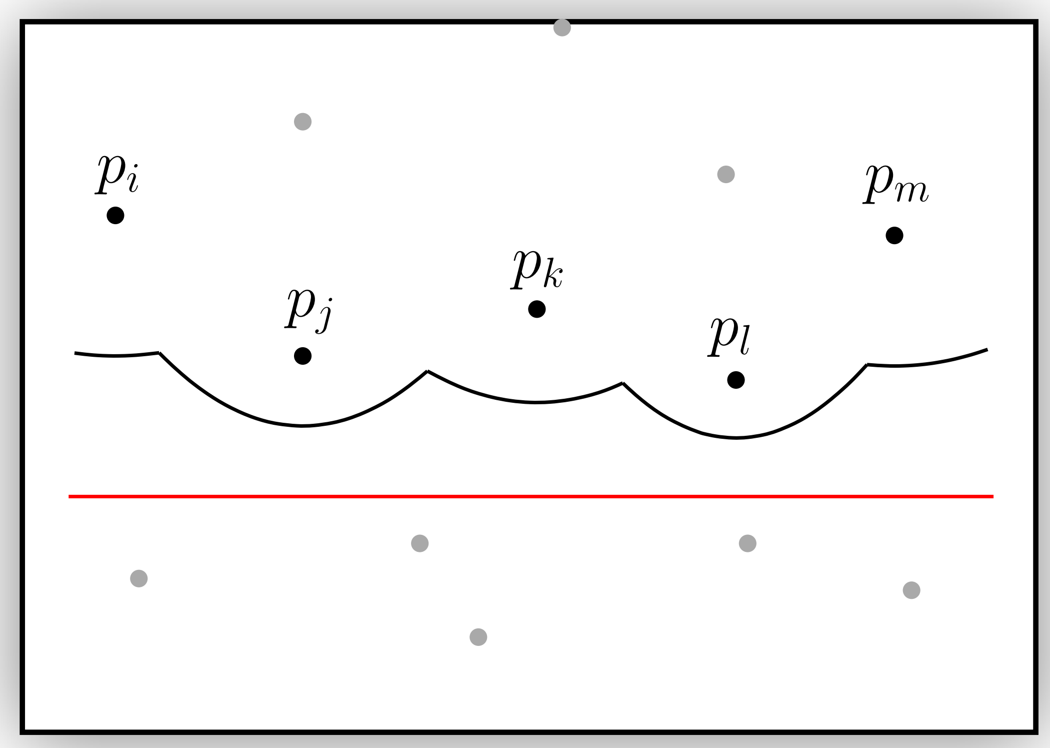

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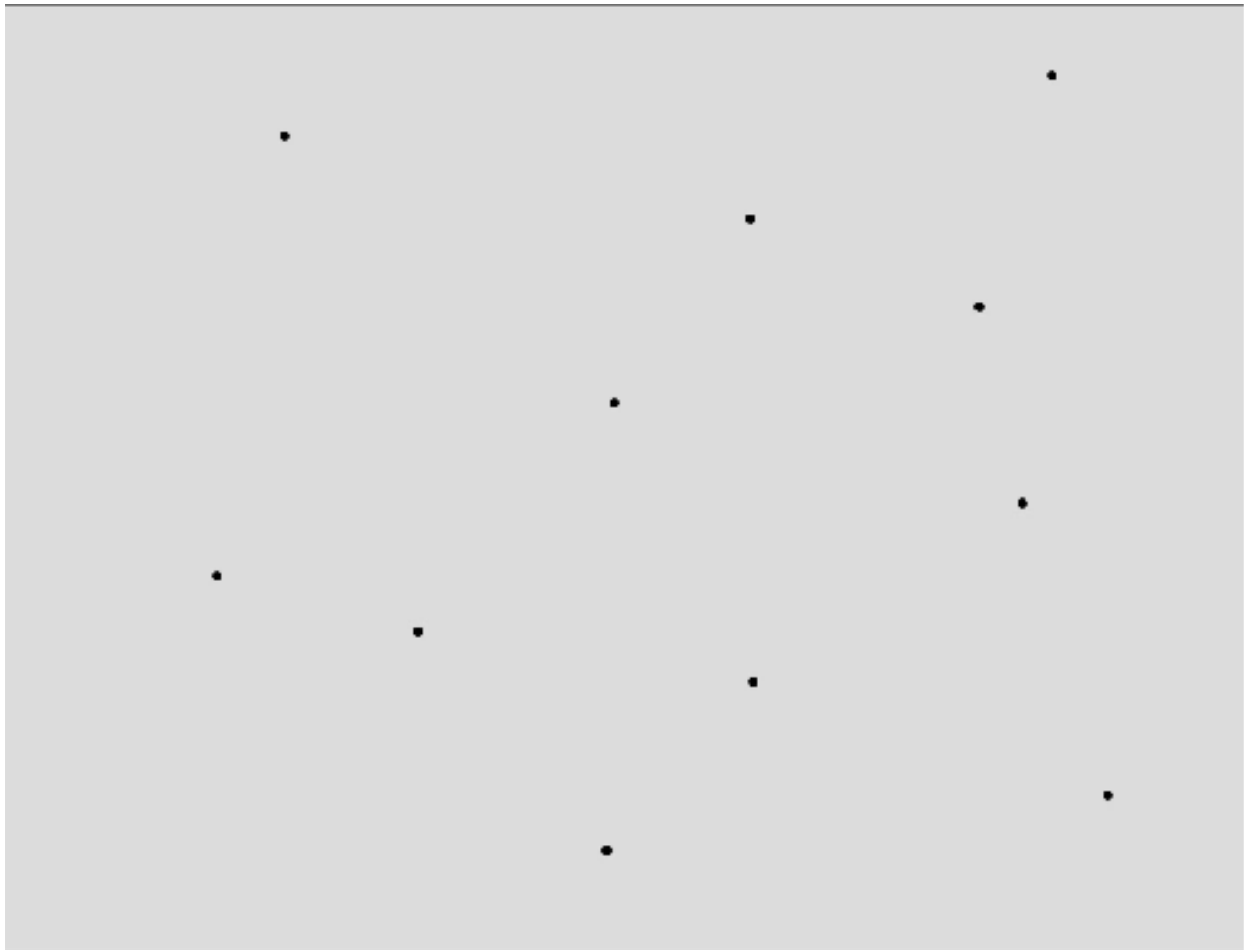

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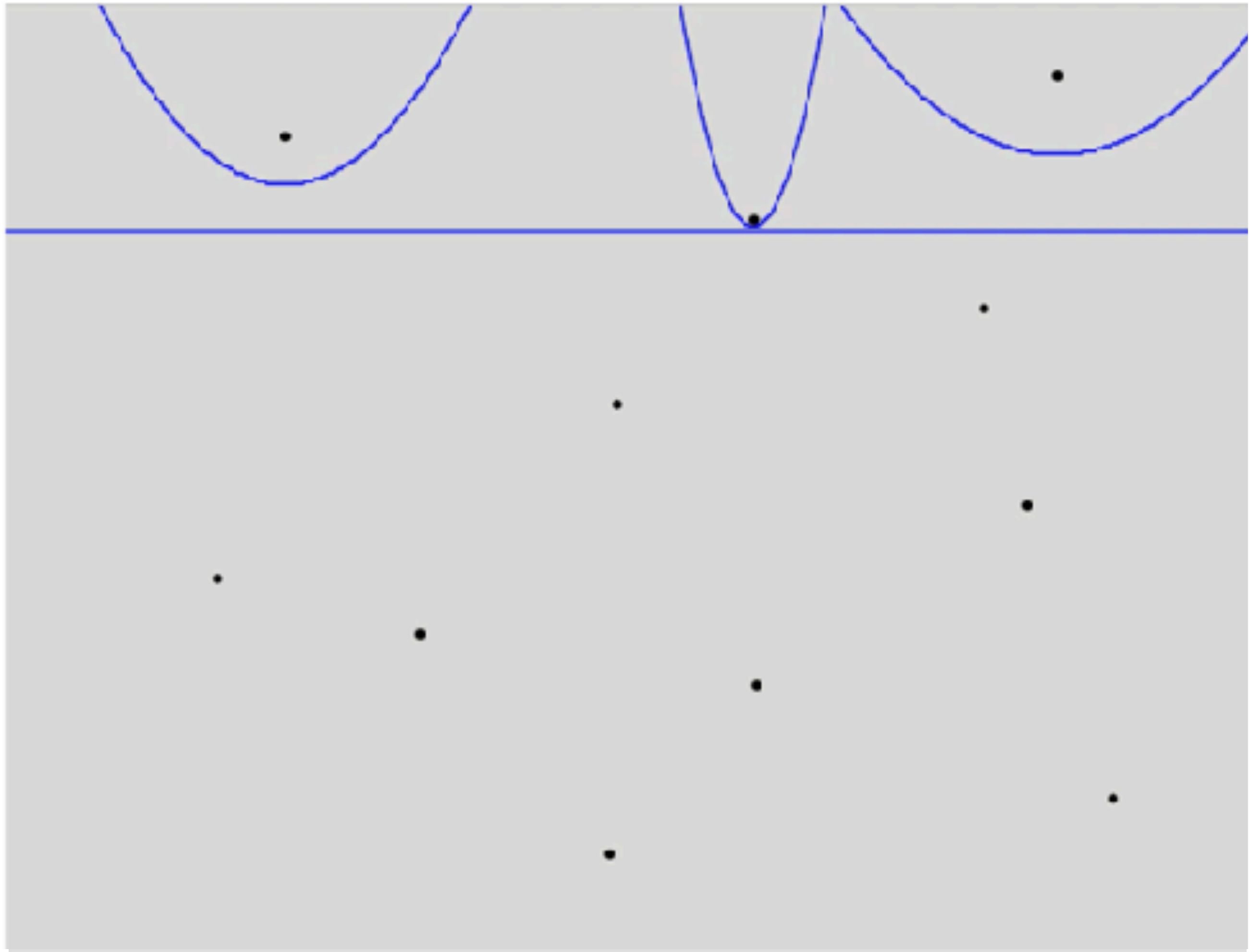

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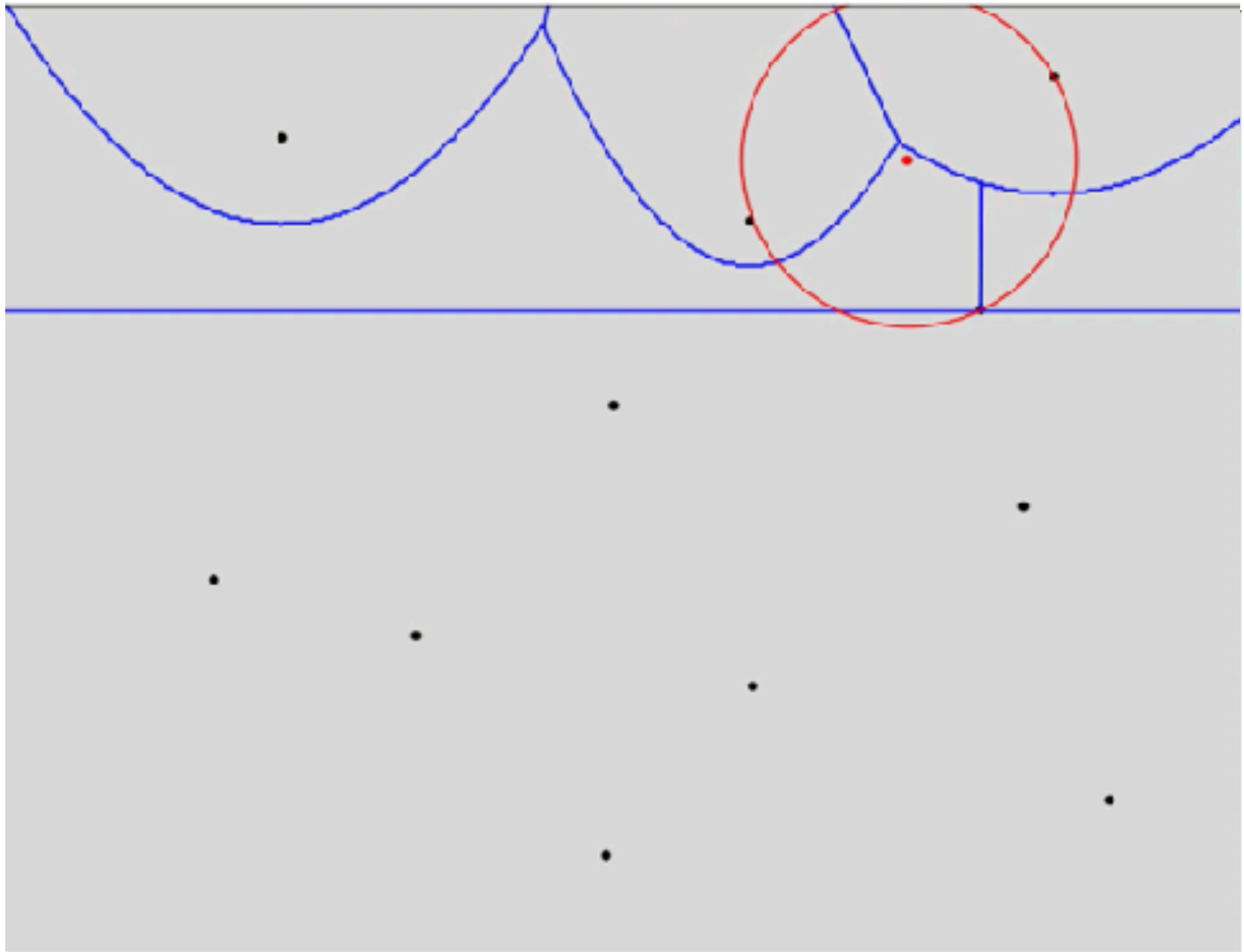

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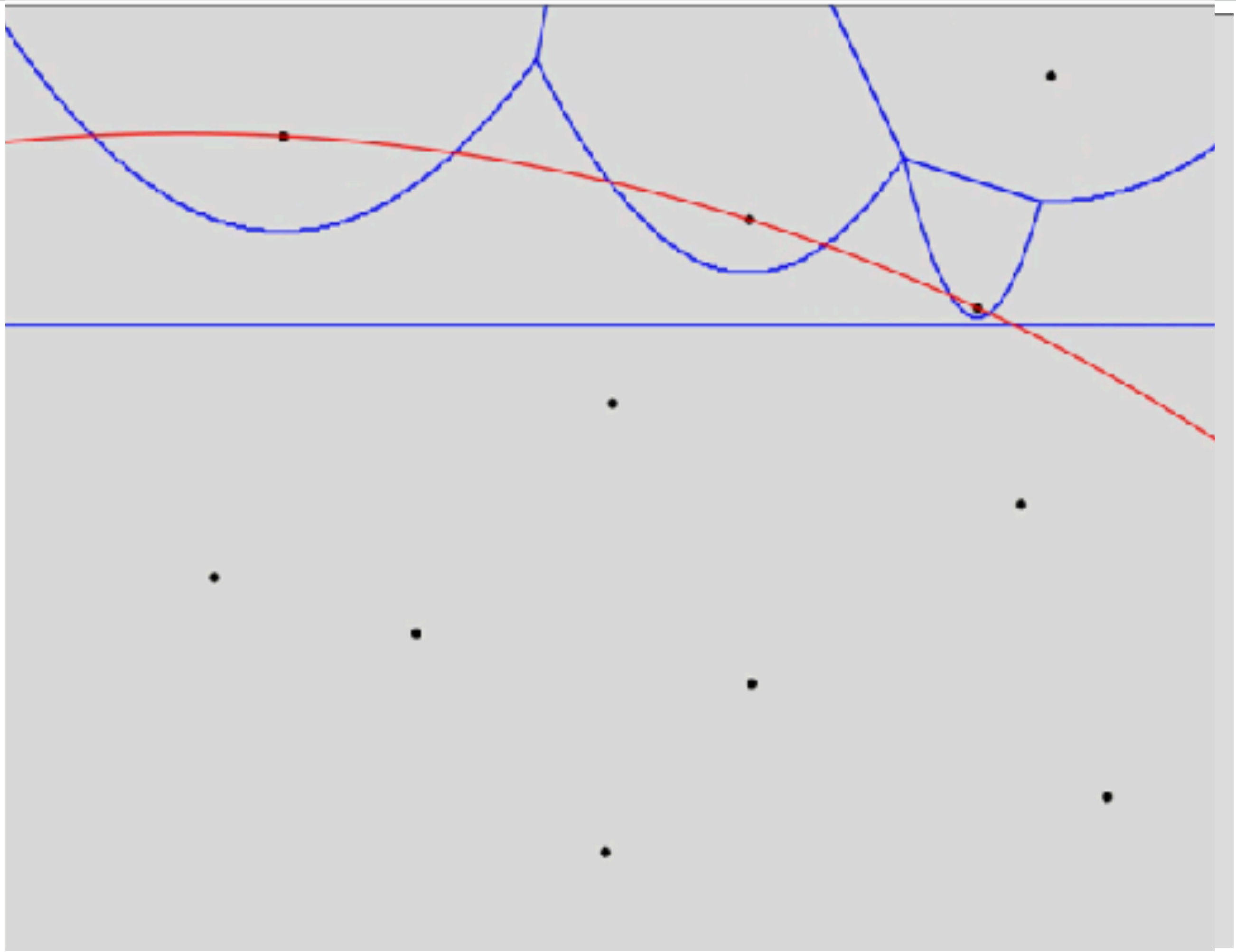

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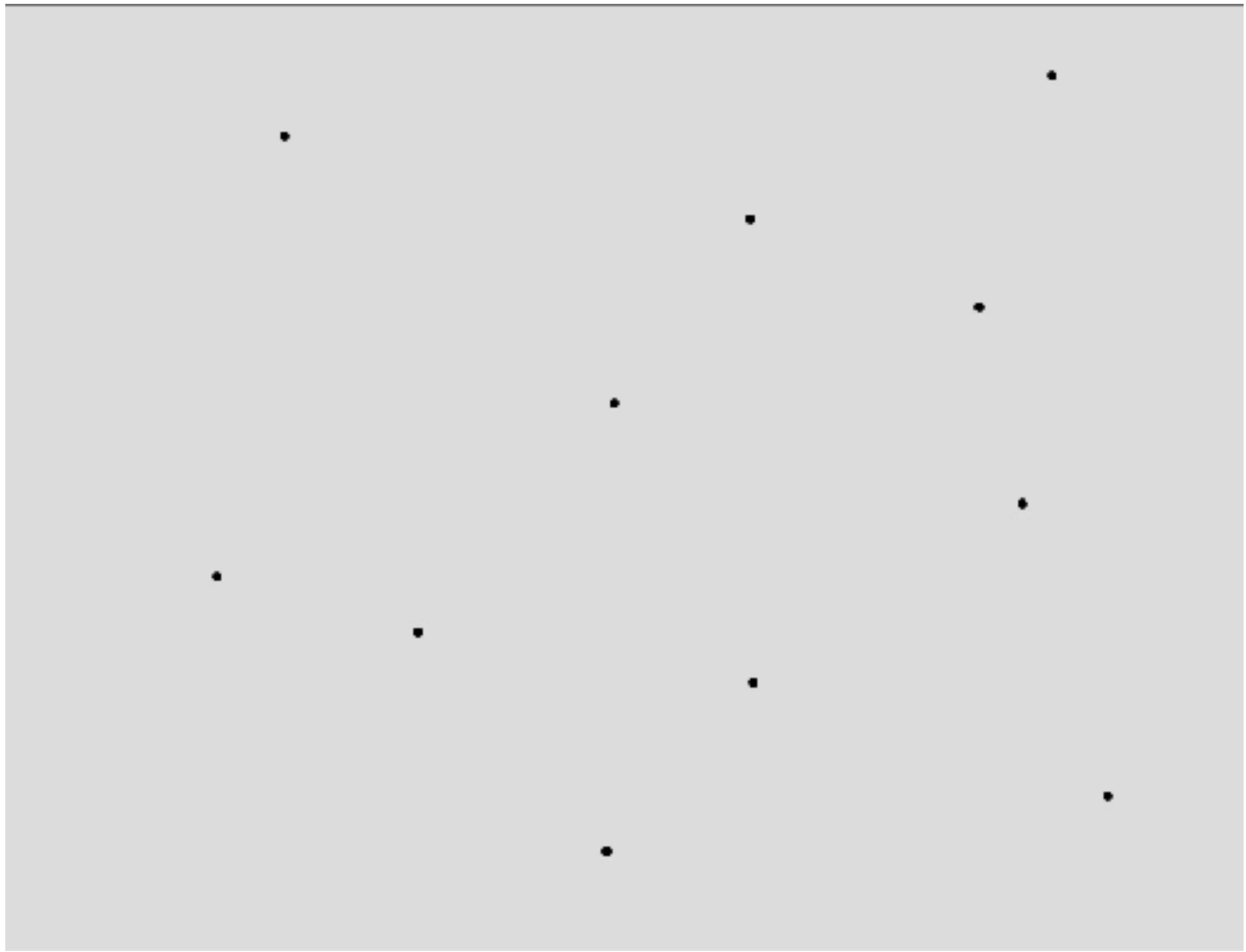

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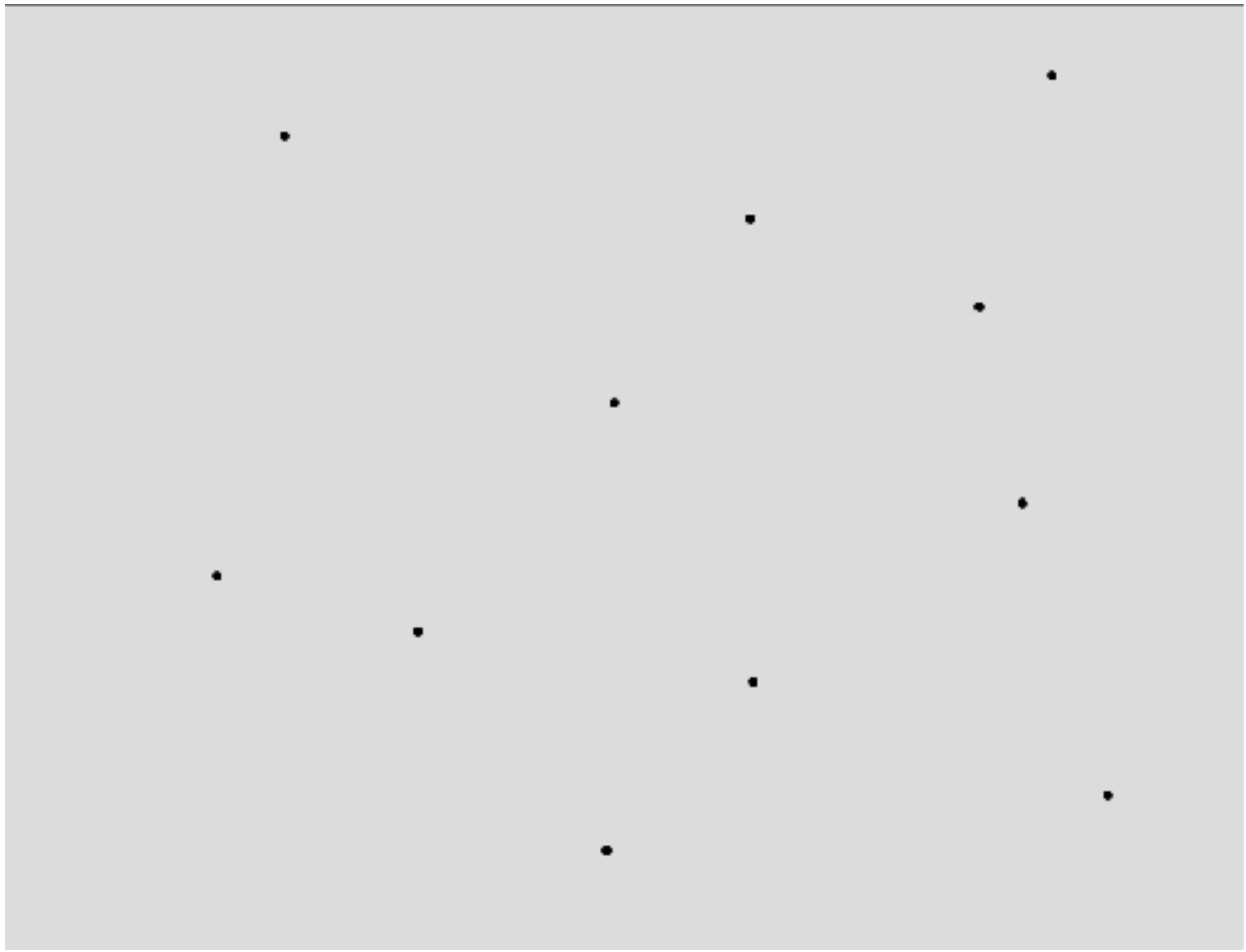

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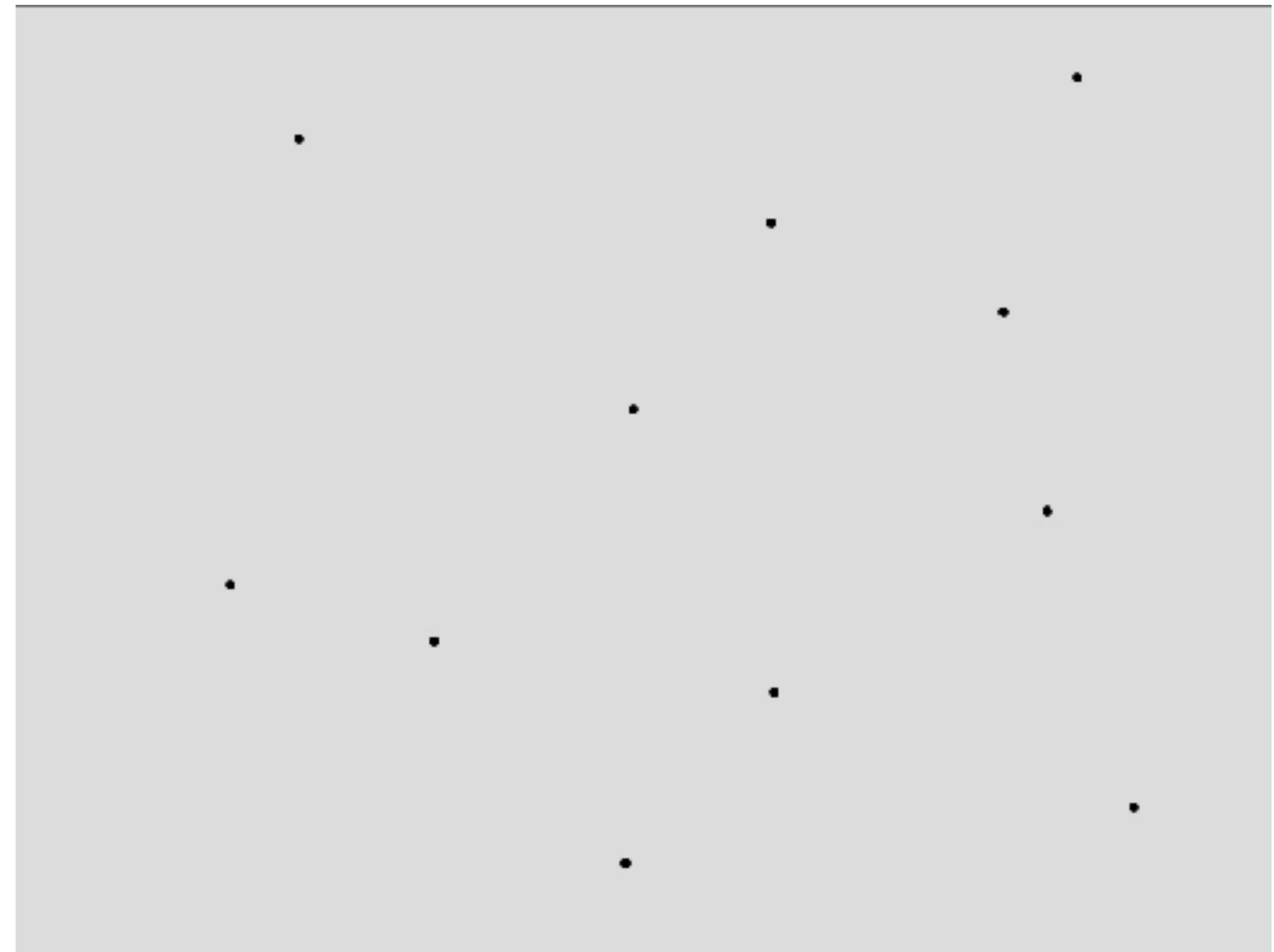

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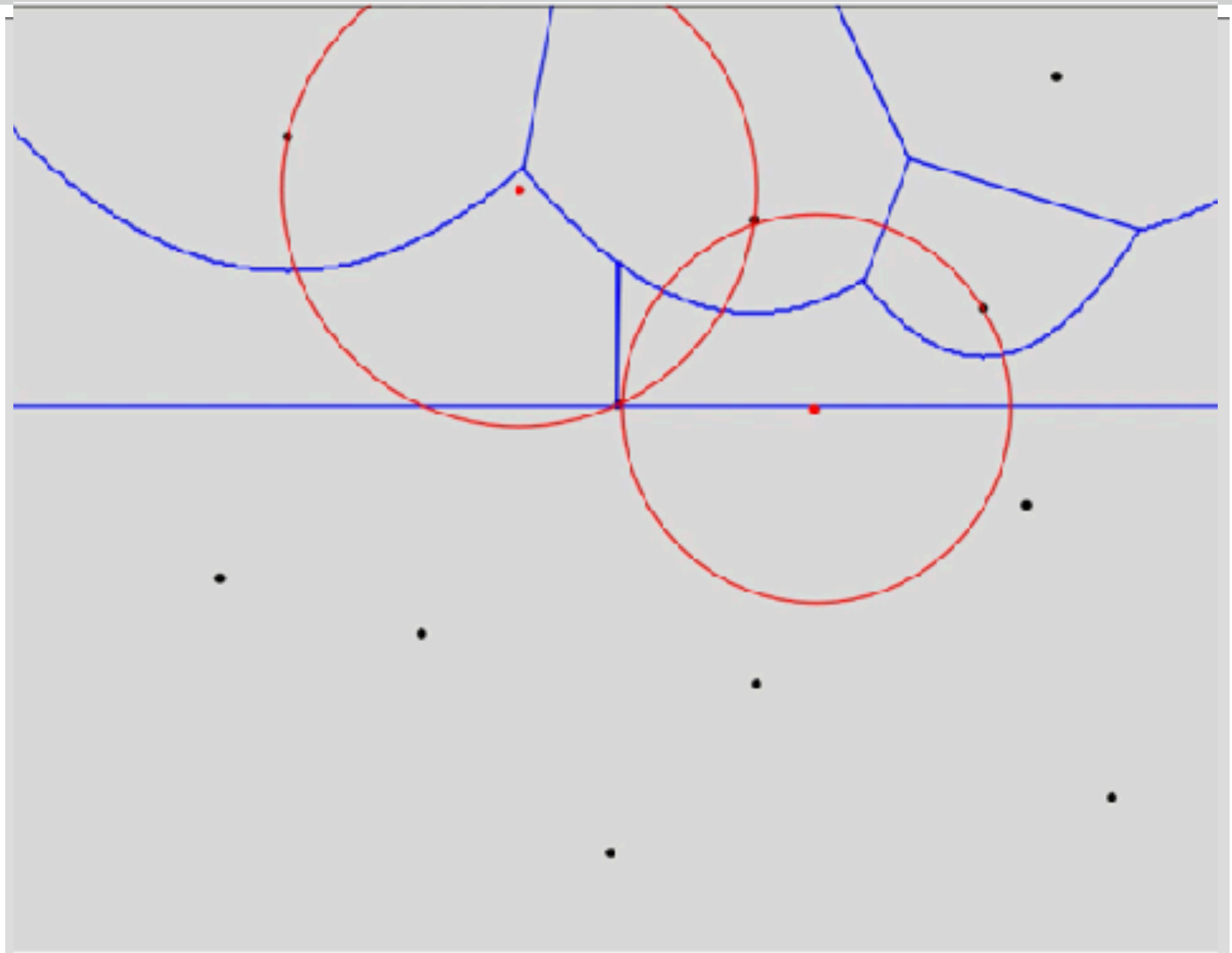

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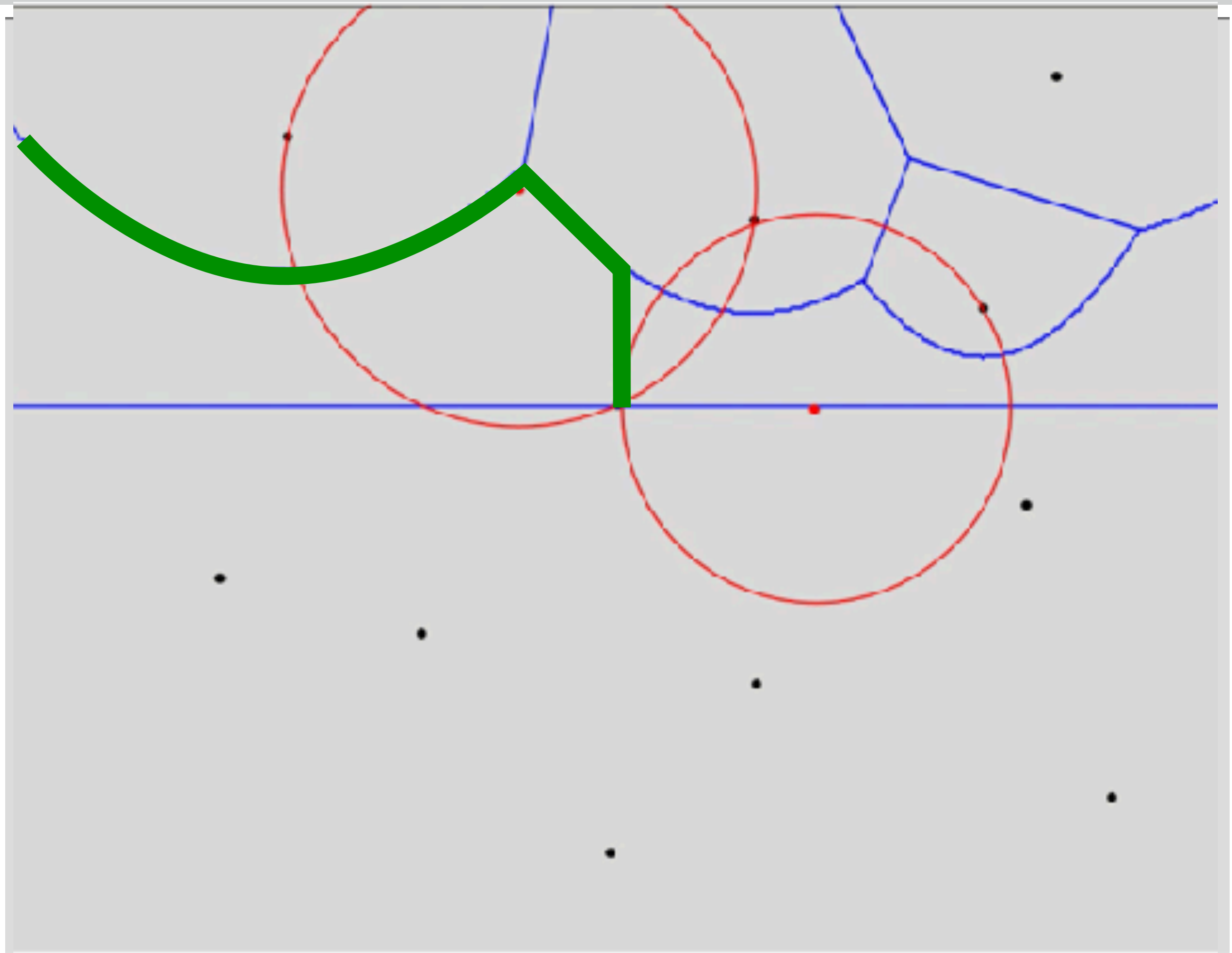

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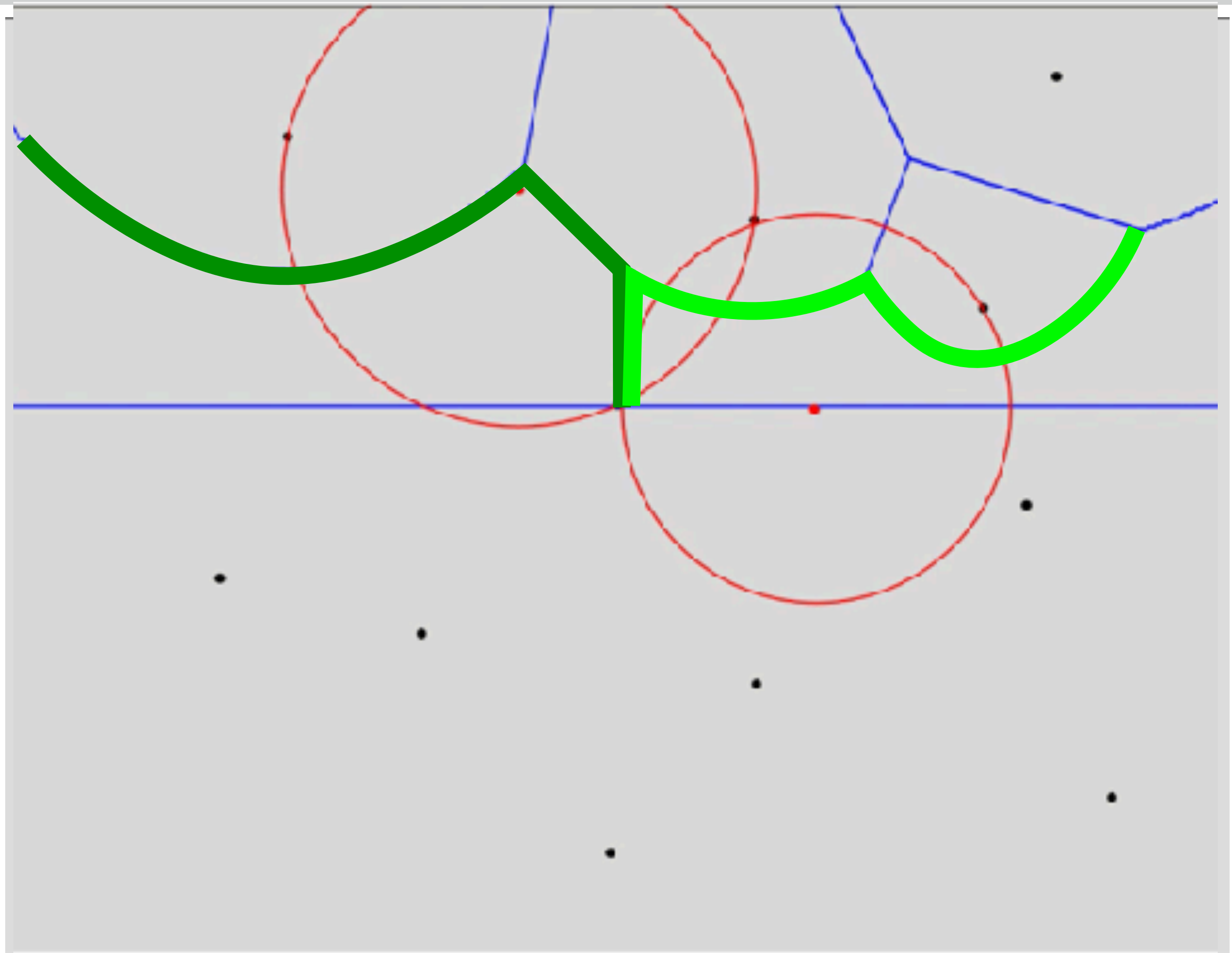

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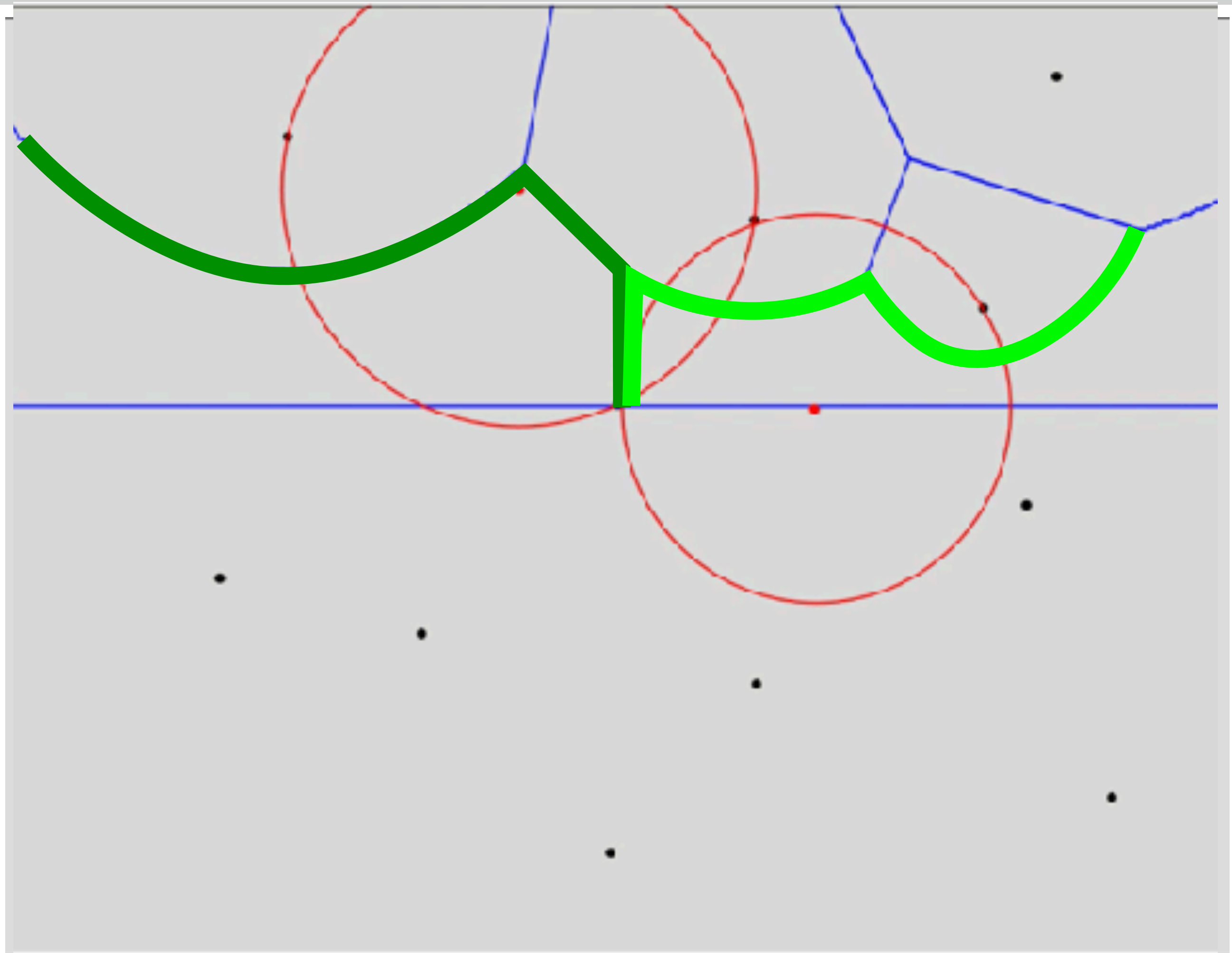

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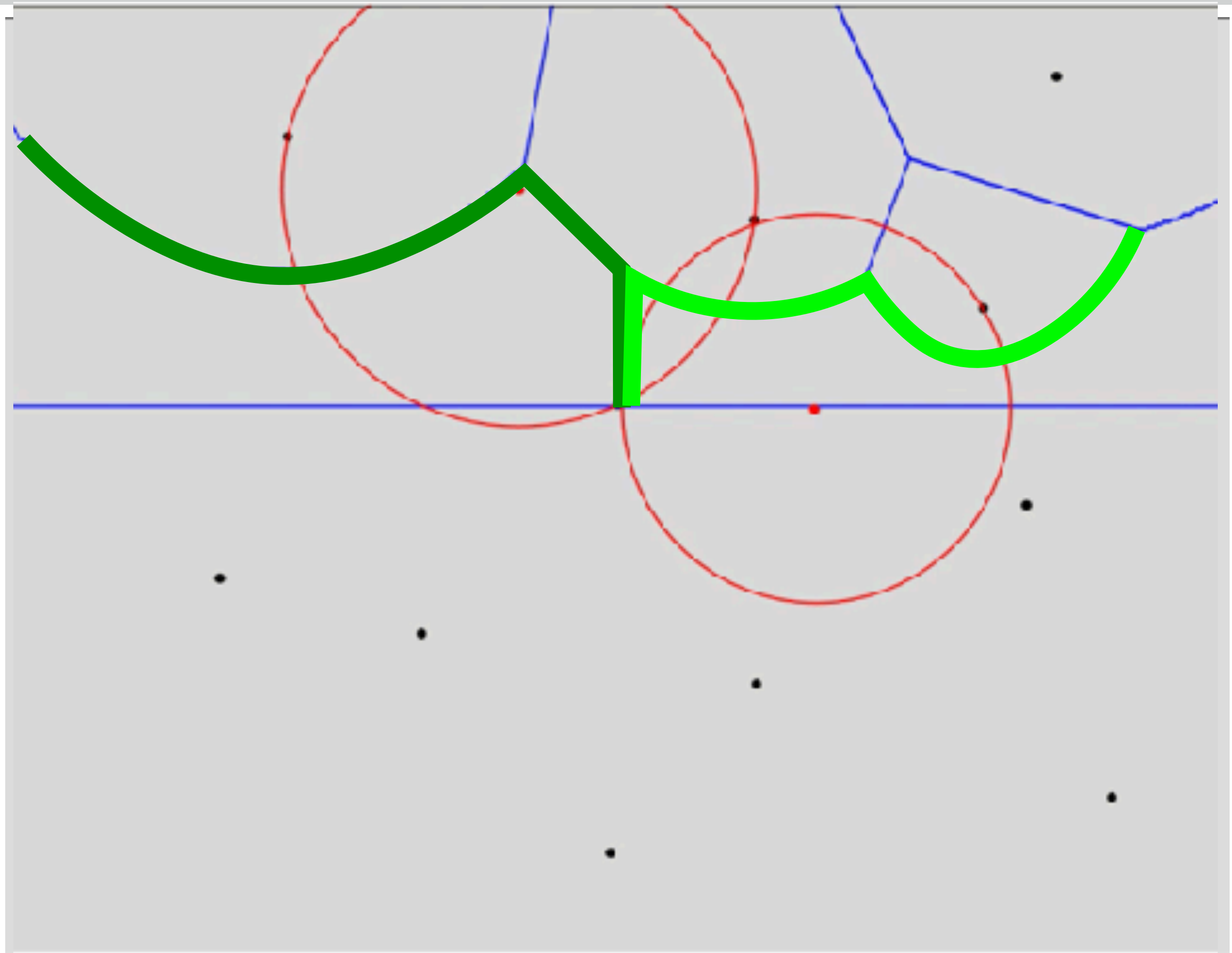
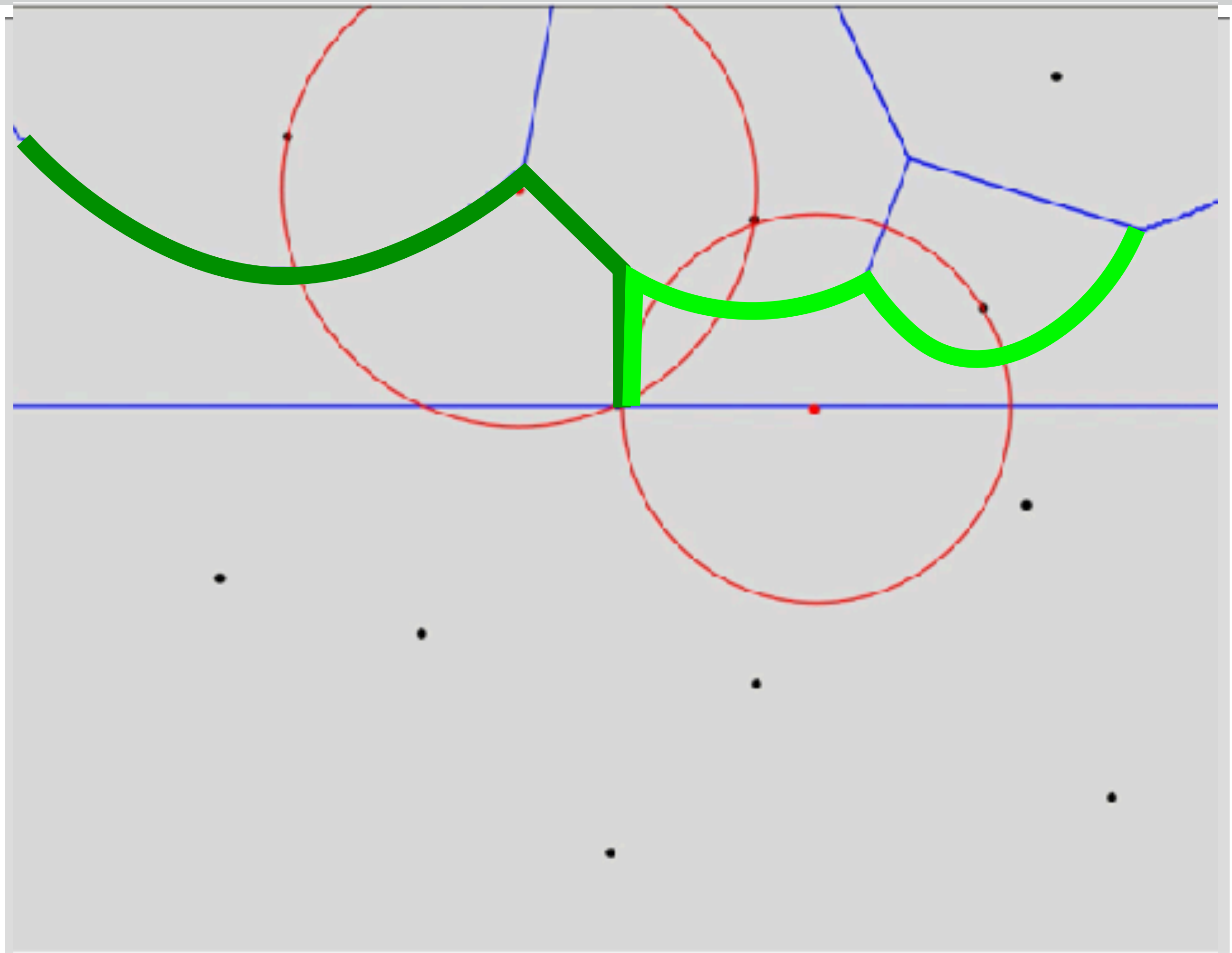
Point events $p_i \rightarrow$ Priority by $p_i.y$.





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  Circle events C .

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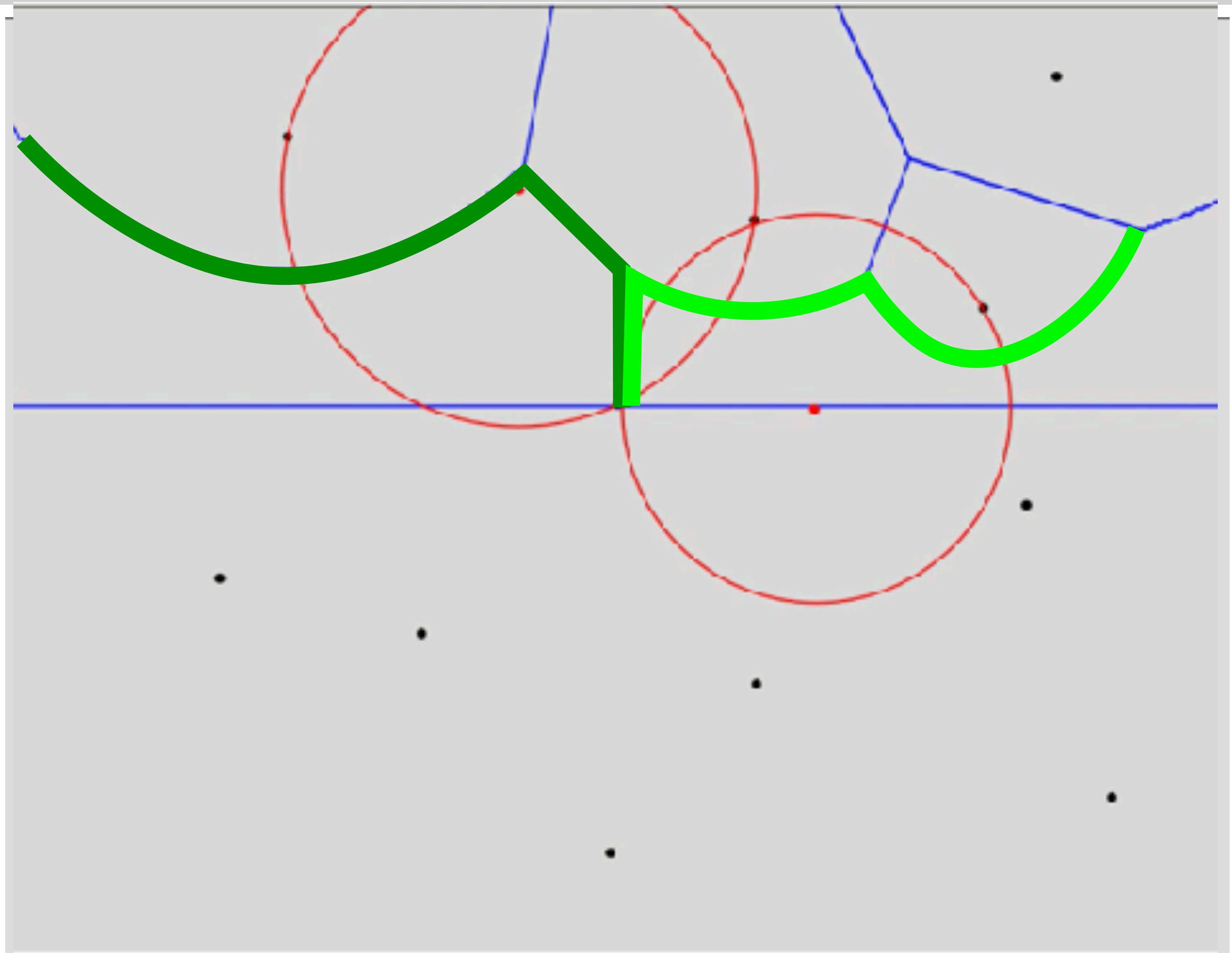

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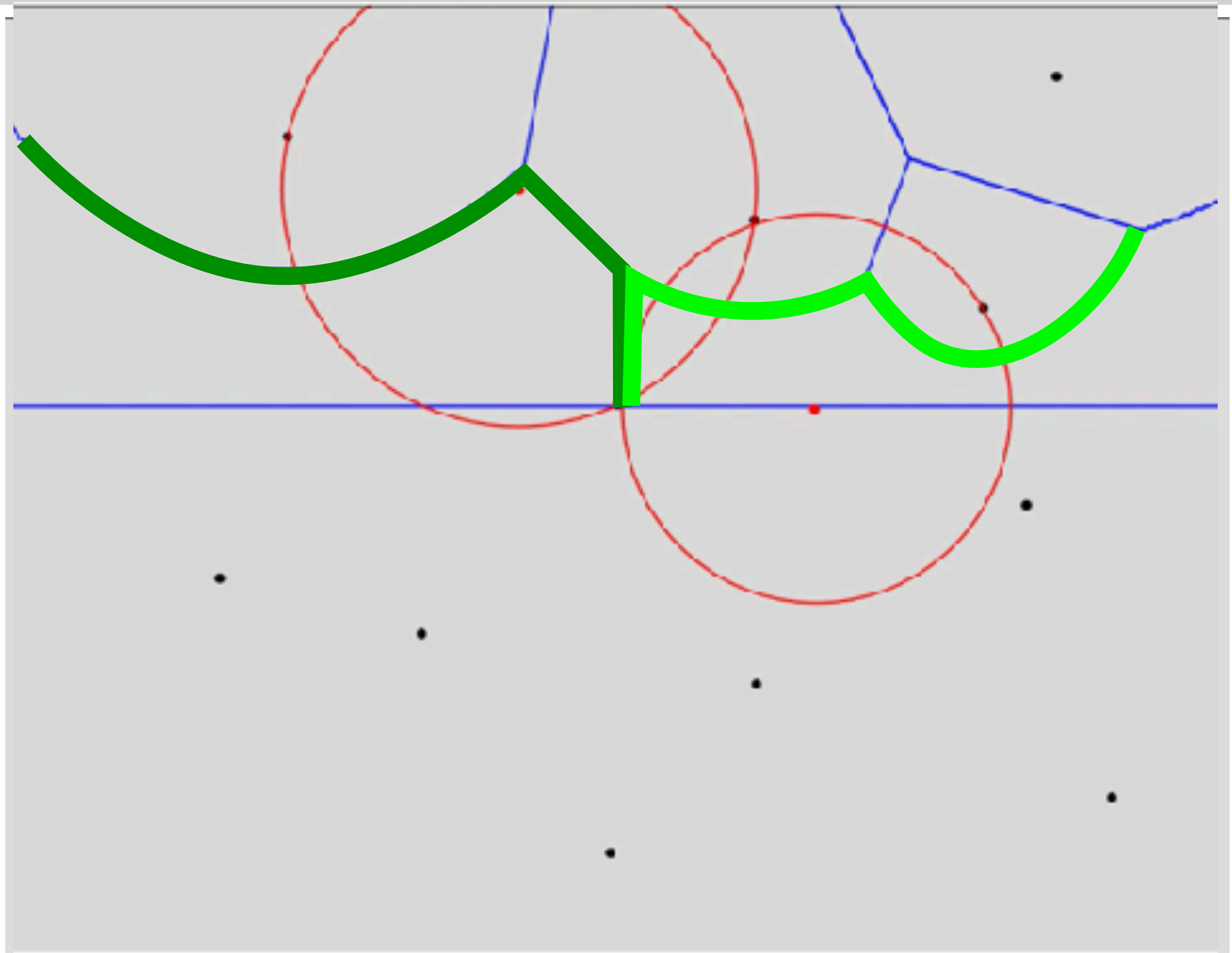
Point events $p_i \rightarrow$ Priority by $p_i.y$.

Circle events $C \rightarrow$ Priority by y-coordinate of lowest point of C.



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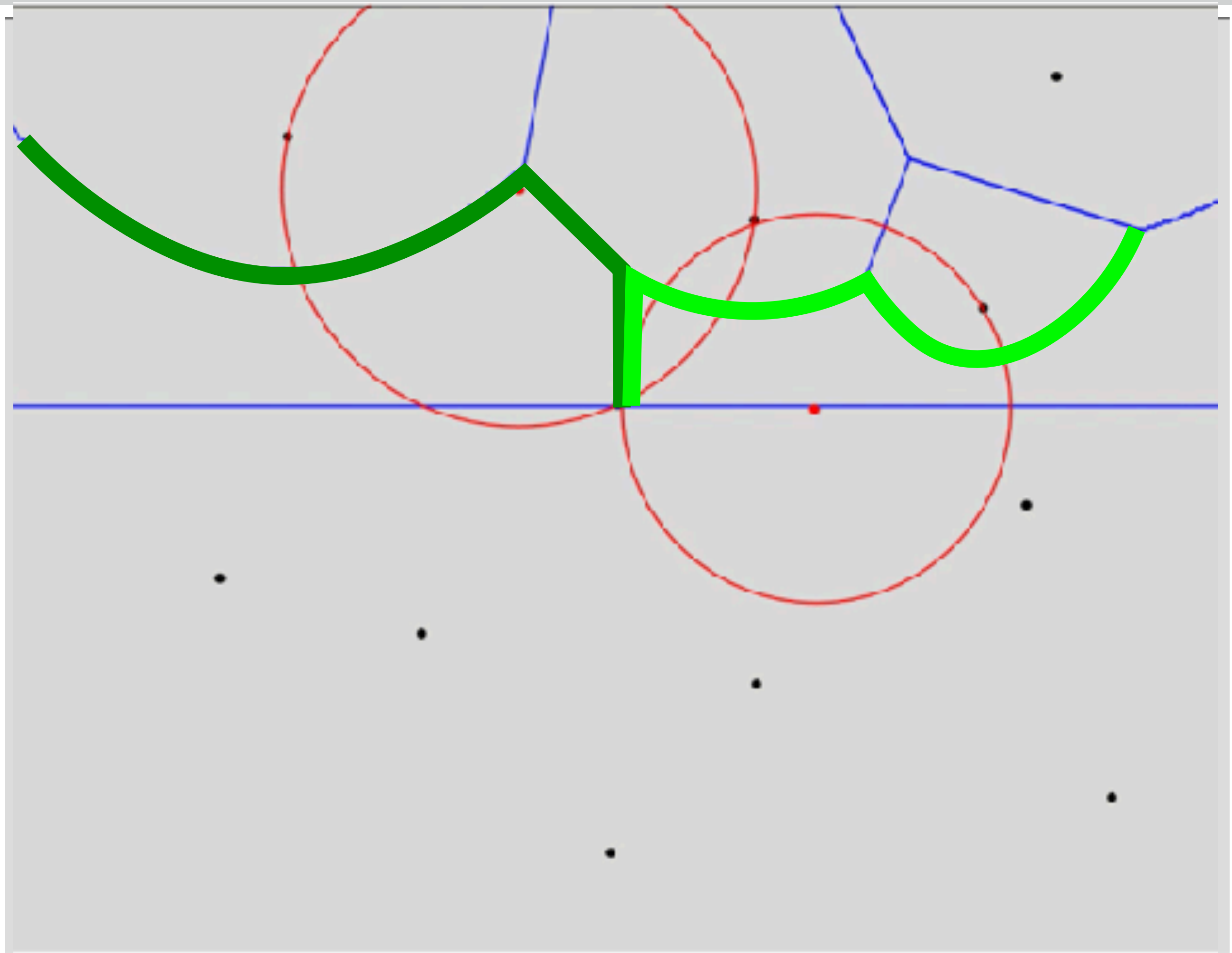
Point events p_i $\rightarrow$ Priority by $p_i.y$.

Circle events C $\rightarrow$ Priority by y-coordinate of lowest point of C.
$\rightarrow$ Pointer to arc (representing leaf $b \in B$) which may disappear.



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  Point events $p_i \to$ Priority by $p_i.y$.

  Circle events $C \to$ Priority by y-coordinate of lowest point of C.
  $\to$ Pointer to arc (representing leaf $b \in B$) which may disappear.

- Leaf $b \in B$ points to circle event $C \in Q$ for which arc β of b may disappear.

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Observation 1:

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- Discovering $p \in C^\circ$

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- Discovering $p \in C^\circ \Rightarrow$ Circle event C becomes obsolete

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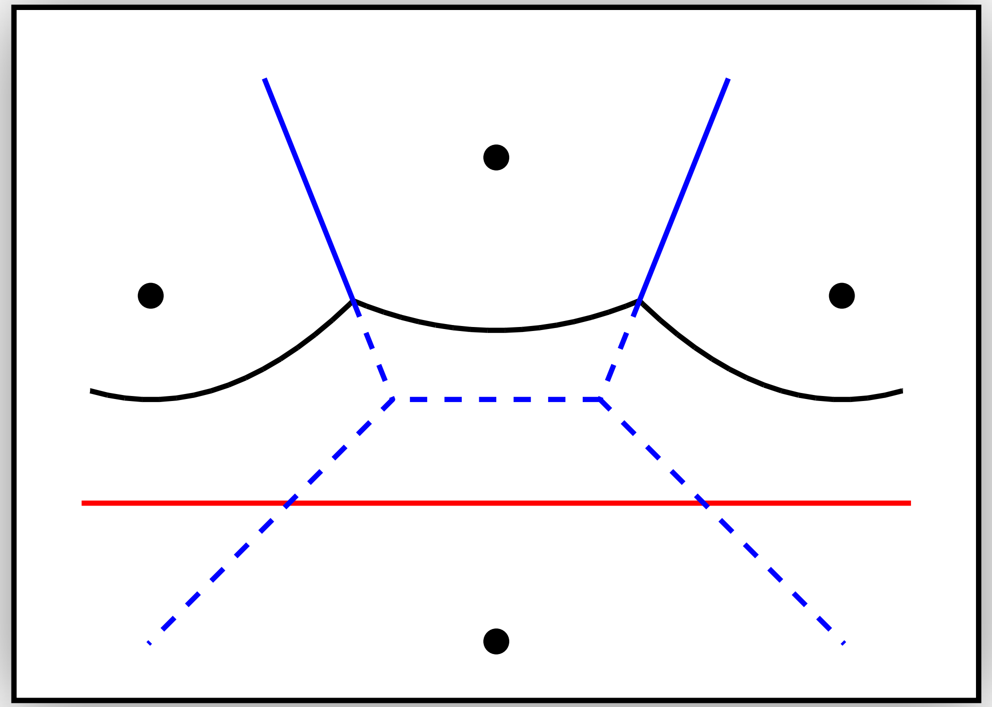

Observation 1:

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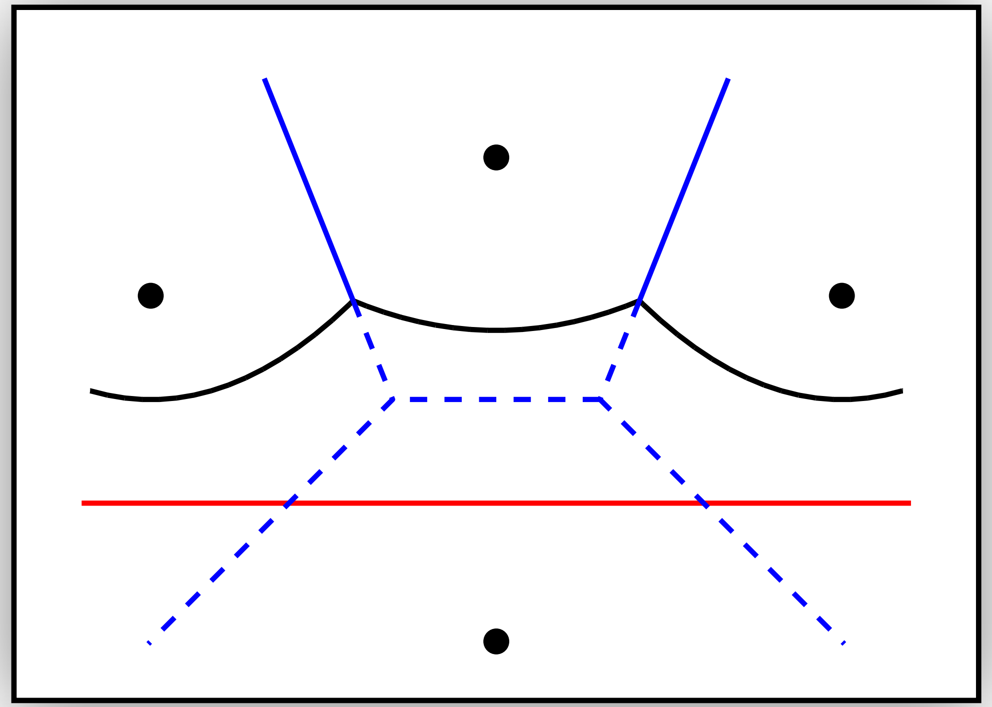
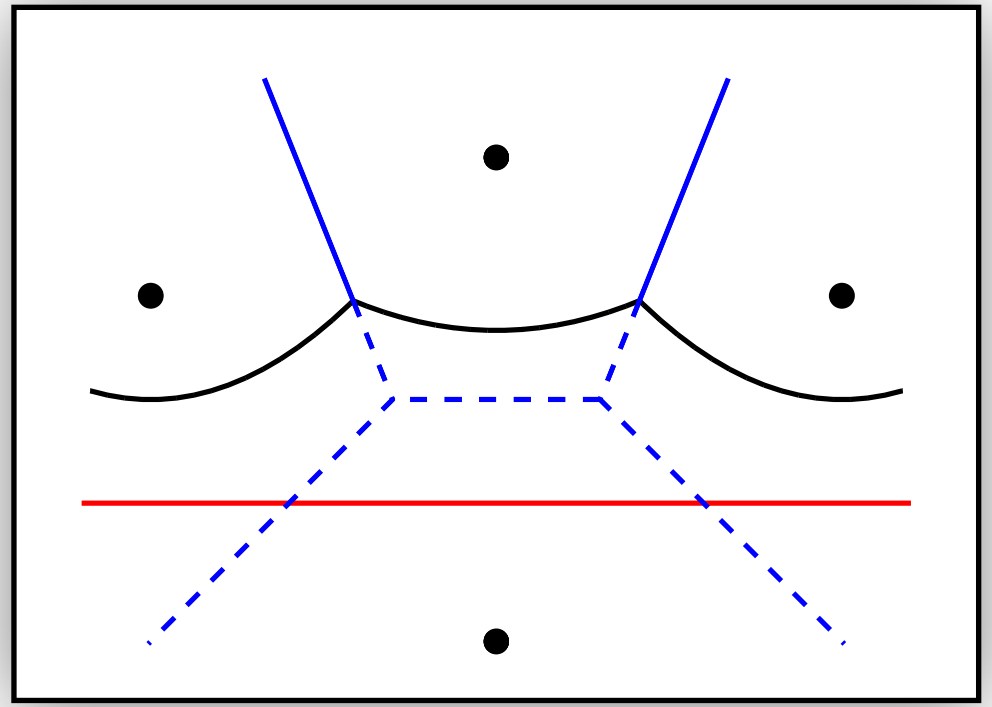
$\rightarrow$ Parabolic arc must know associated circle event.





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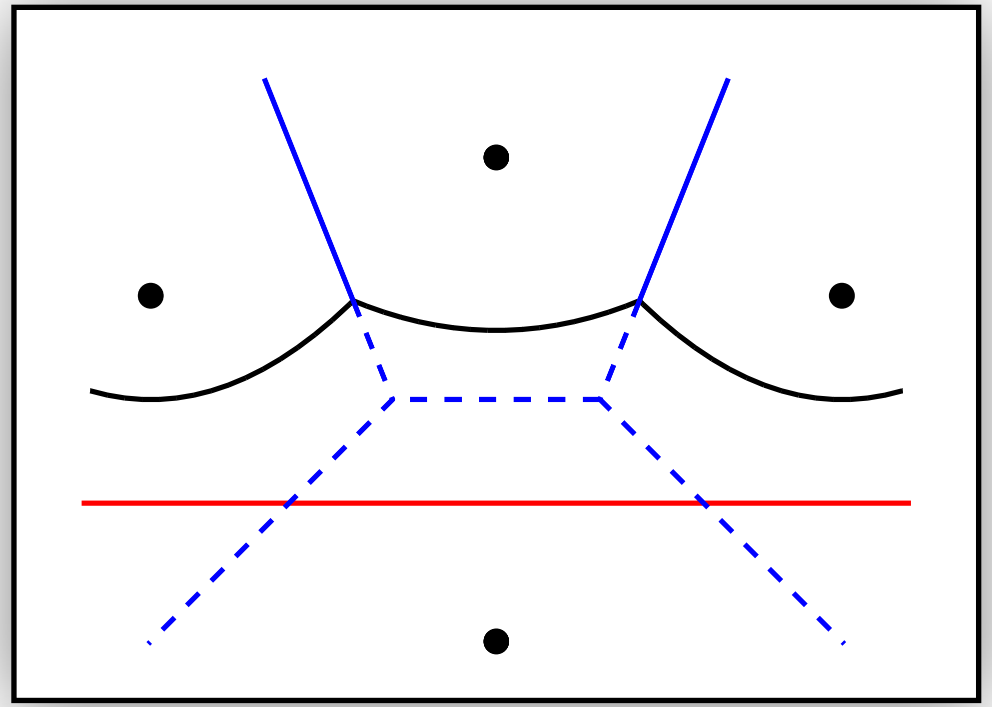
Observation 2:



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- Triple of adjacent arcs do not always define a circle event.

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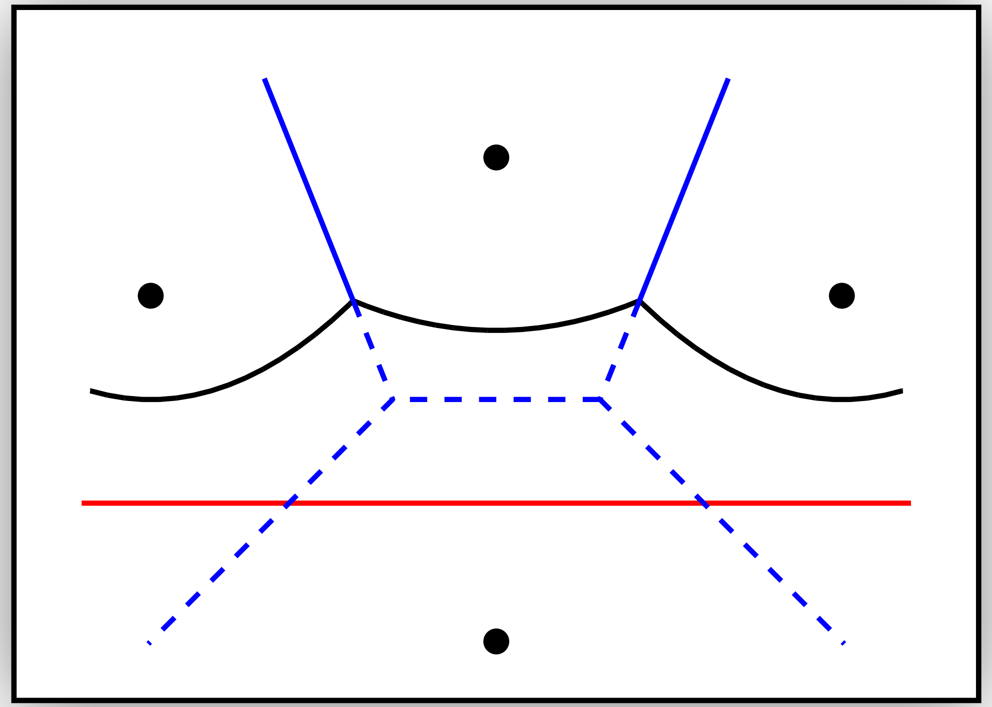

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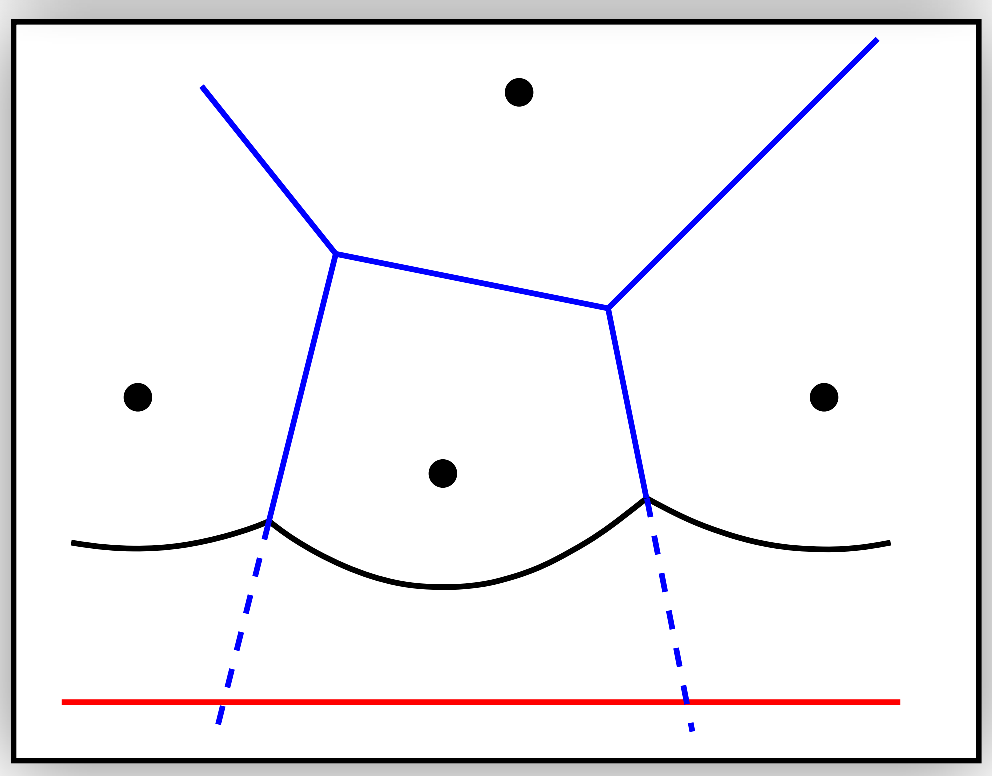
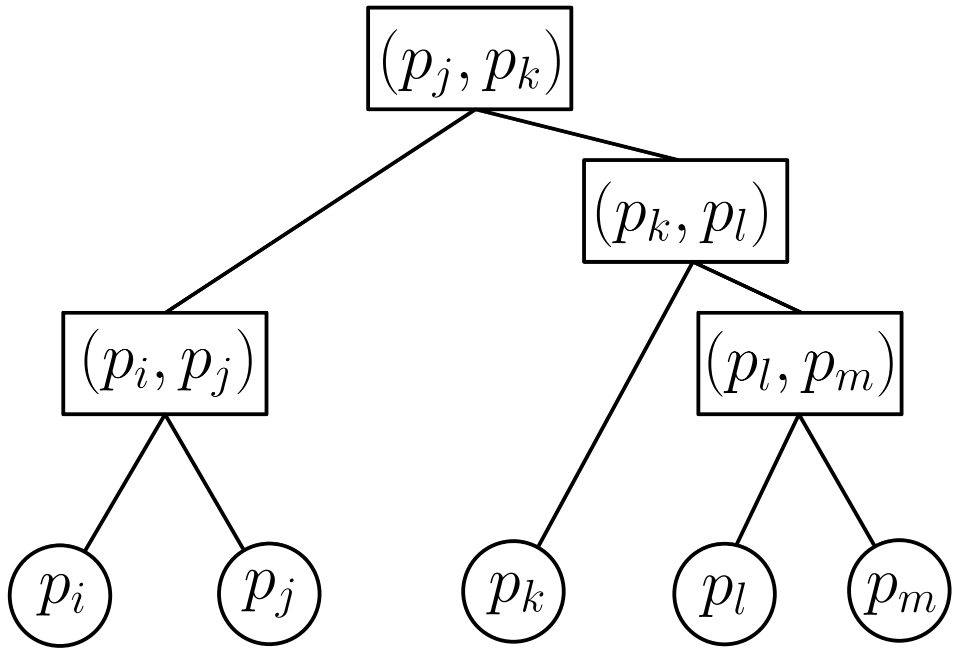
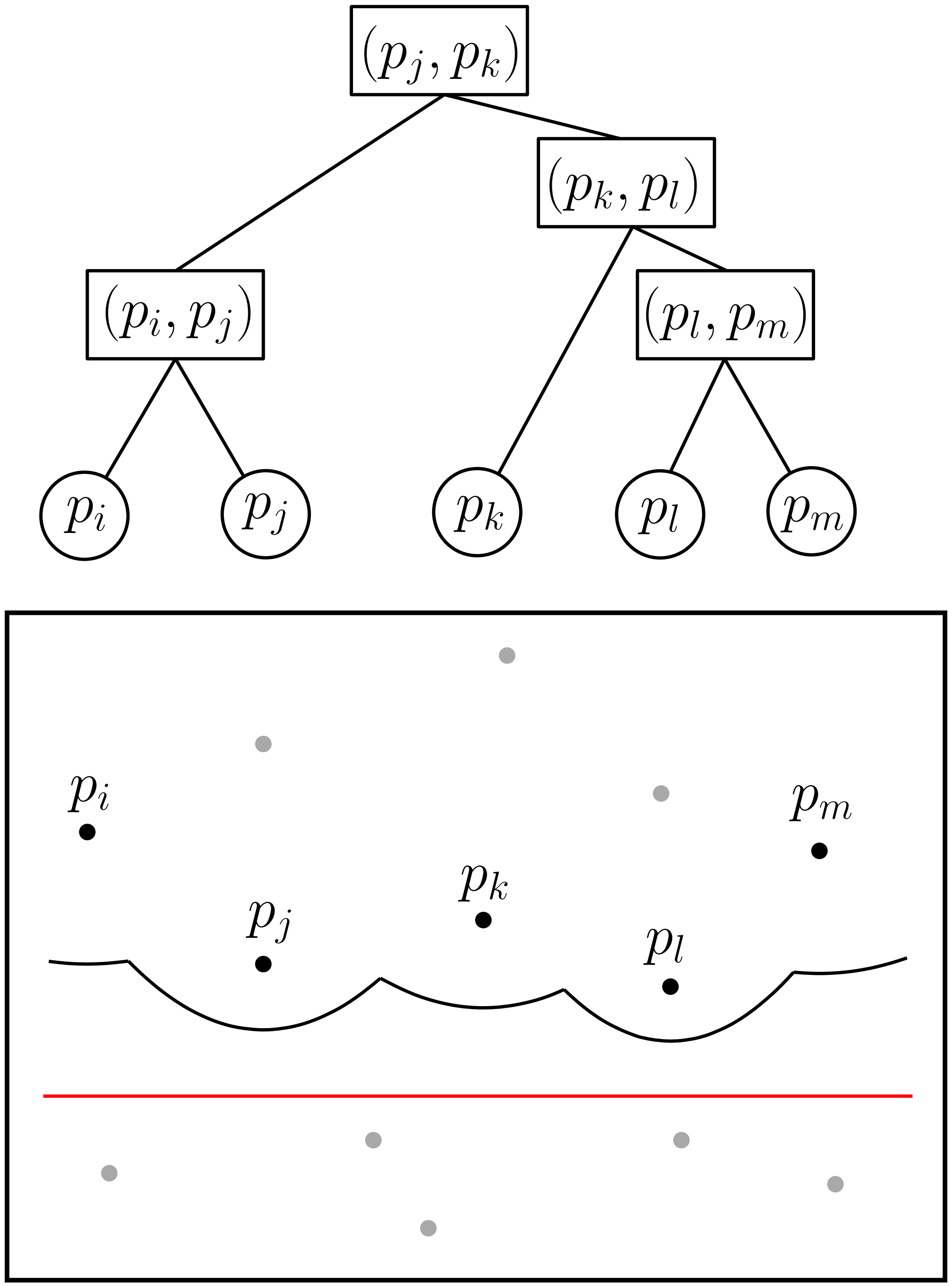



(p_j, p_k)
(p_k, p_l)
(p_i, p_j)
(p_l, p_m)
p_i
p_j
p_k
p_l
p_m



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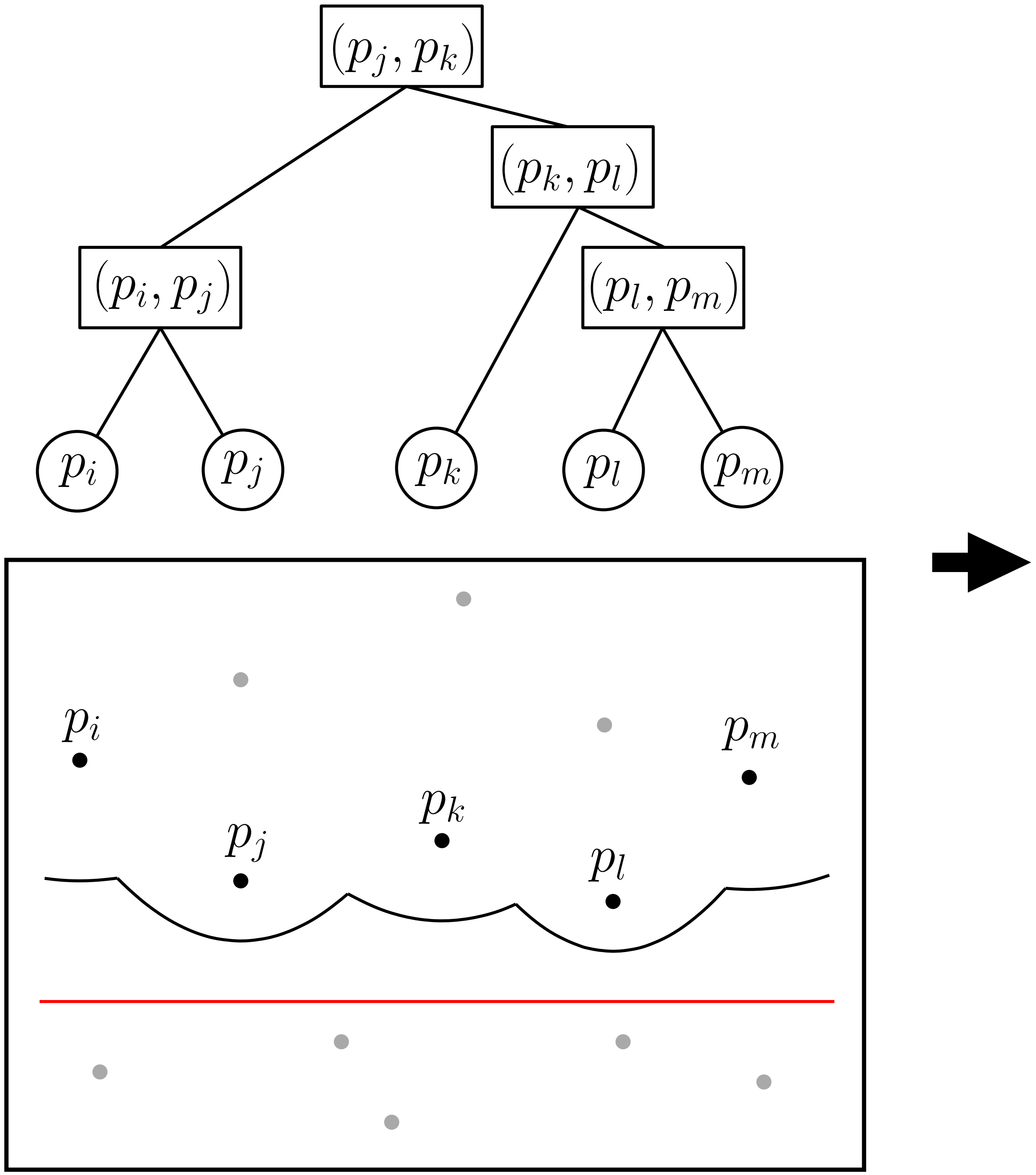
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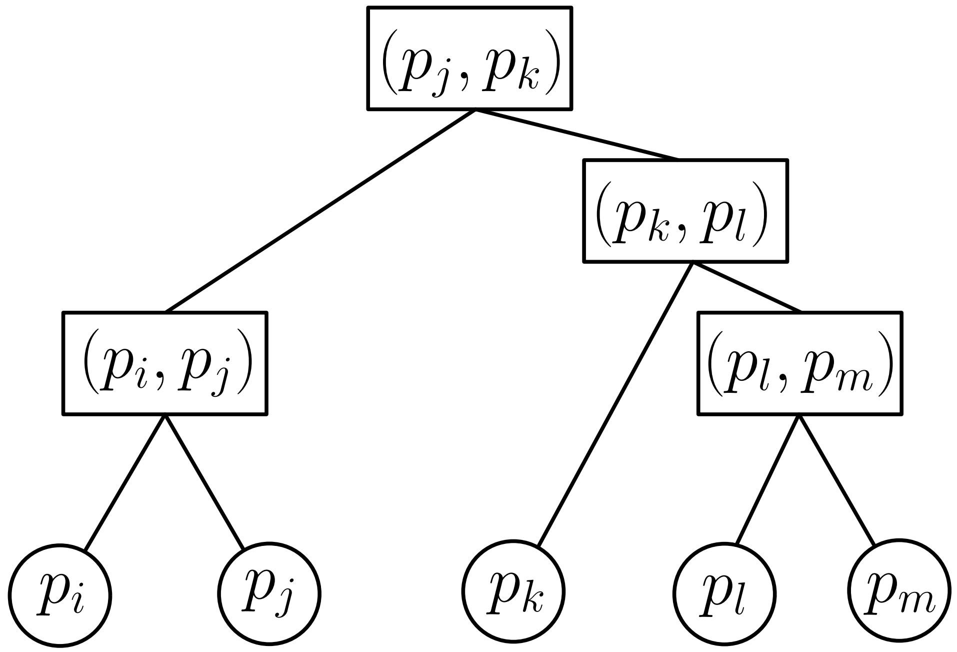
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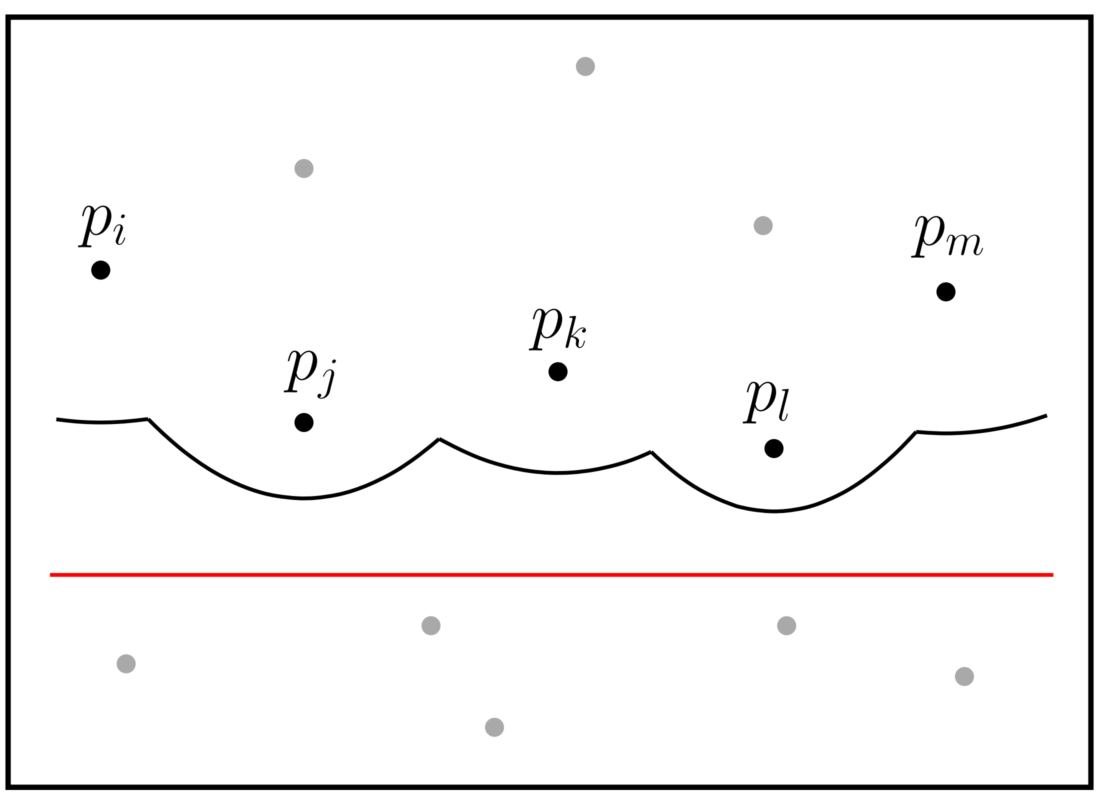
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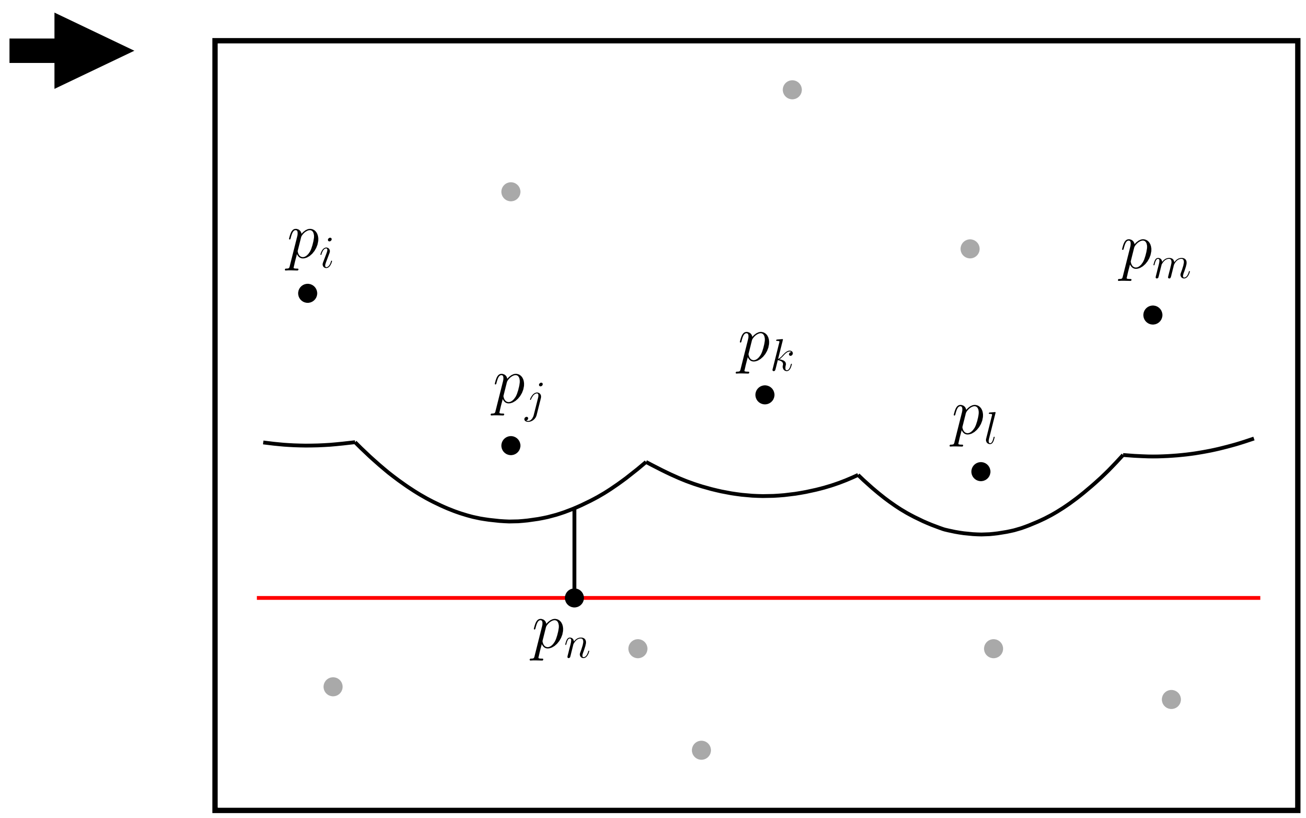
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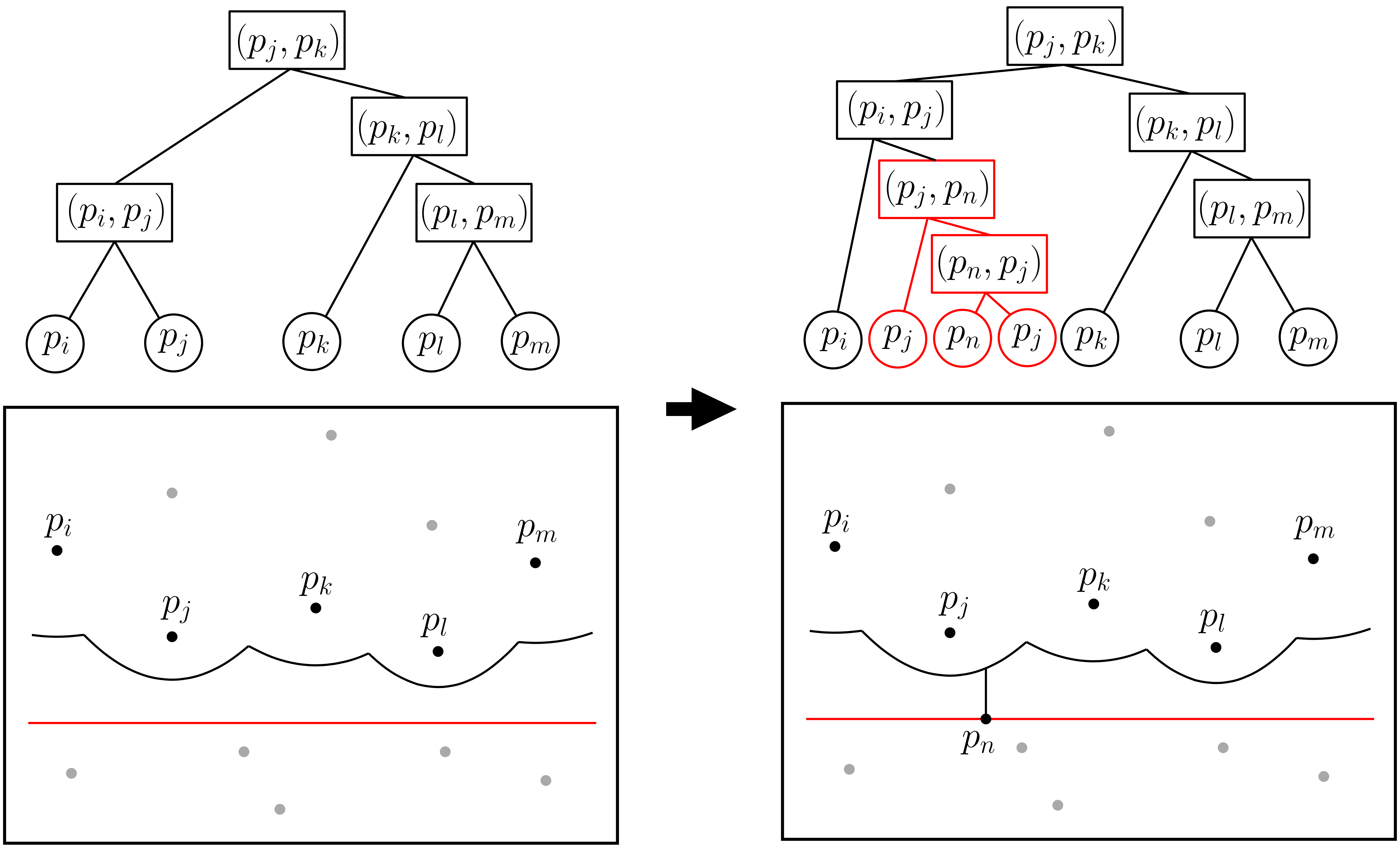
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Introducing an arc

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- Defined by point p_n

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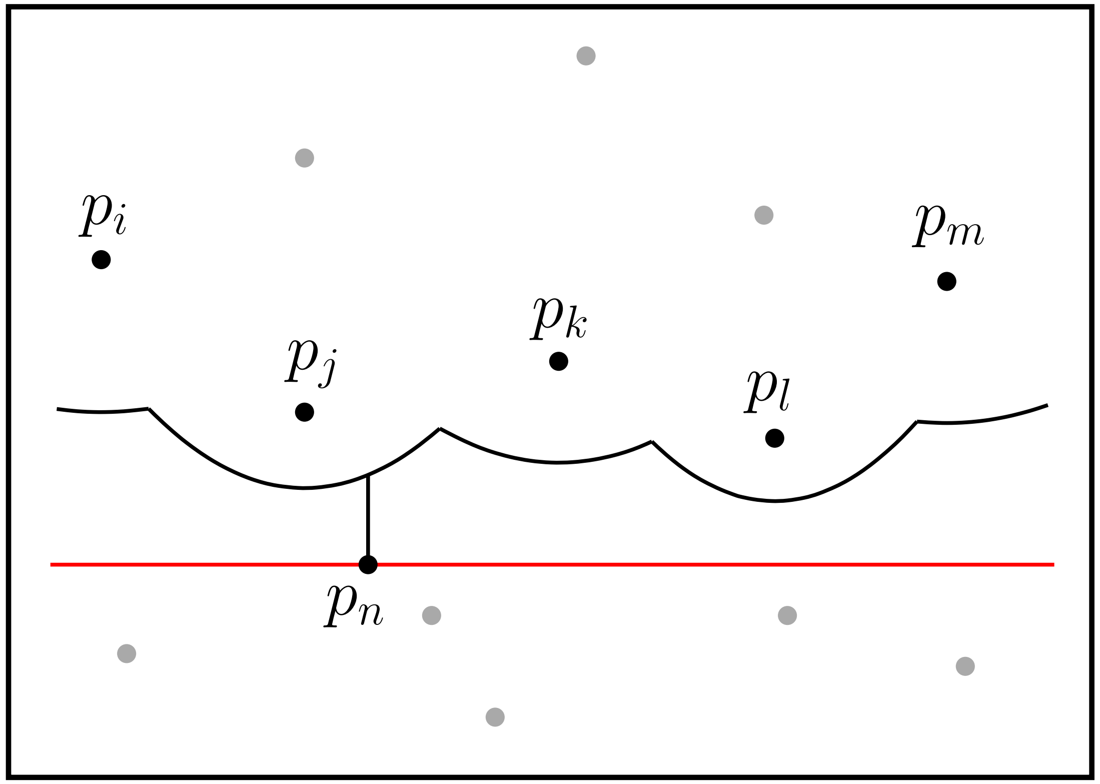

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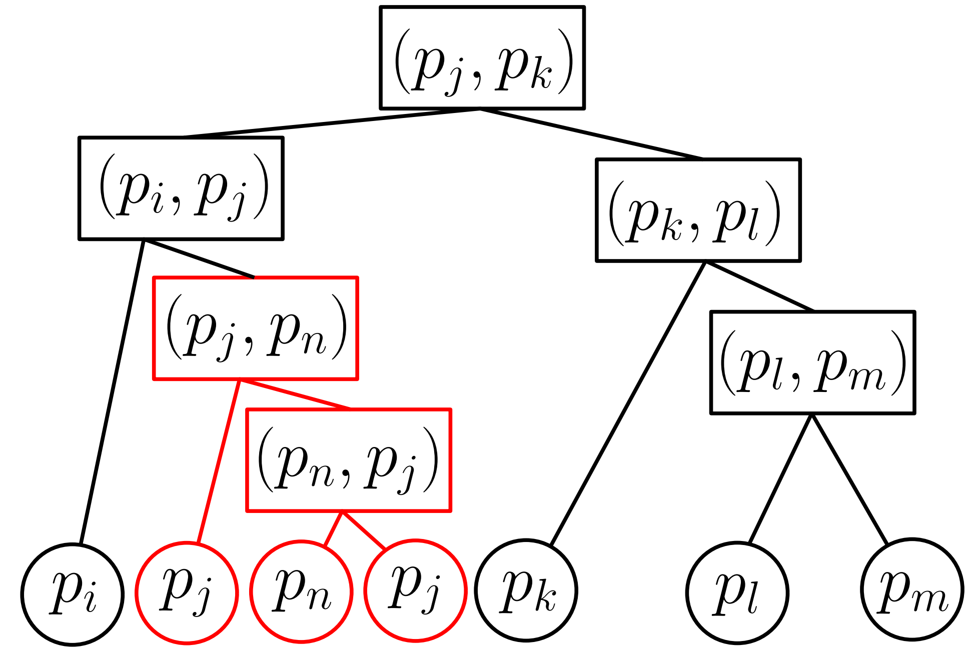
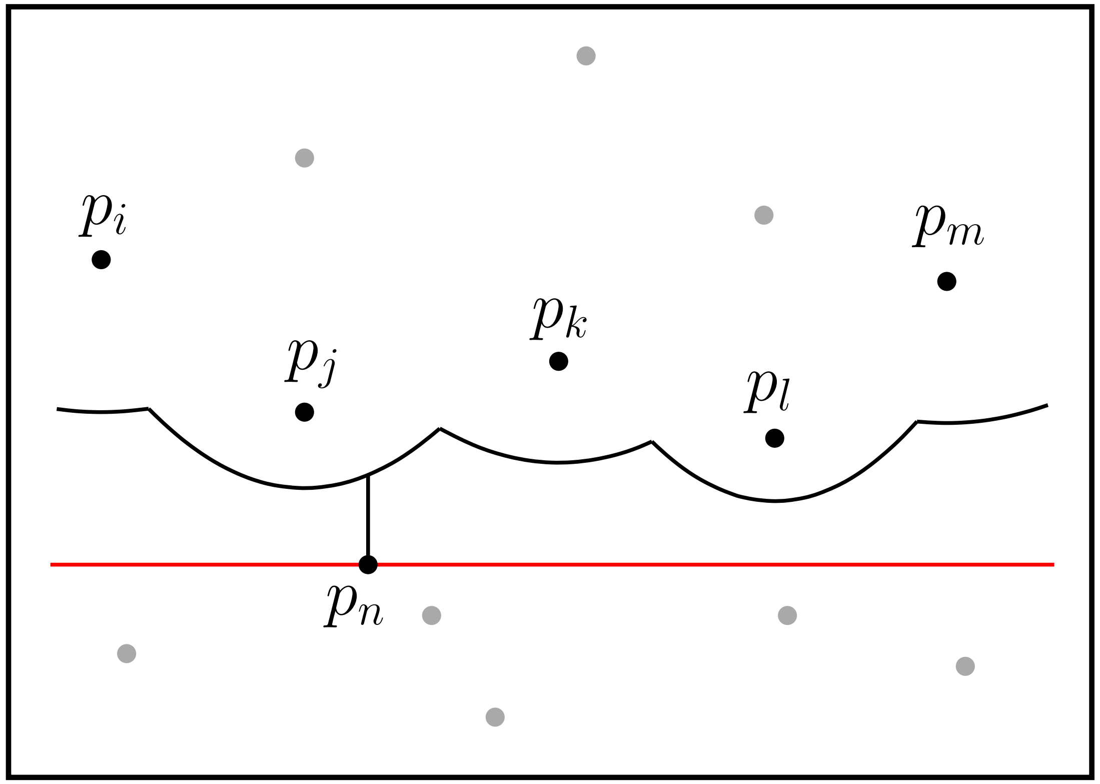
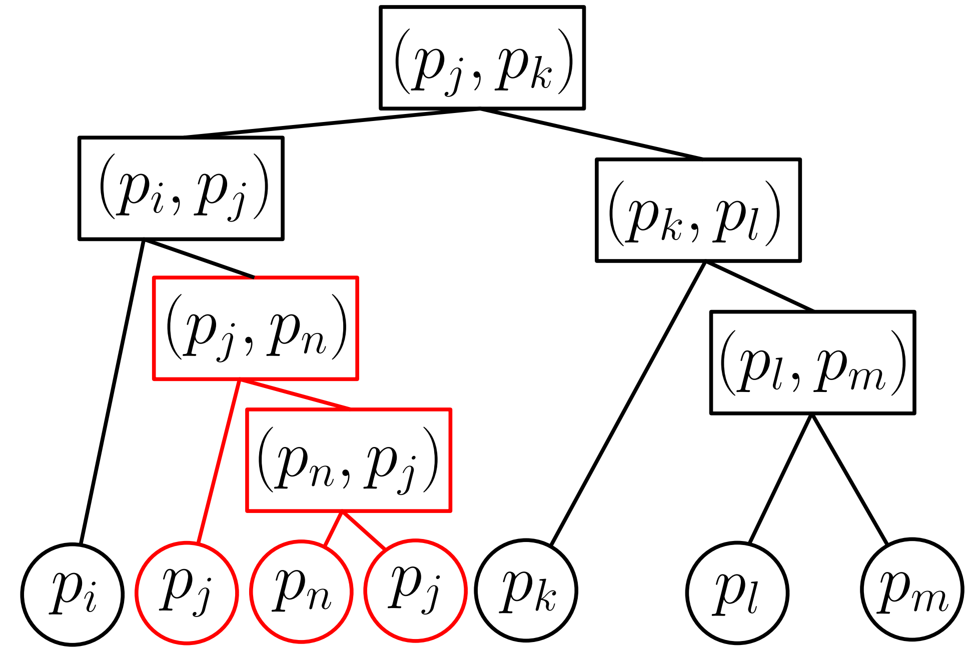
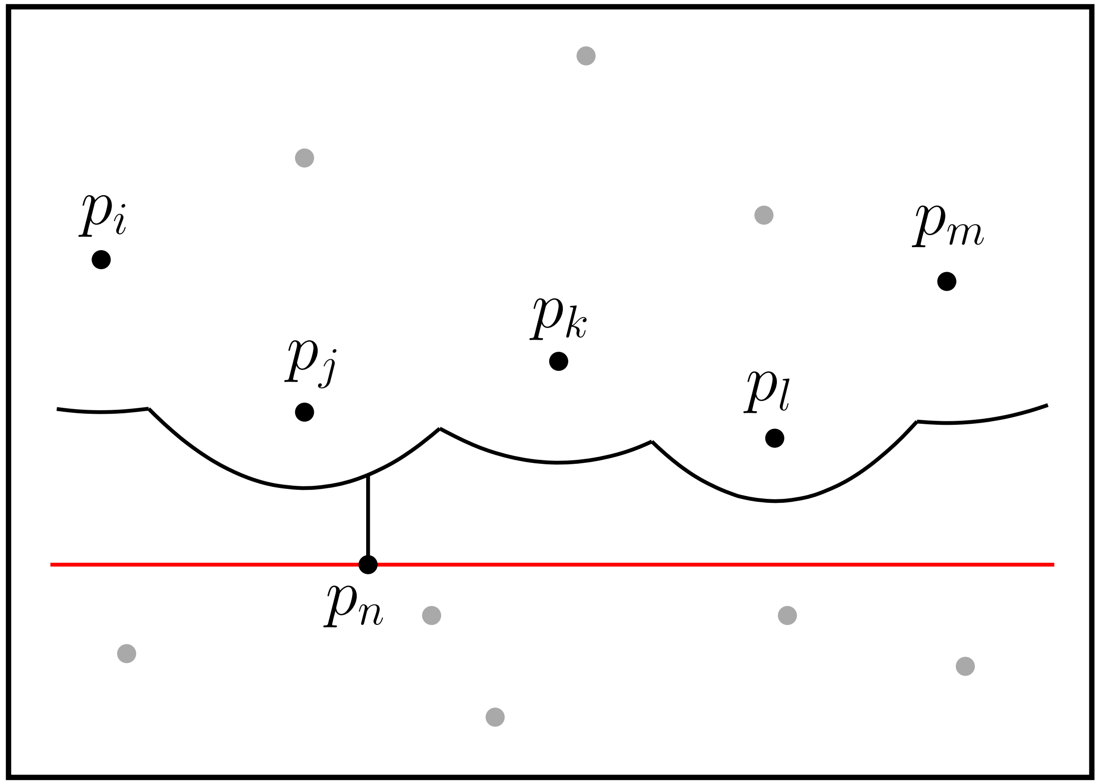






Introducing an arc

- Defined by point p_n
- Search for B at insert position.



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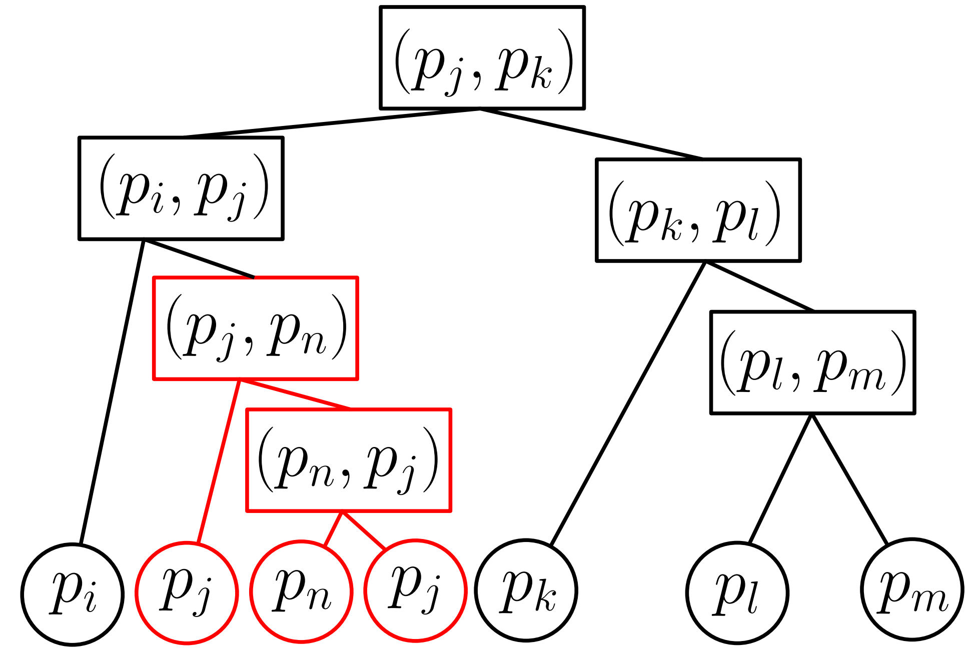
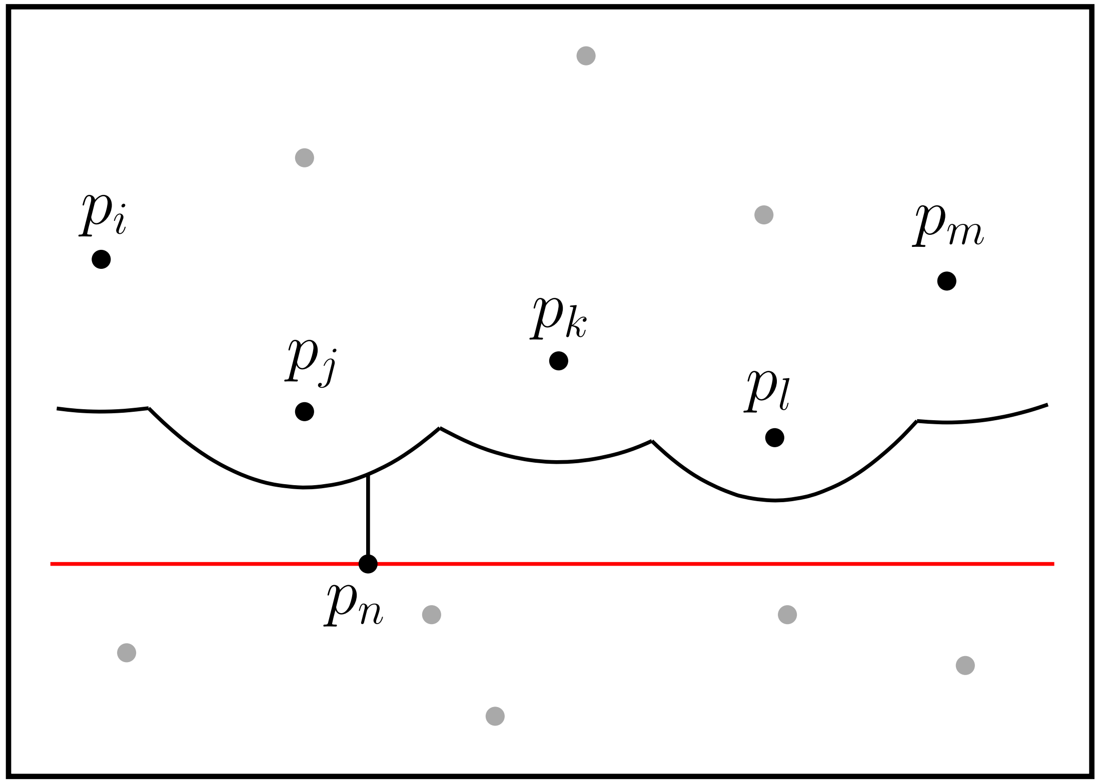
$\rightarrow$ Intersection points stored implicitly





Introducing an arc

- Defined by point p_n
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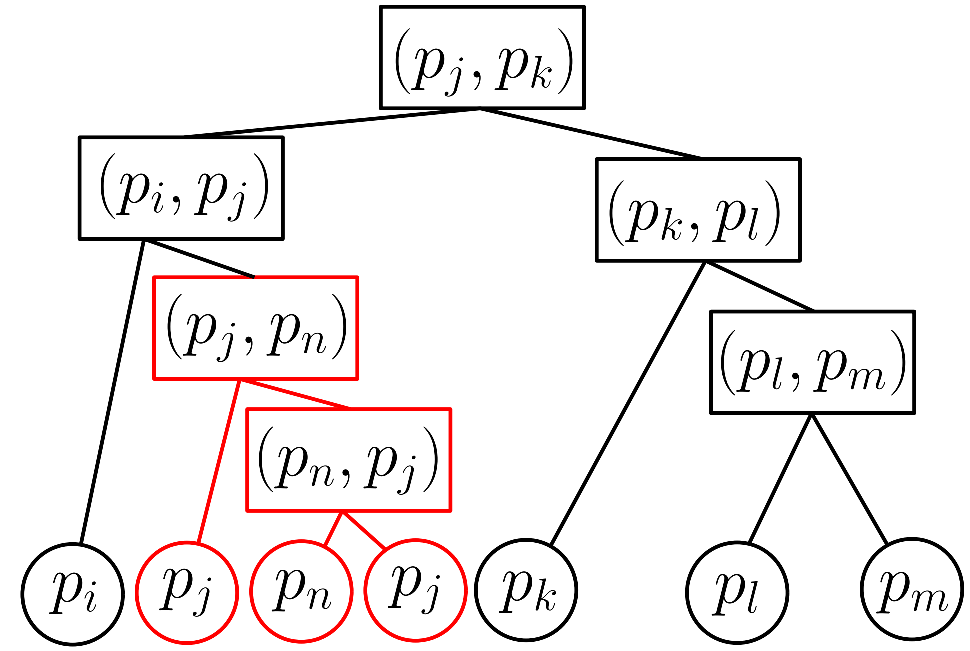
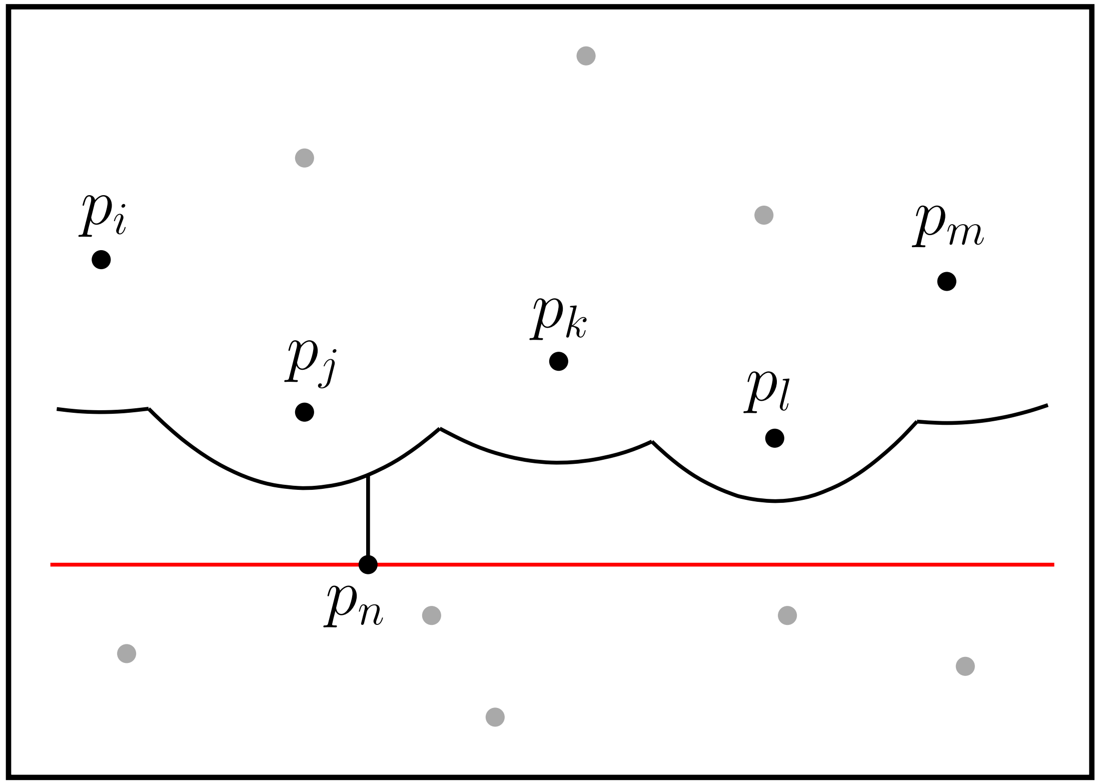
- Splitting an arc β
  (Special case: p_n below arc intersection)





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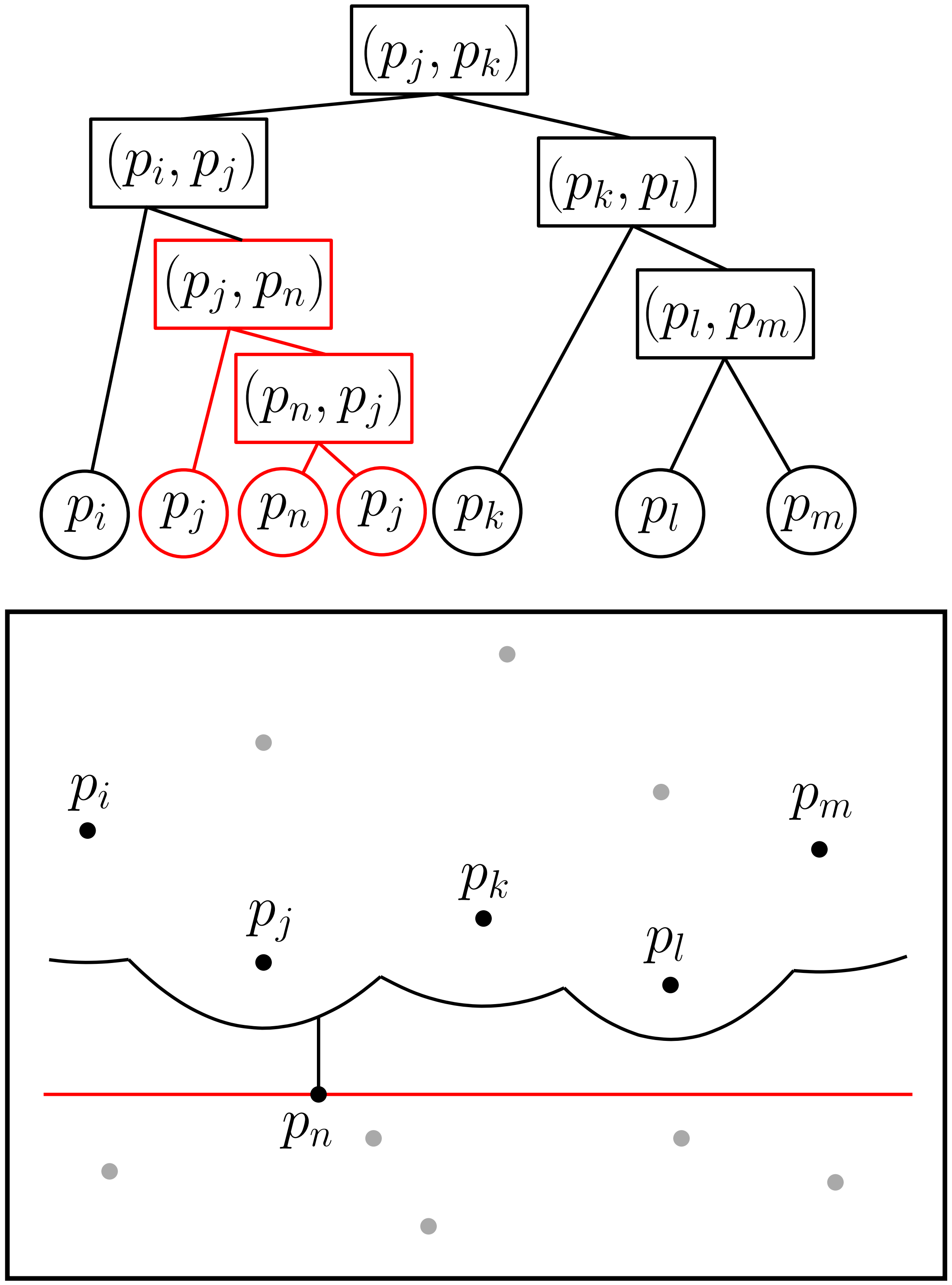
- Splitting an arc β
  (Special case: p_n below arc intersection)
- Splitting: $\beta \rightarrow \beta_1, \beta_n, \beta_2$
  (rebalancing if necessary)



Generating circle events:

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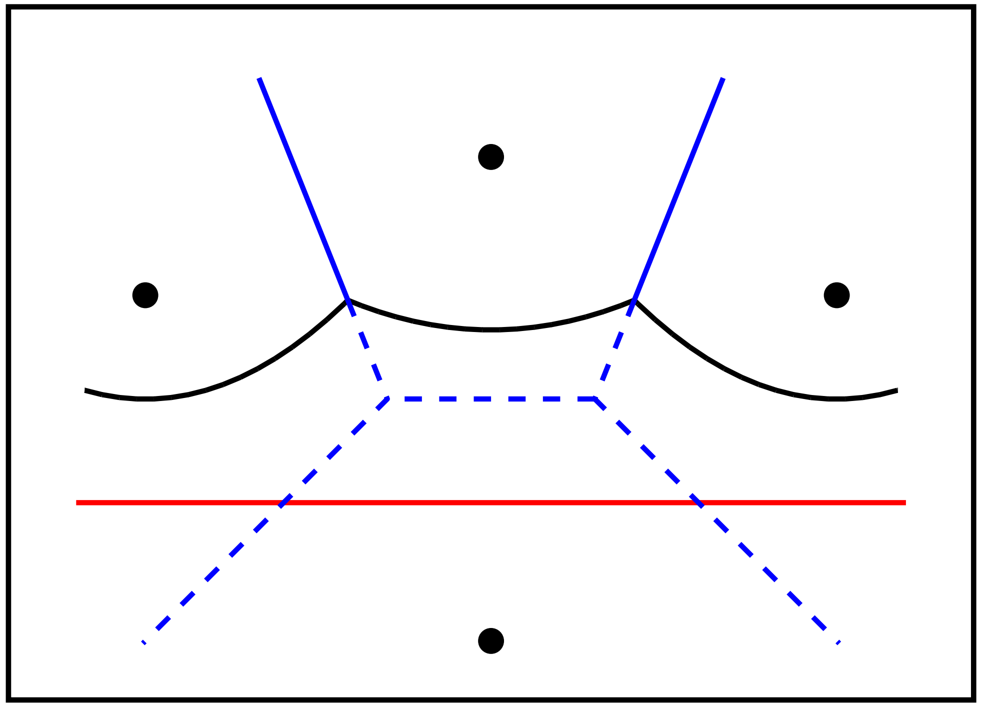
- Before insertion of β_n(defined by p_n):

$$\dots \beta_i \beta_j \beta_k \beta_l \dots$$



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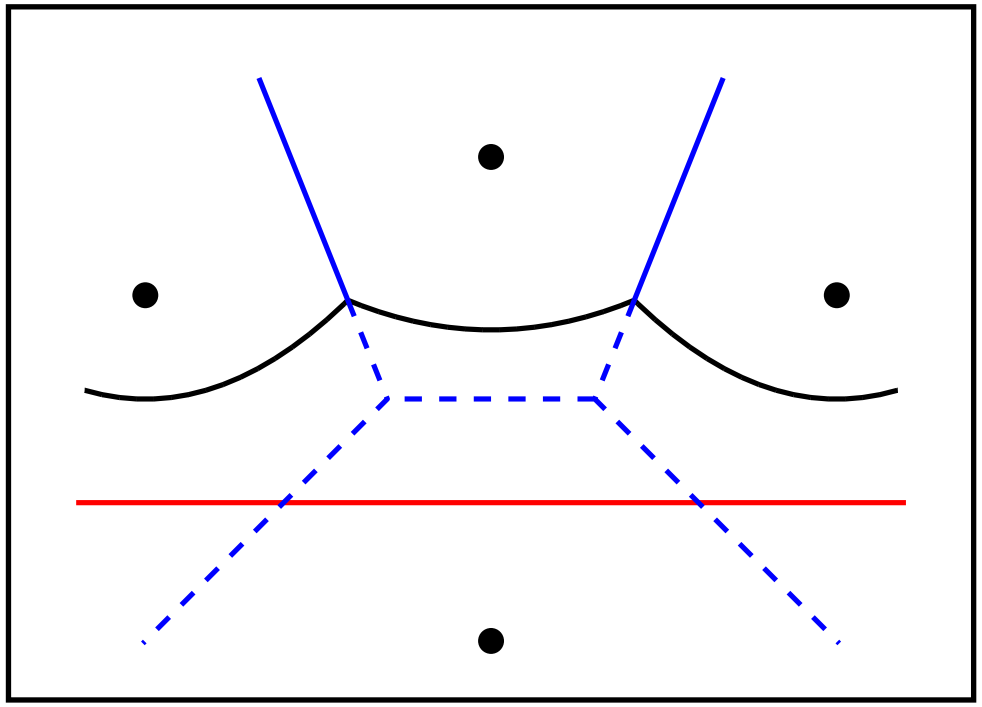

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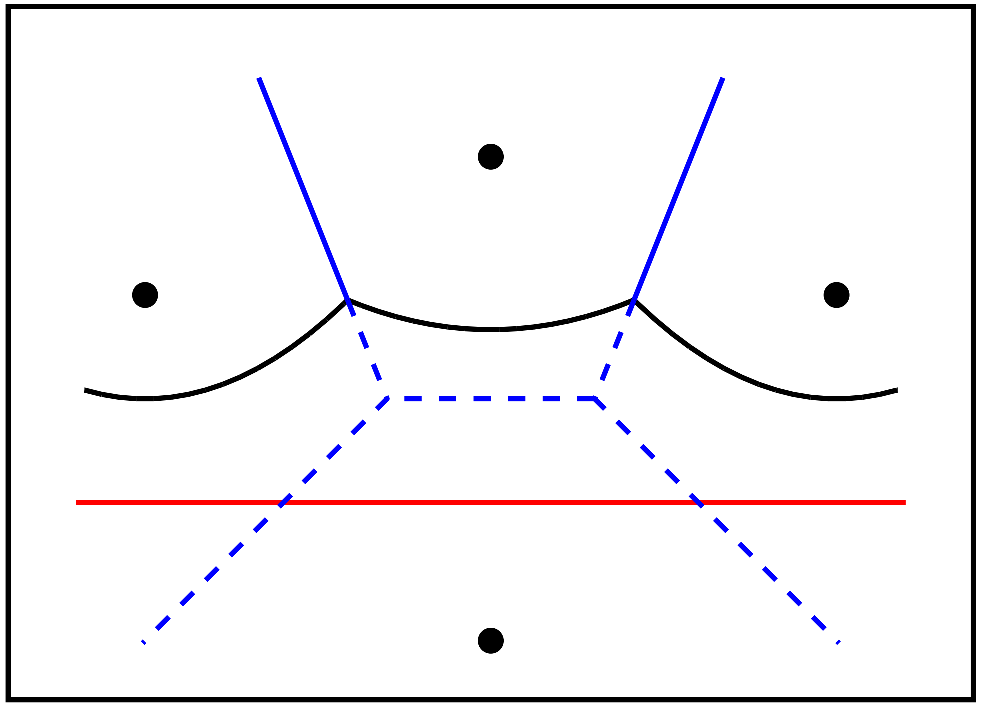
- Possibly deletion of circle events (e.g., defined by $(\beta_i, \beta_j, \beta_k)$).



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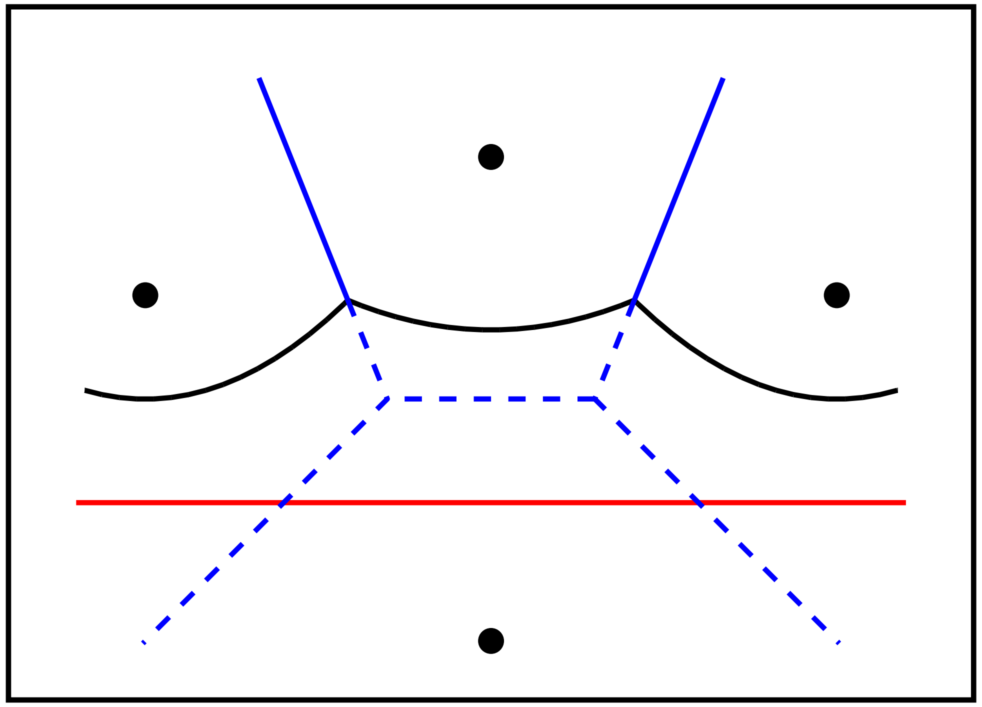

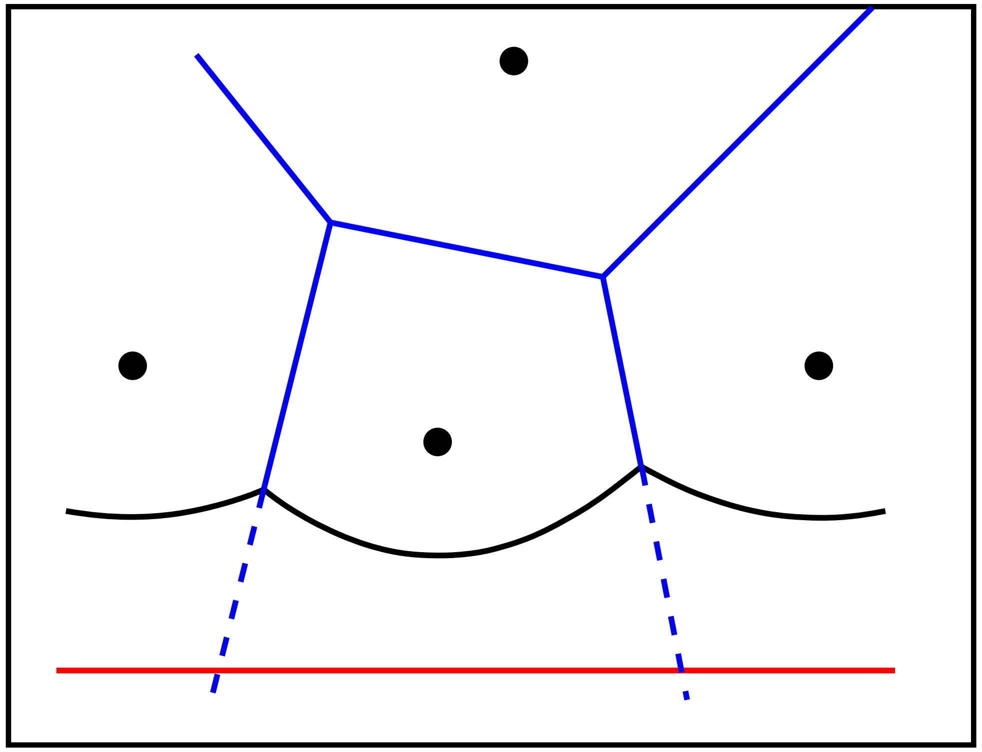

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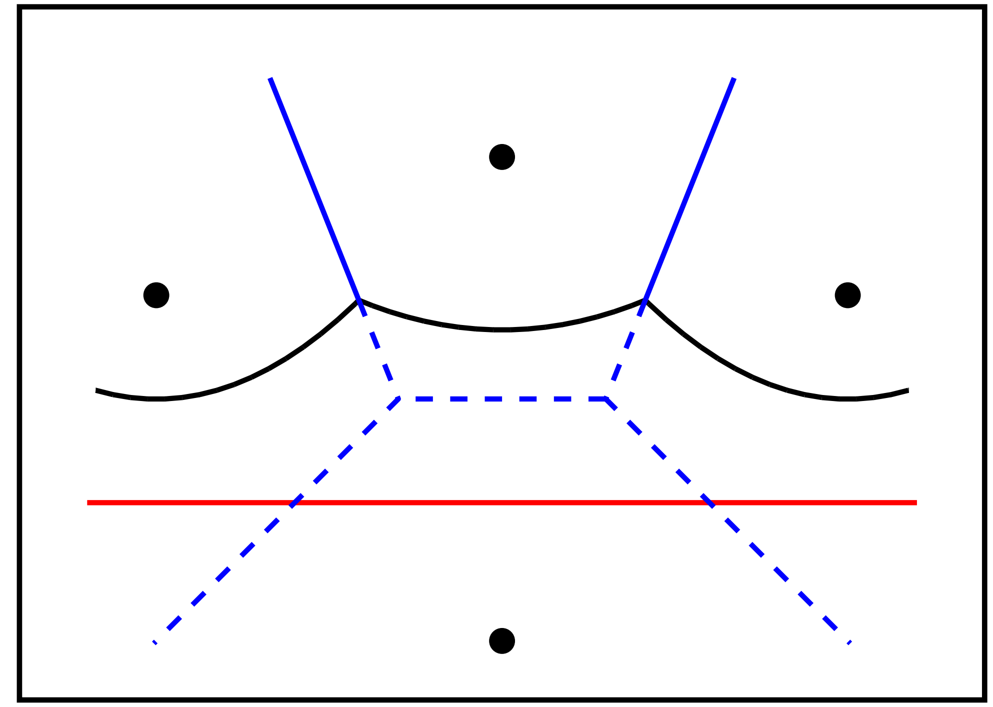
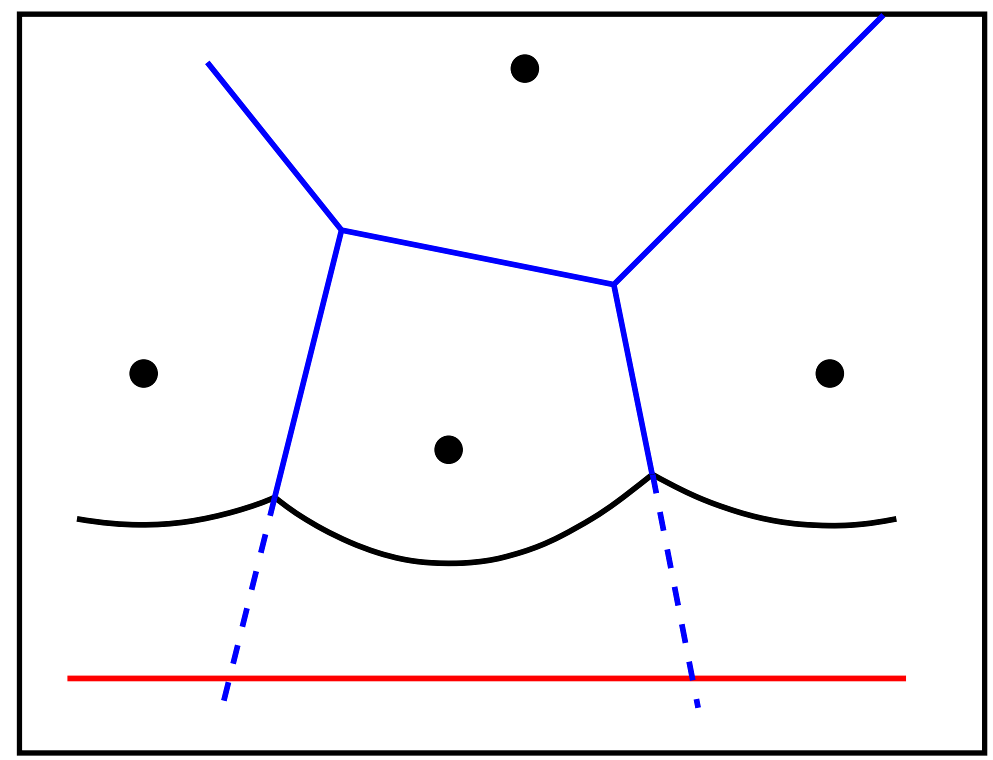
- Test all newly adjacent triples.





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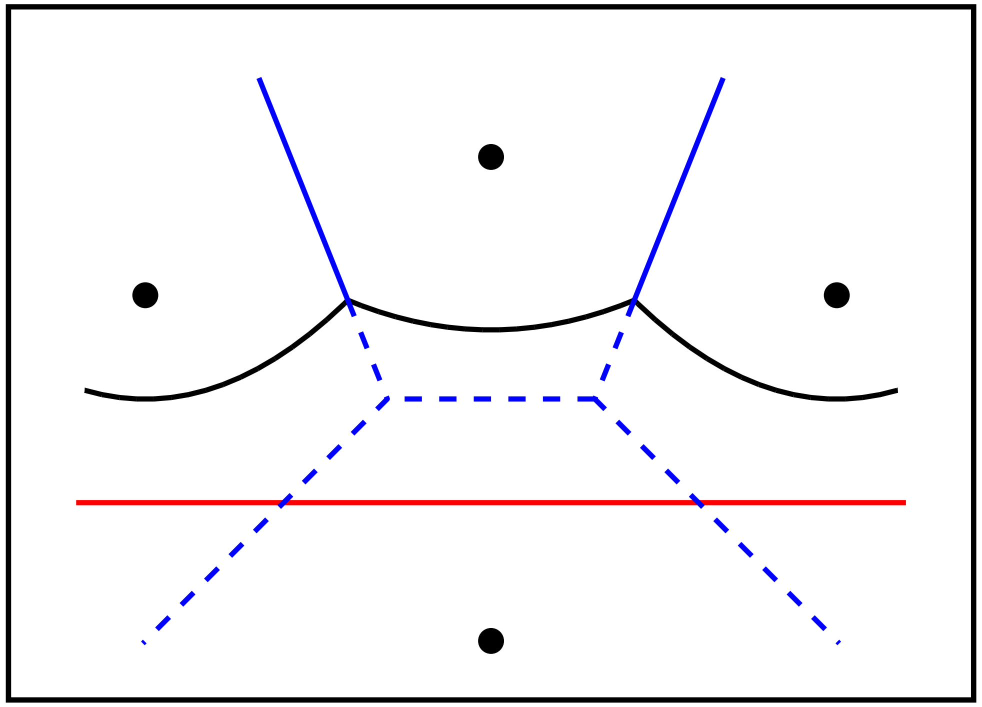
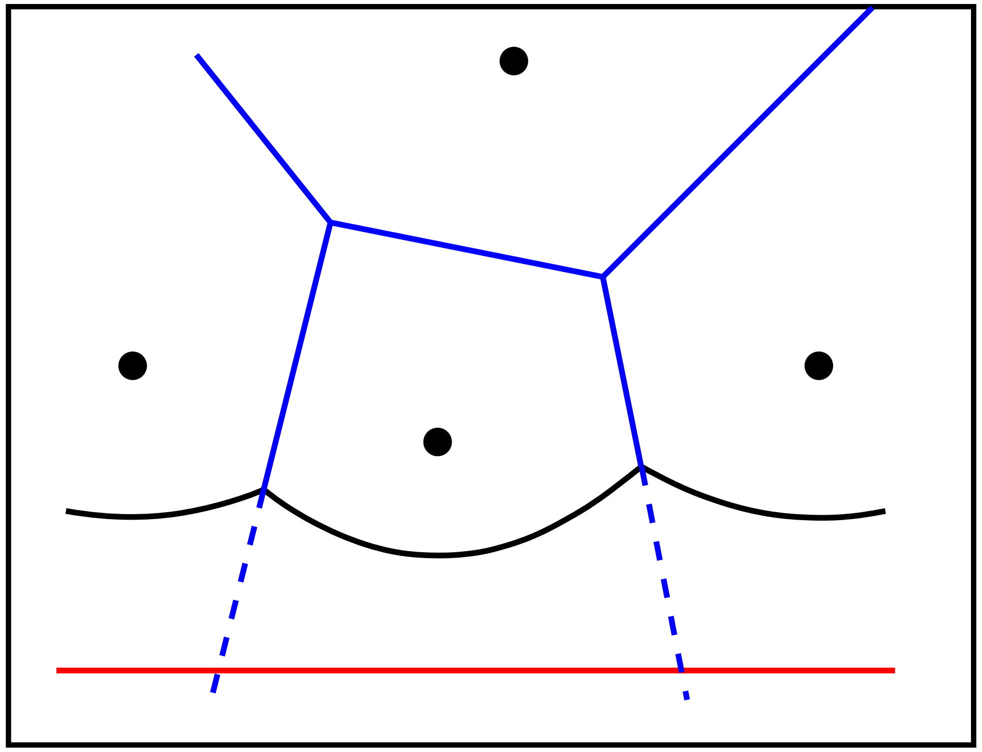
- Possibly deletion of circle events (e.g., defined by $(\beta_i, \beta_j, \beta_k)$).

- Test all newly adjacent triples.

- Insert new events into Q.

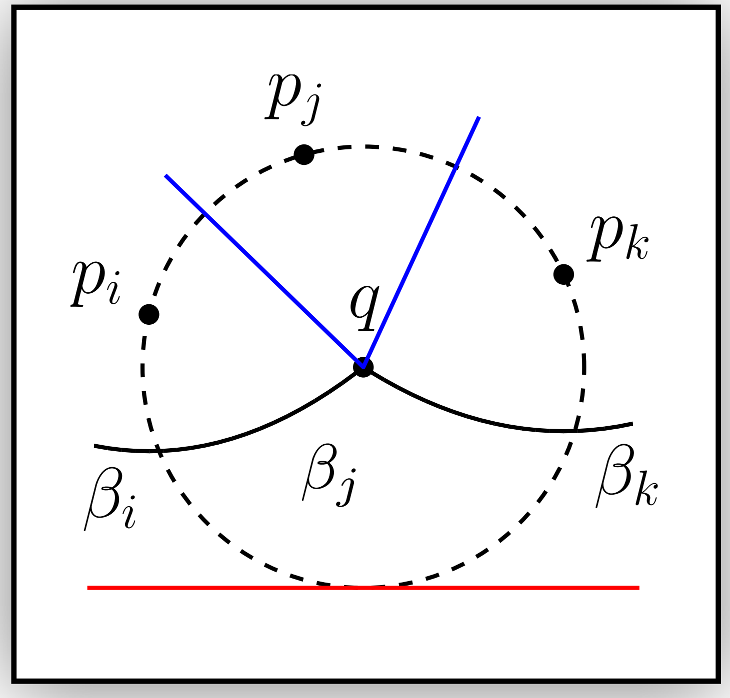




Processing Circle Events - I

Updates

Updates



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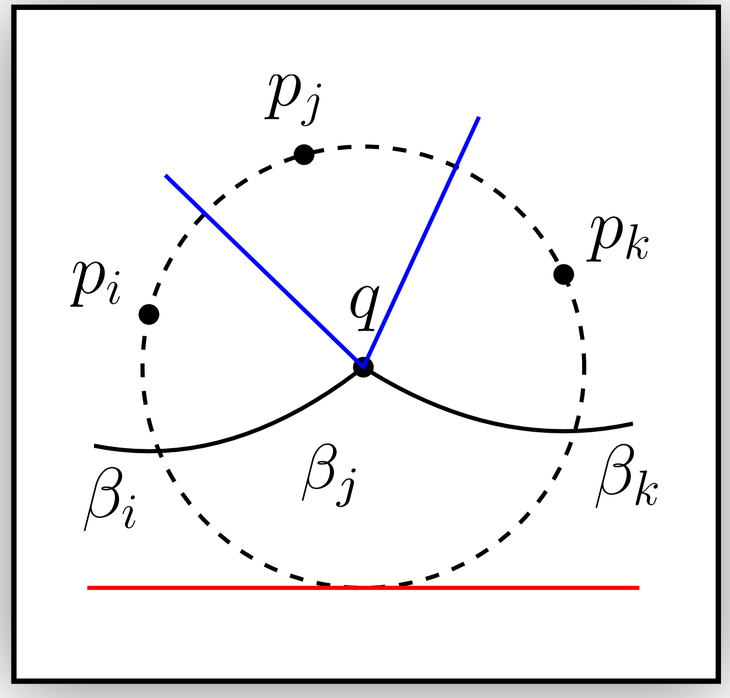
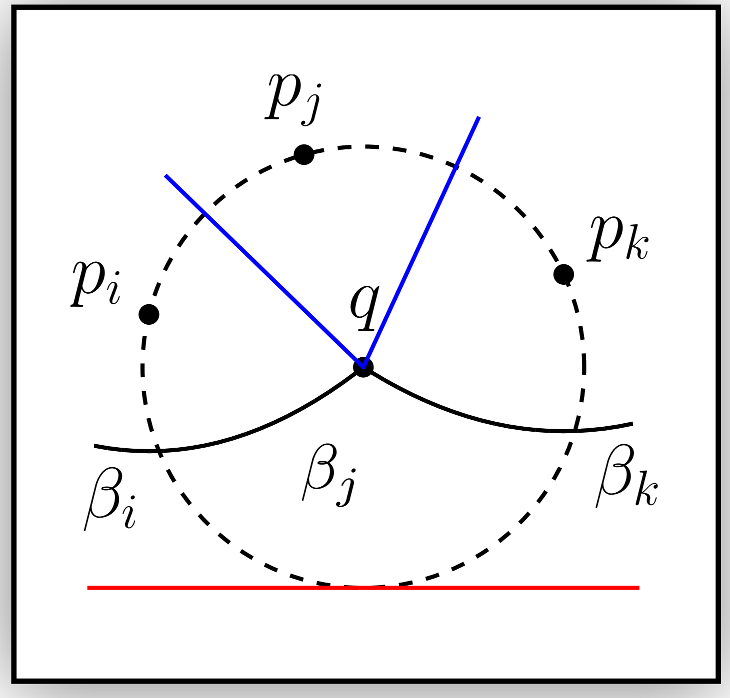
Updates

- Delete β_j from B.



Updates

- Delete β_j from B.
  $\rightarrow$ Rebalance if necessary



Updates

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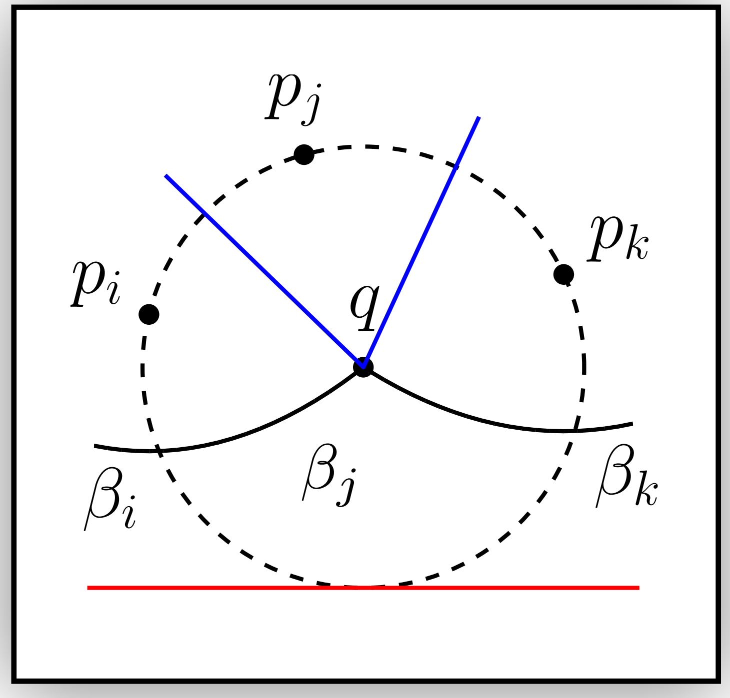

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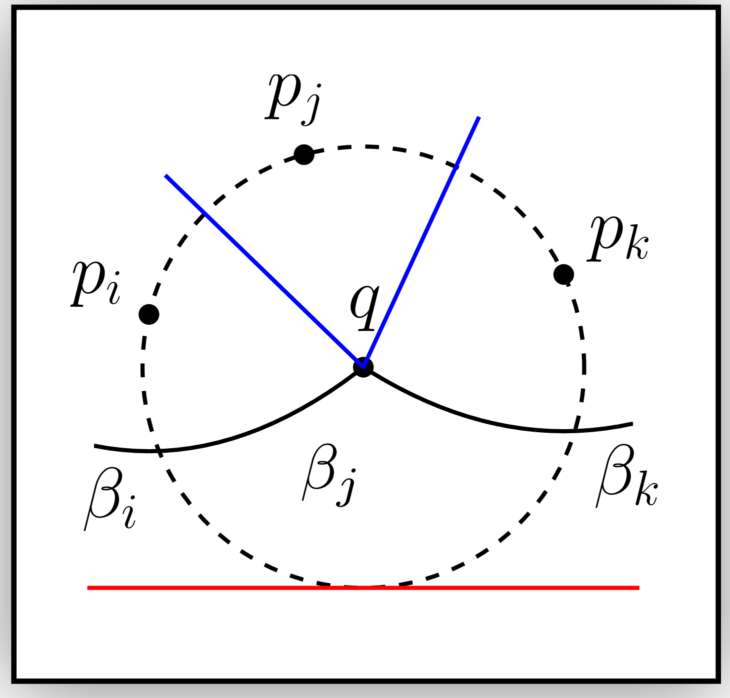
  $\rightarrow$ Rebalance if necessary
- Delete circle events defined by β_j
- New adjacent triples
  $\rightarrow$ Possibly insert new circle events



Correctness - I

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- Theorem 4.20 $\Rightarrow \exists p_i, p_j, p_k \in \mathcal{P}$:

$$p_i, p_j, p_k \in \partial C$$

$$C \cap \mathcal{P} = \{p_i, p_j, p_k\}$$

for circle C with center q and radius $d(p_i, q) = d(p_j, q) = d(p_k, q)$.

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Theorem 4.20:

1. $x \in \mathbb{R}^2$ Voronoi vertex
$\Updownarrow$
Largest circle C with
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three points on its boundary
and

2. $p_i, p_j \in \mathcal{P}$ define
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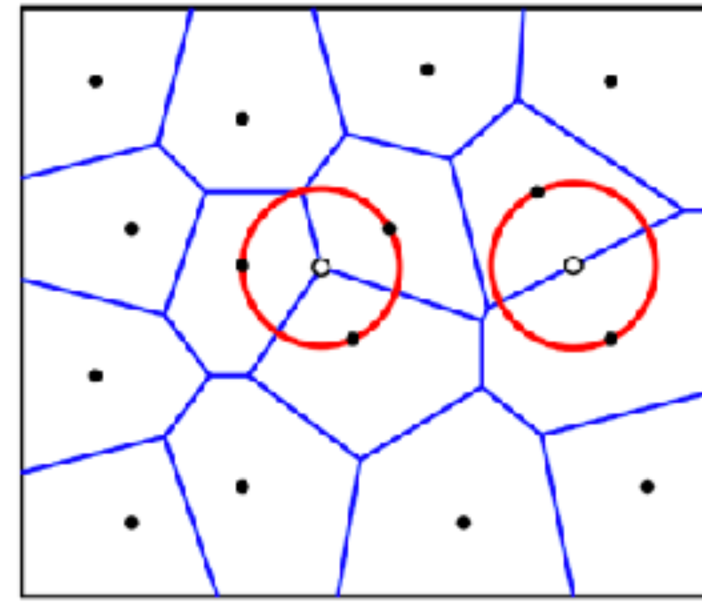

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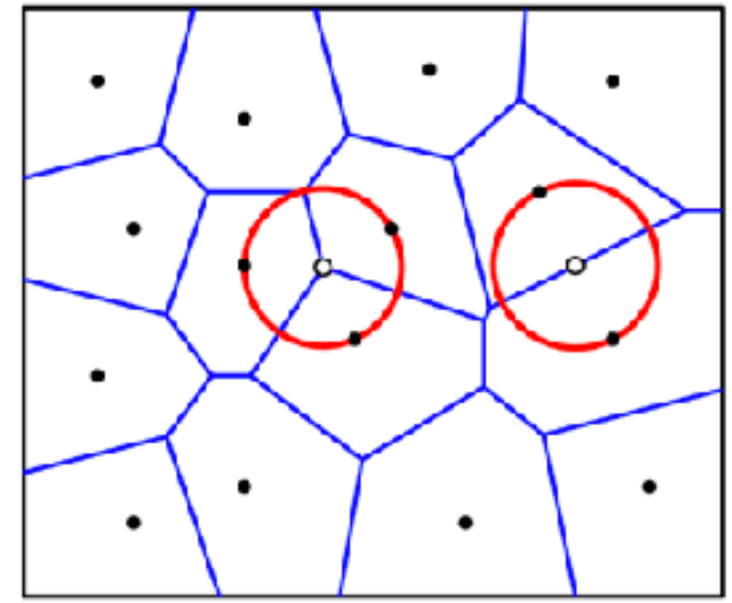

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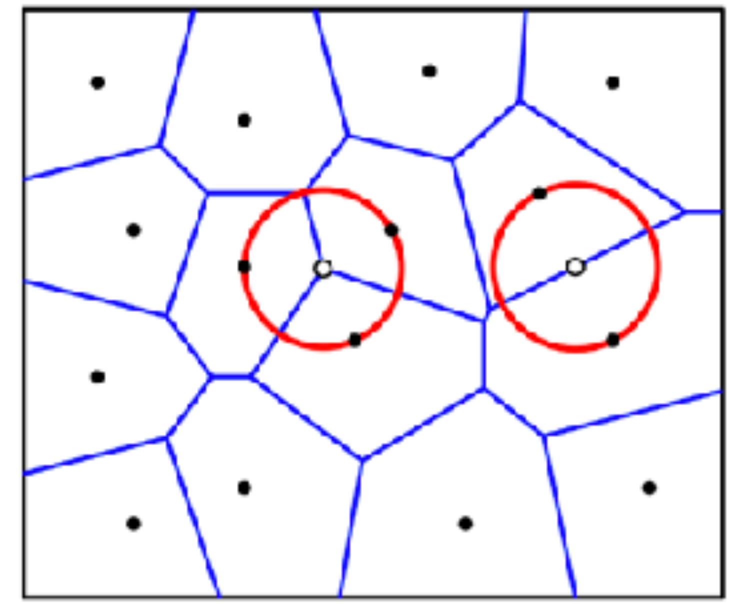

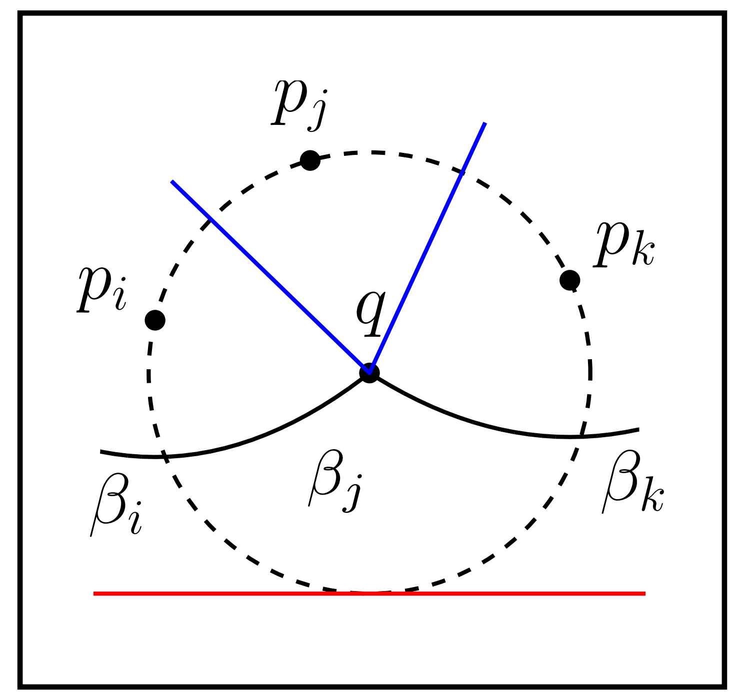

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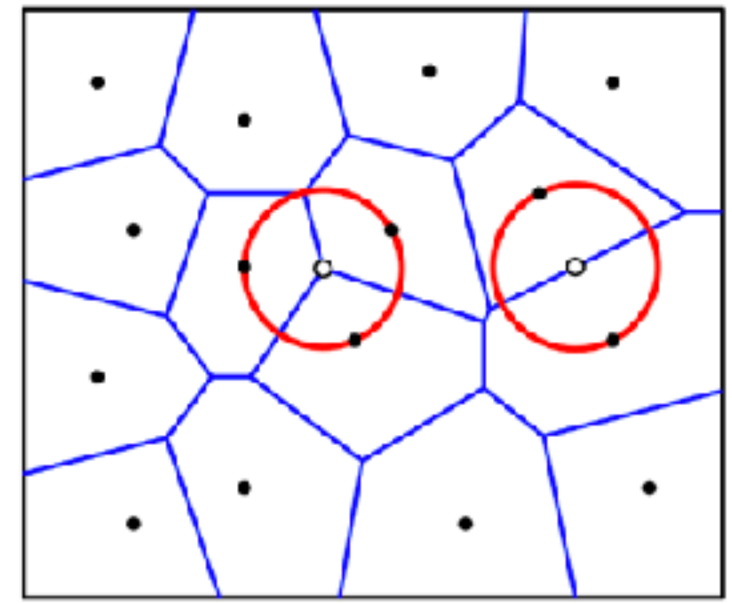

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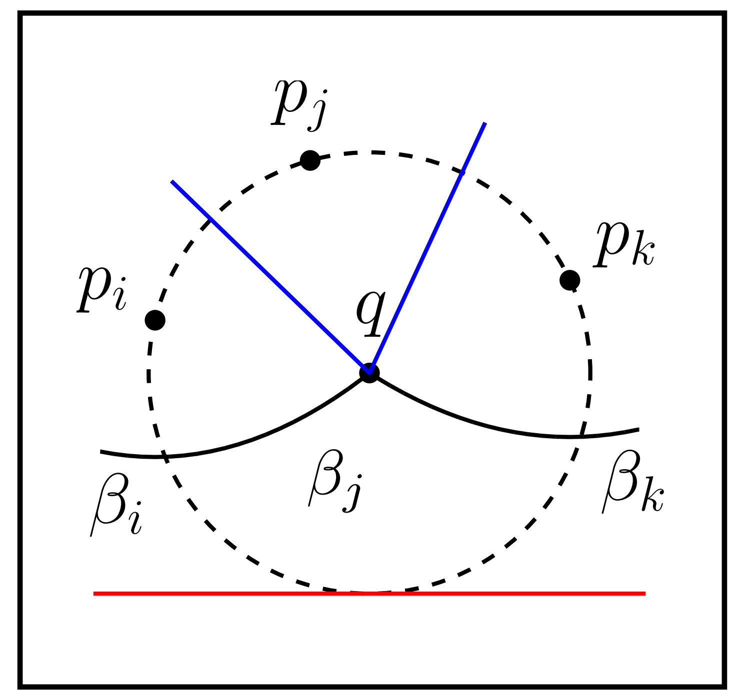

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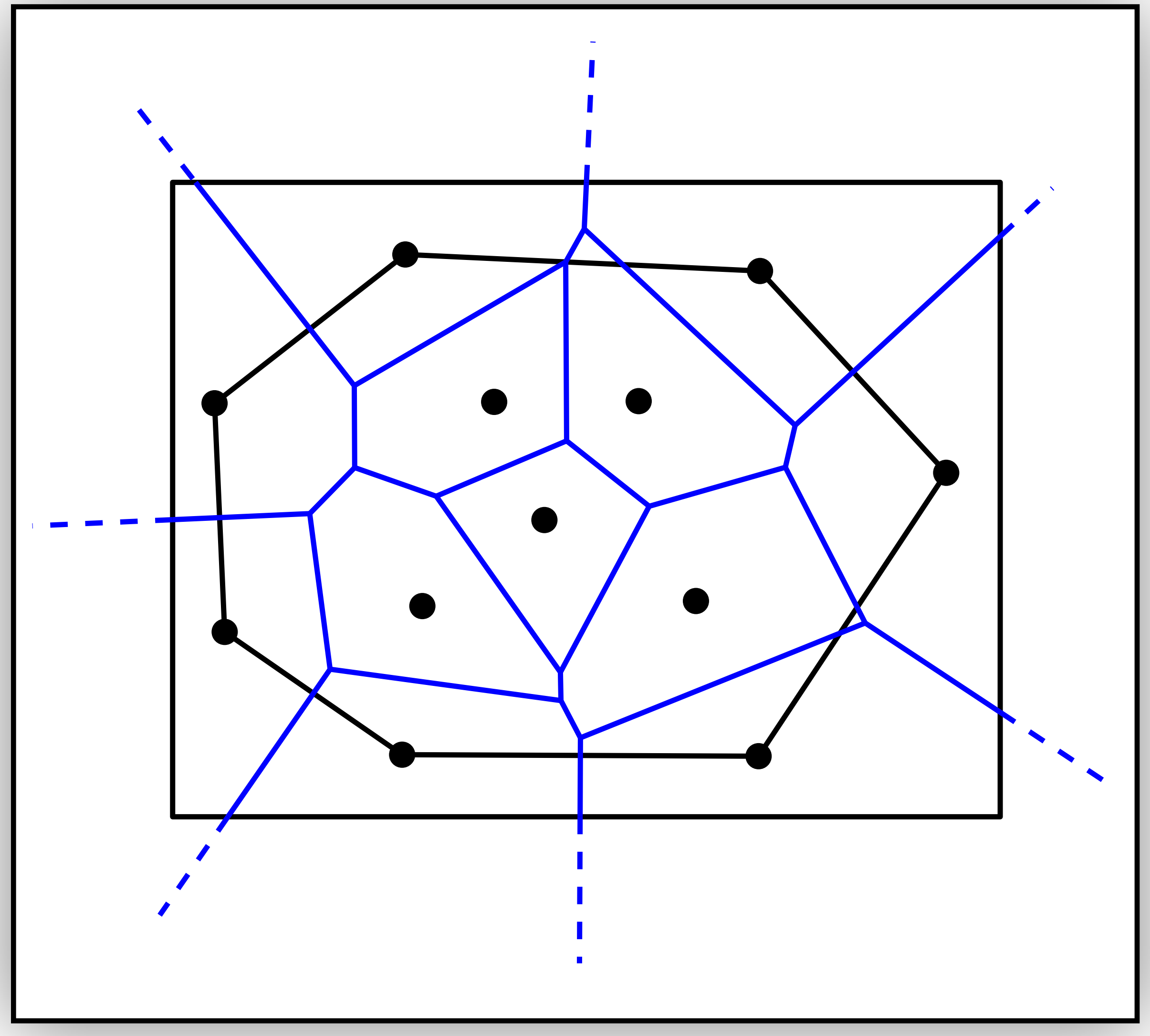

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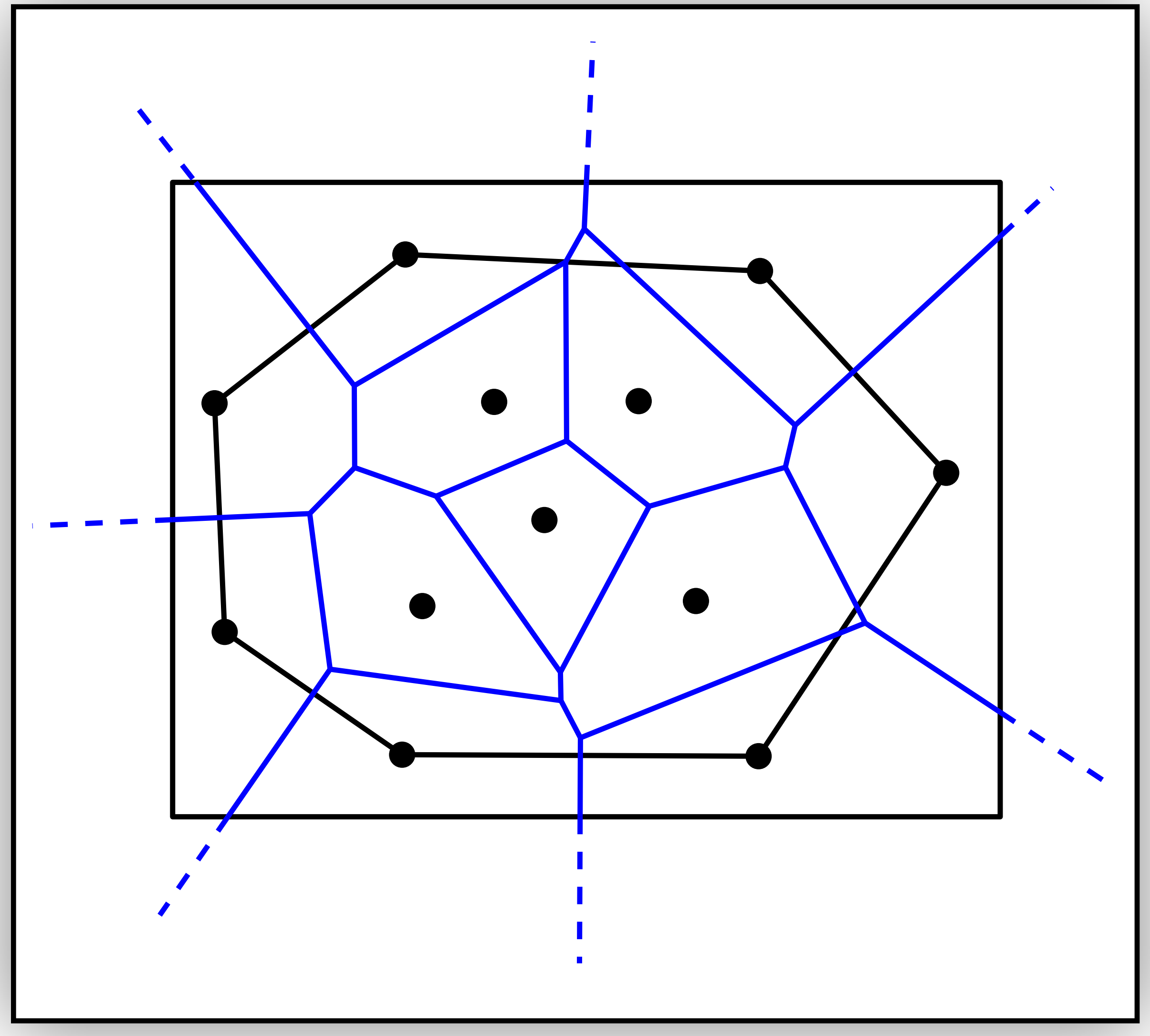

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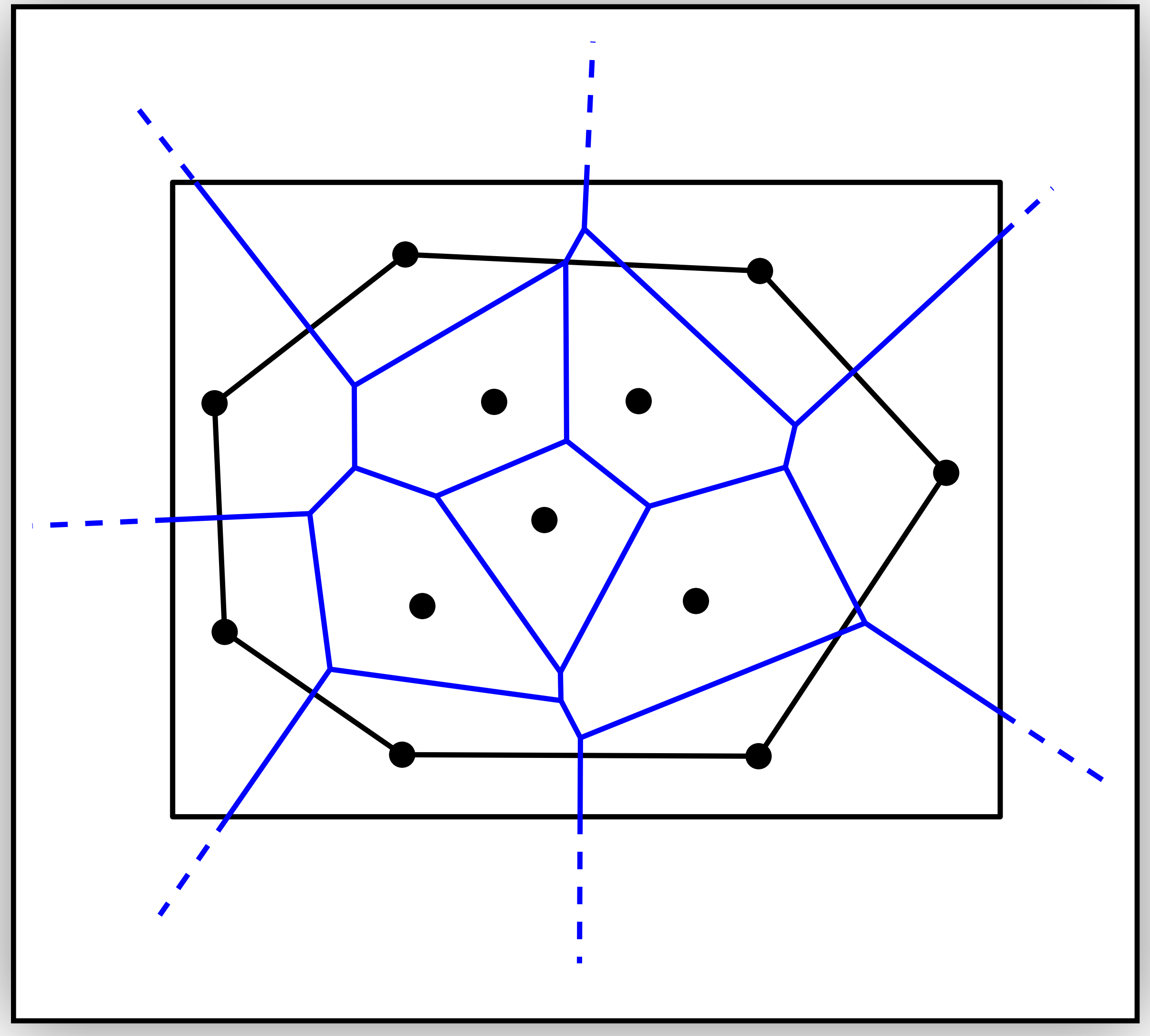

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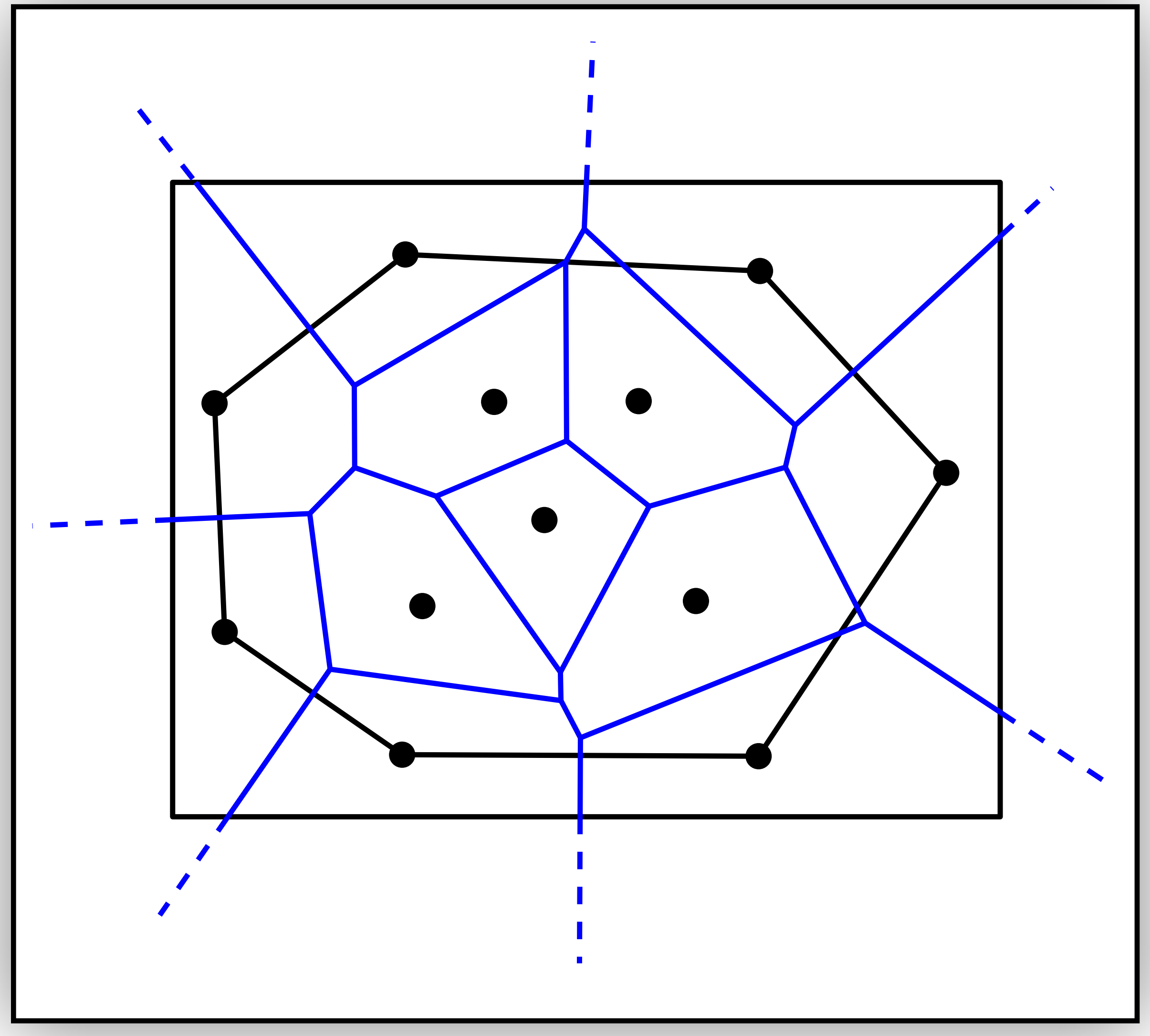
- Constructing $Vor(\mathcal{P})$: second run



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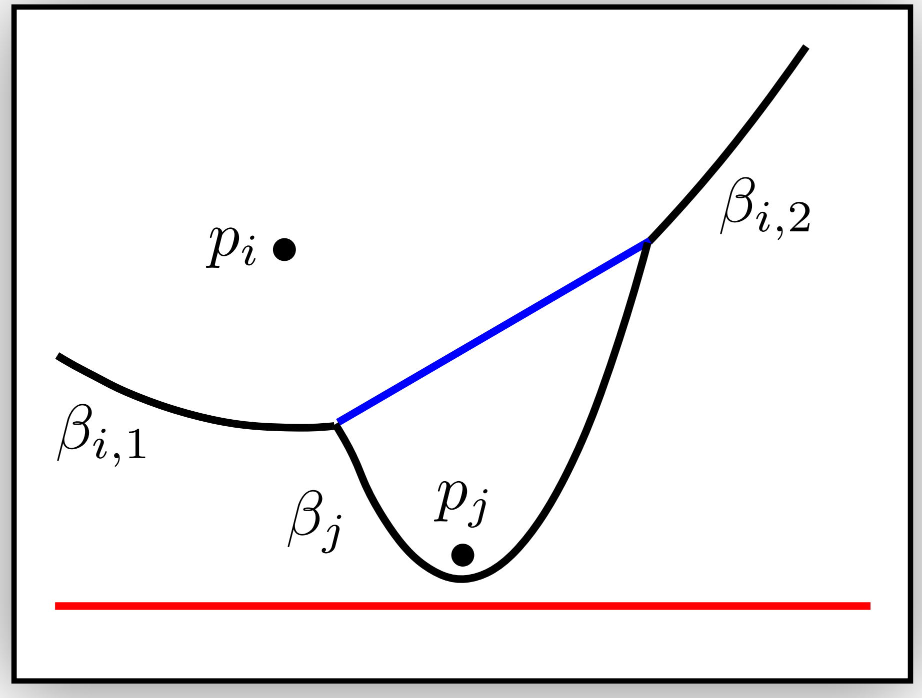

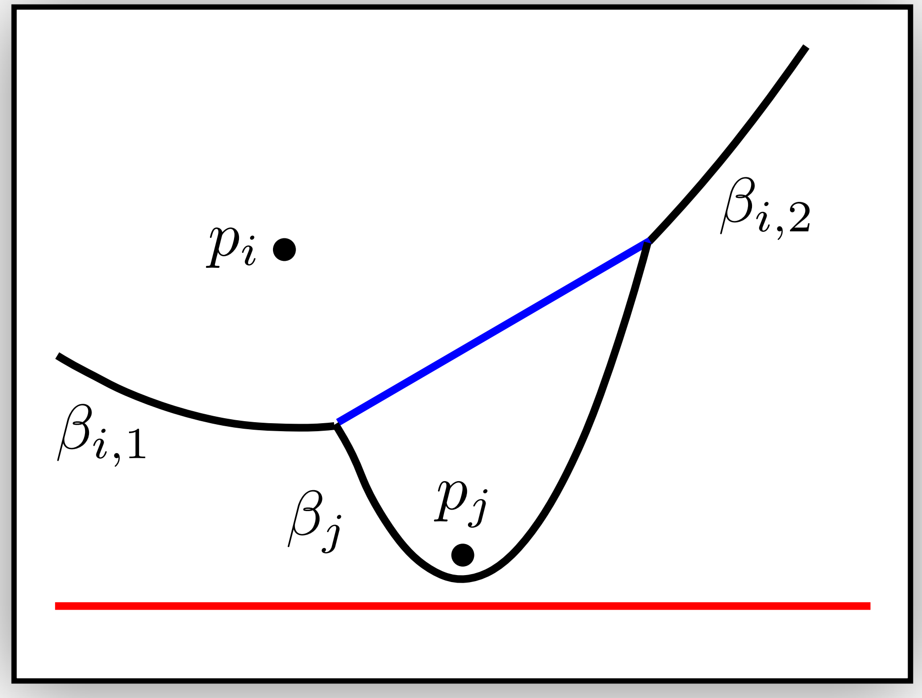

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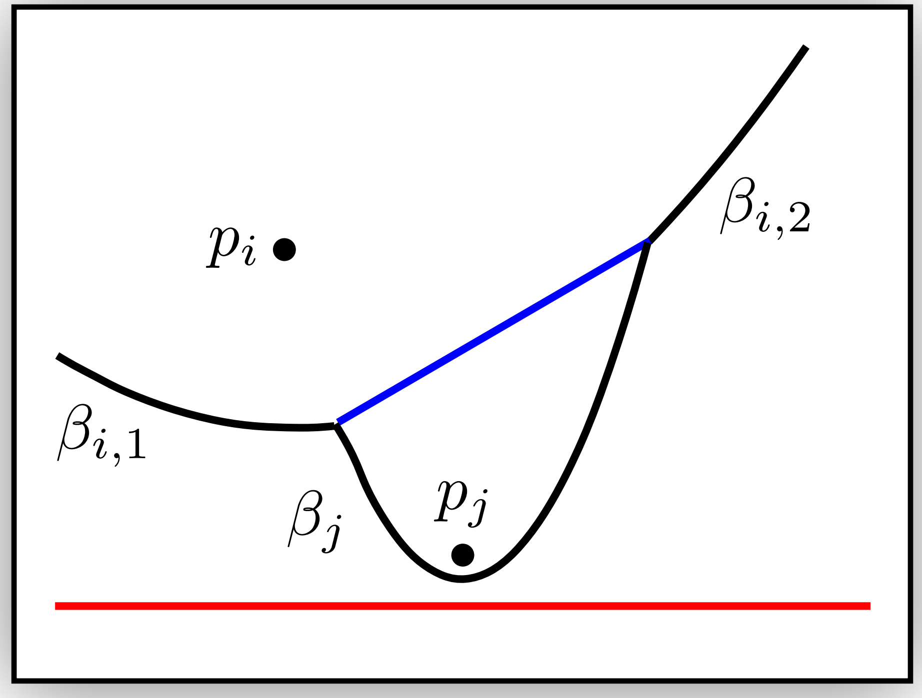

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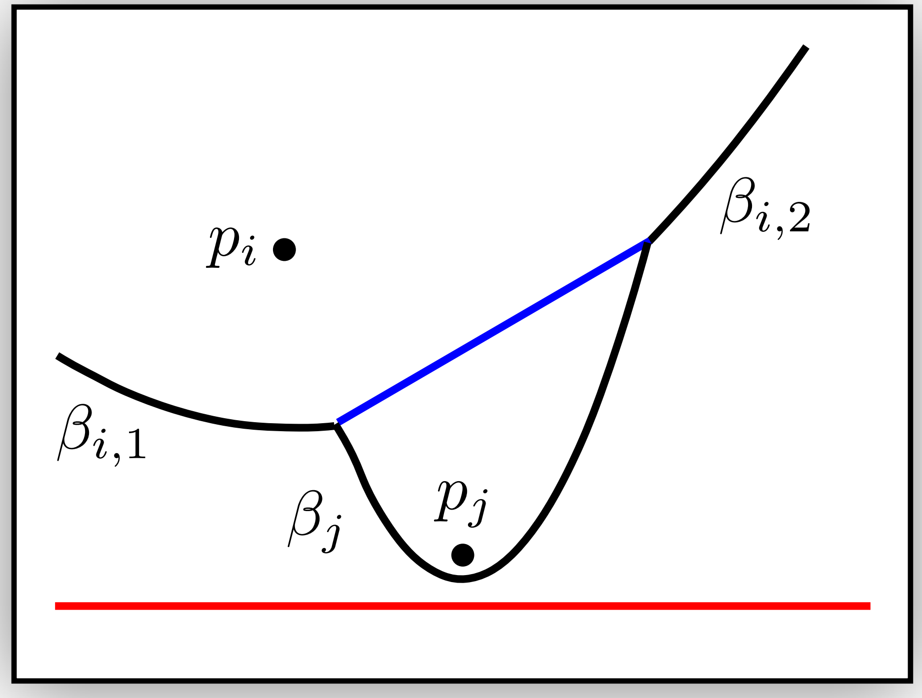

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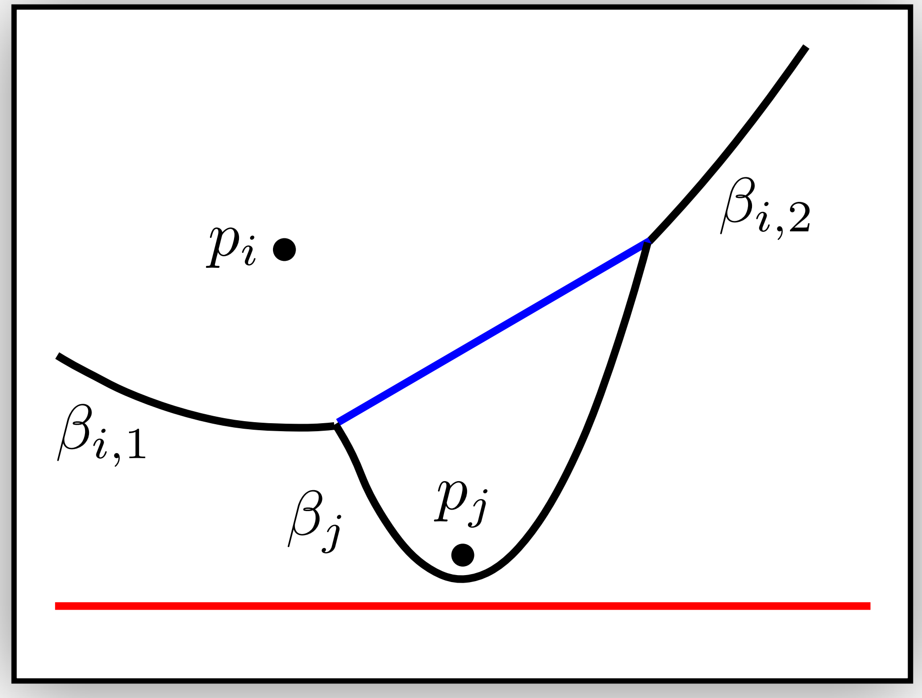

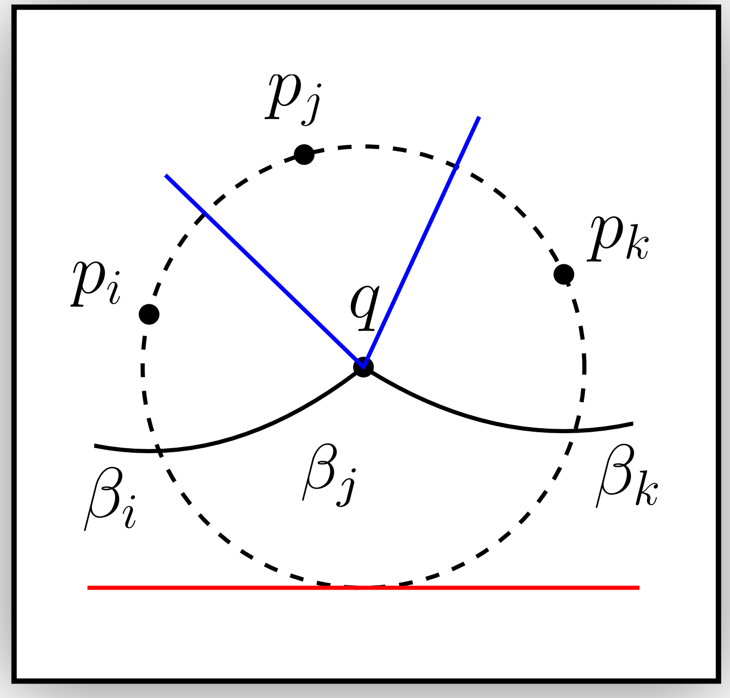

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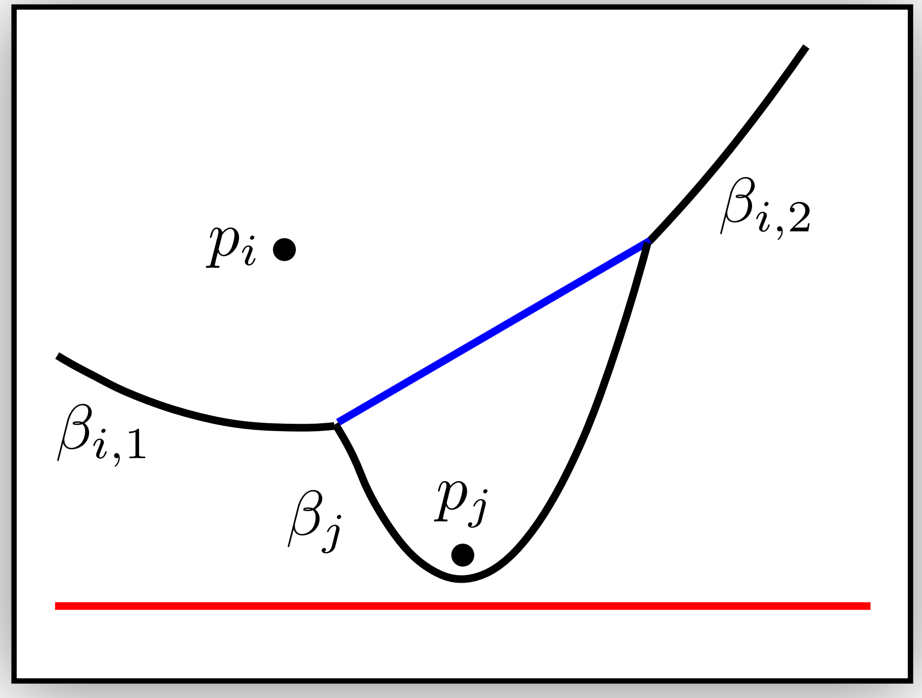
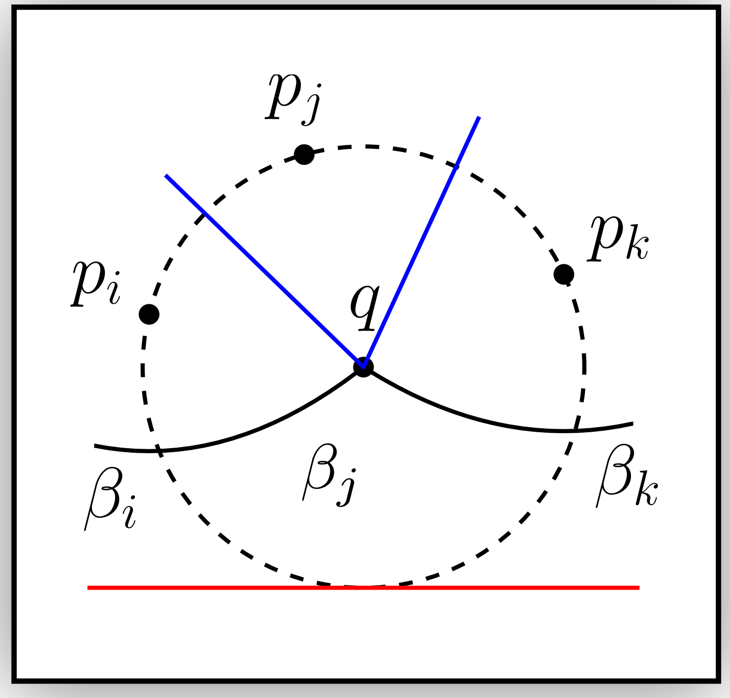
  $\rightarrow$ new edge $e_3 \subset B(p_i, p_k)$ incident to q.





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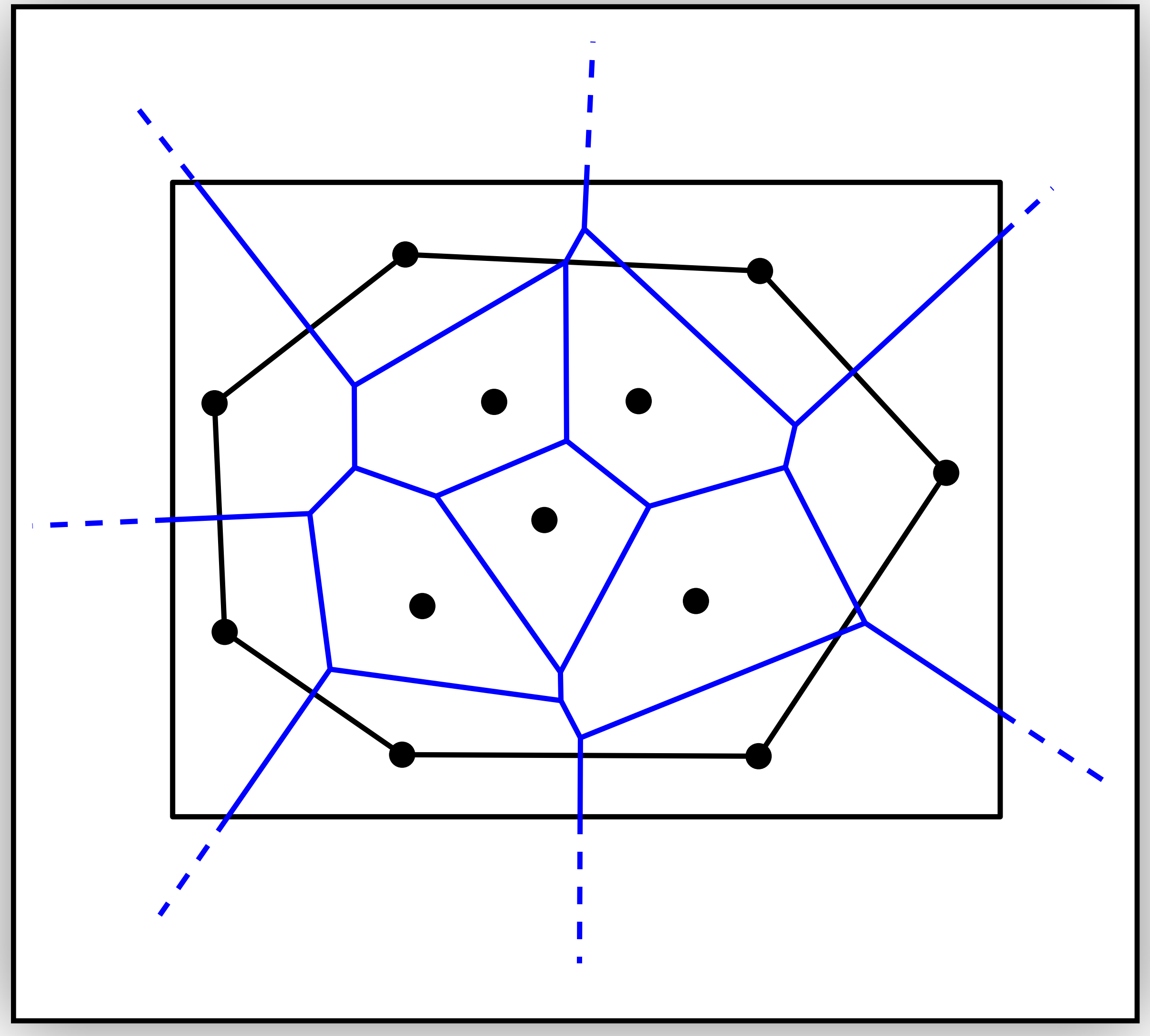

Unbounded edges:

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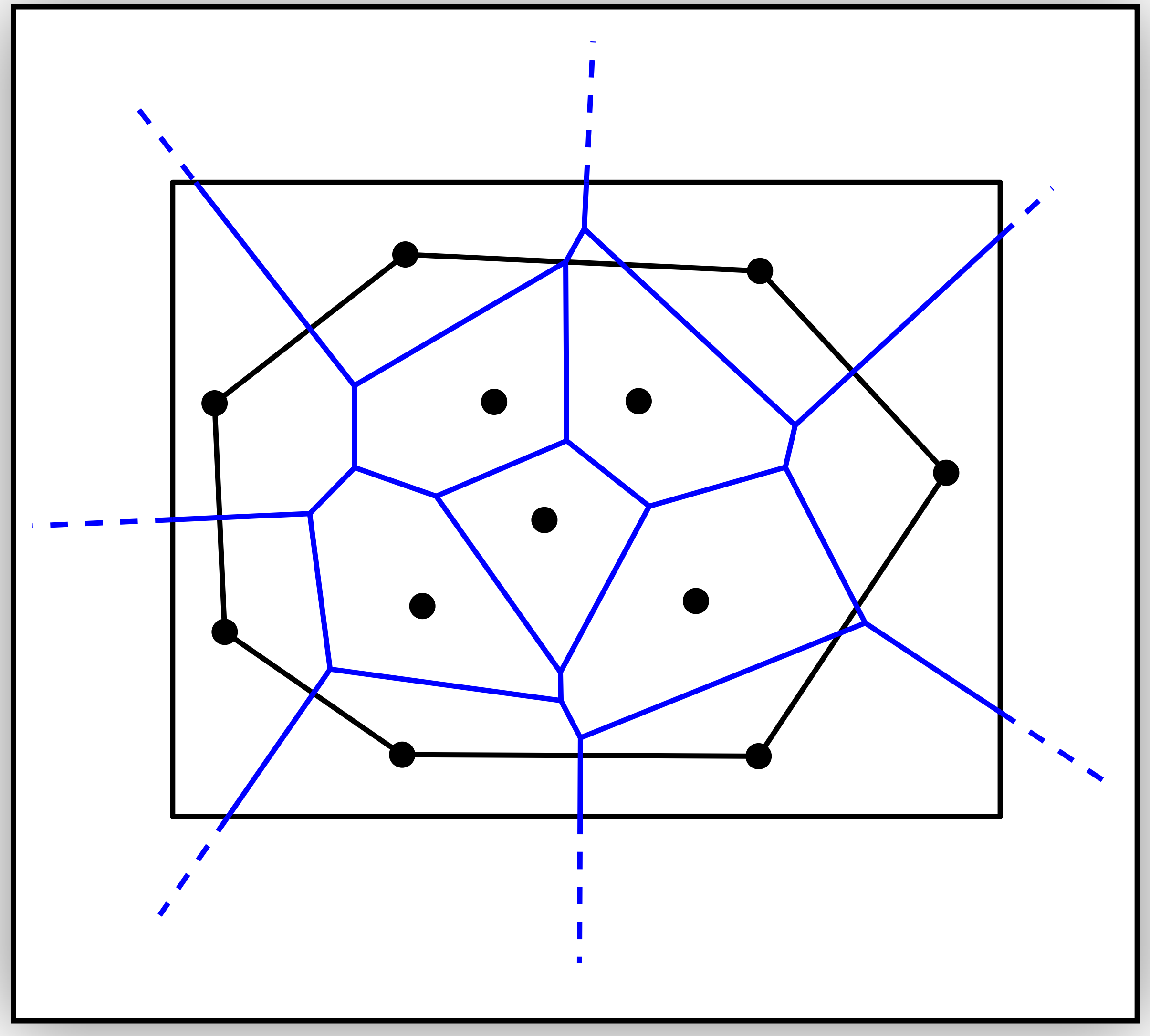

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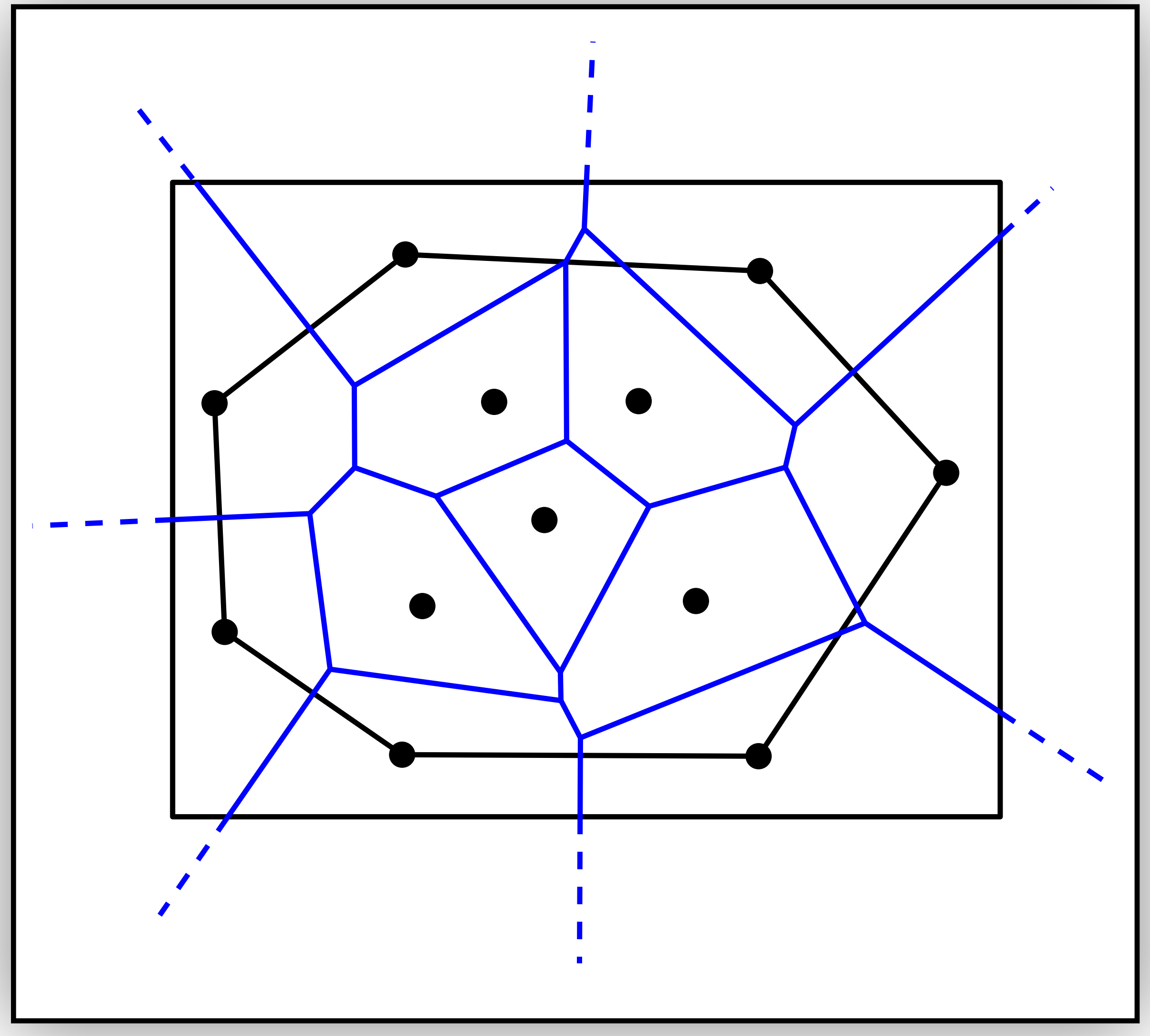

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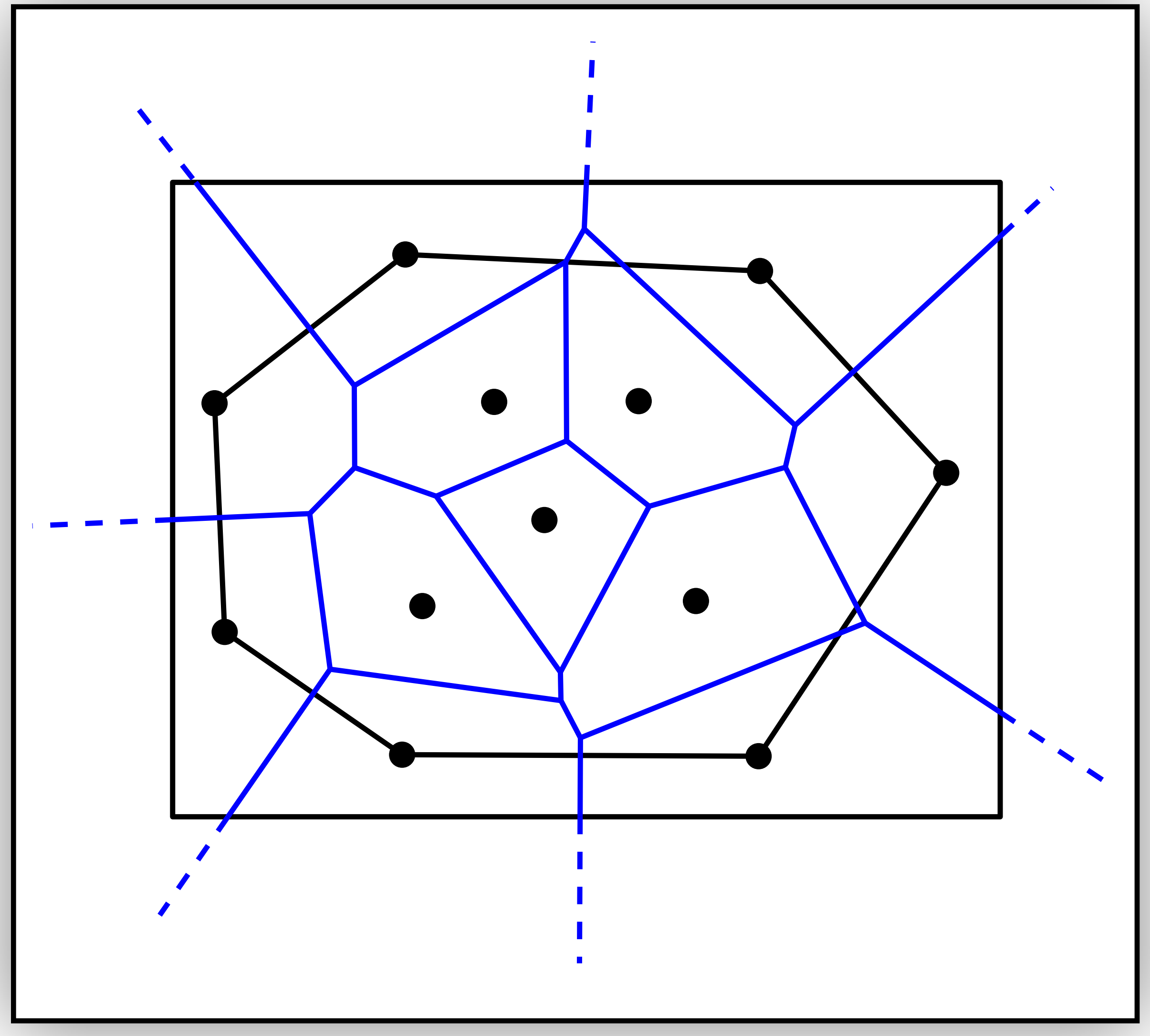
- Connect such edges with the bounding box.



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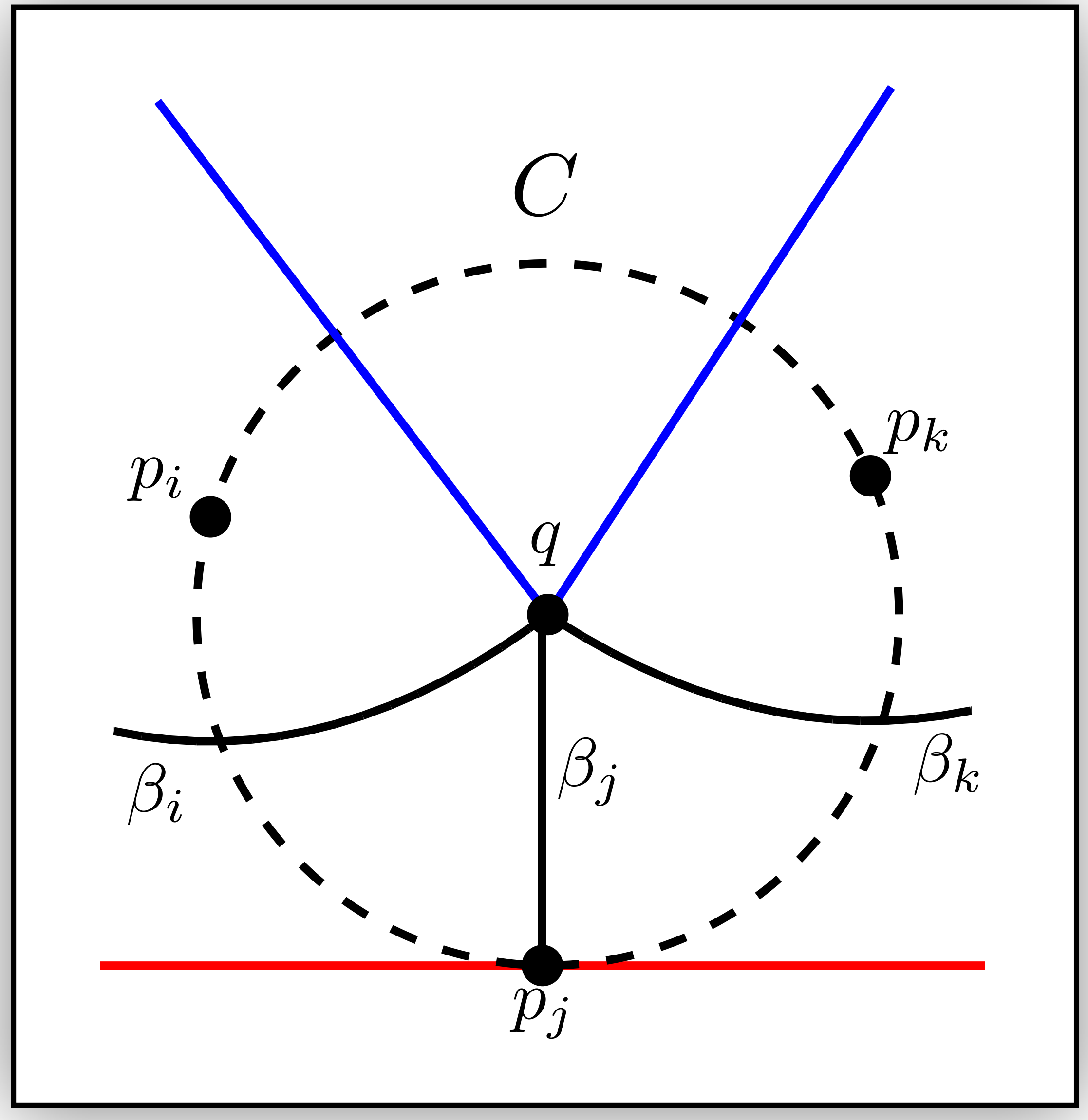

Degenerate situation:

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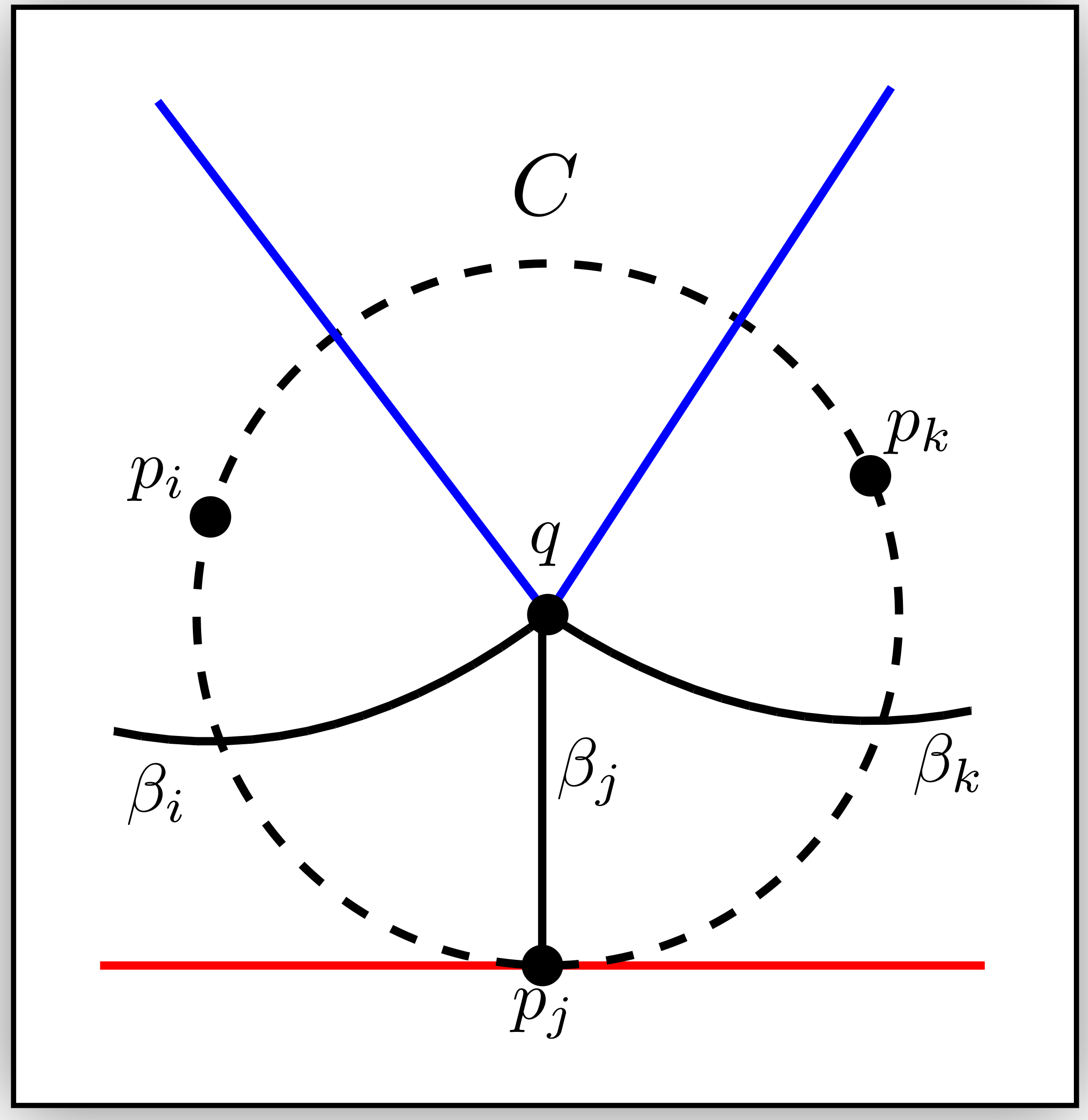

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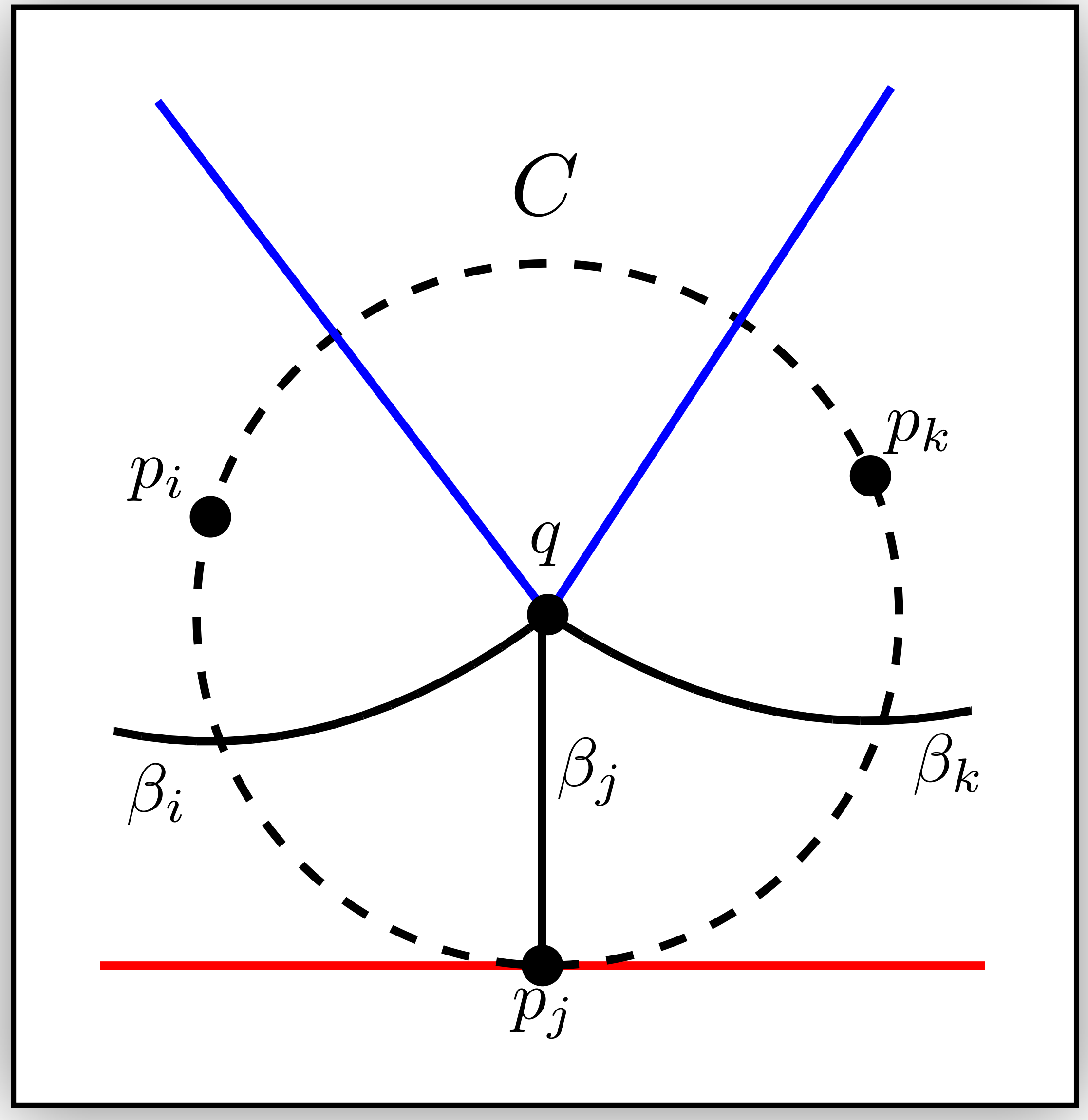

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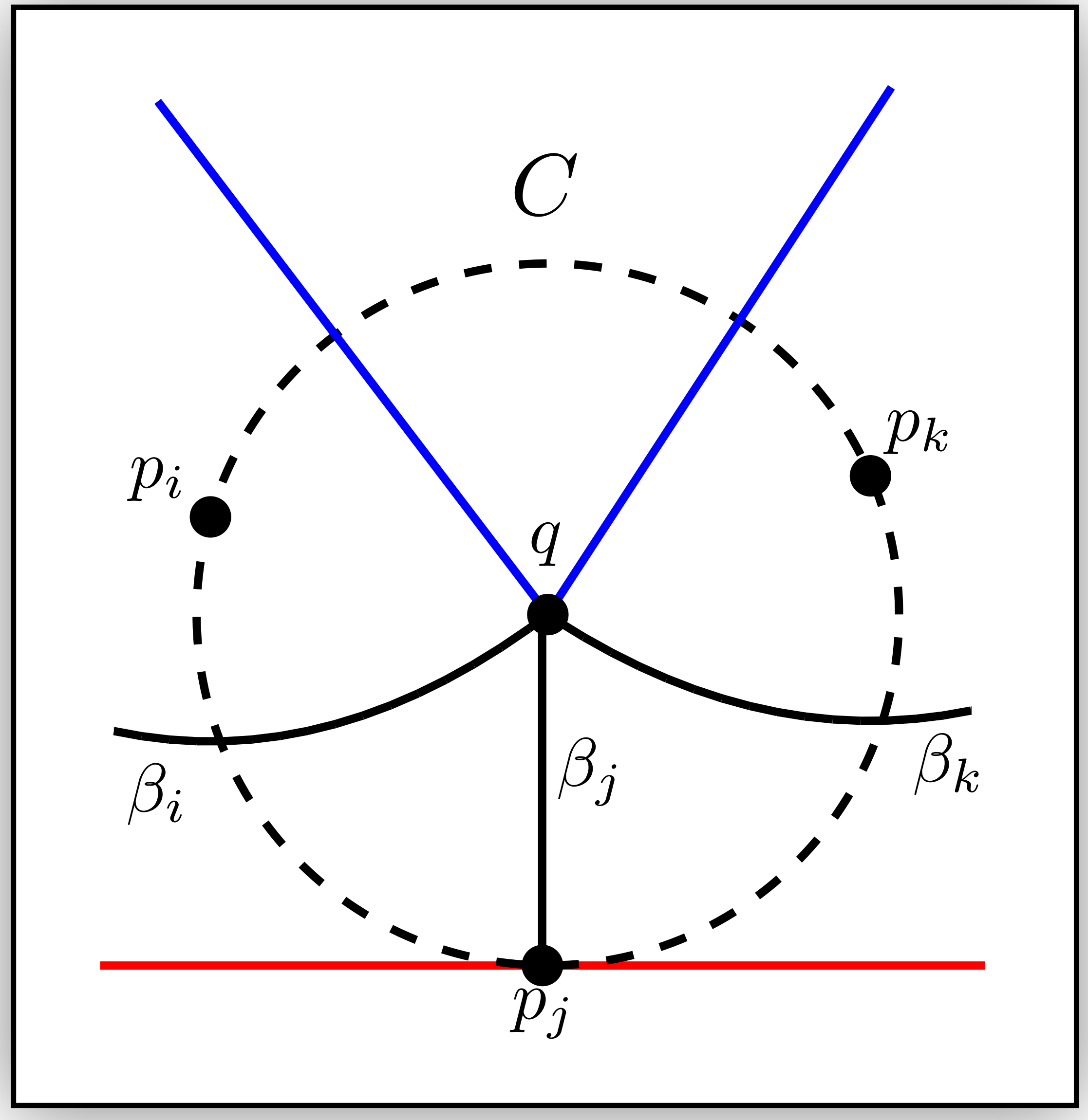

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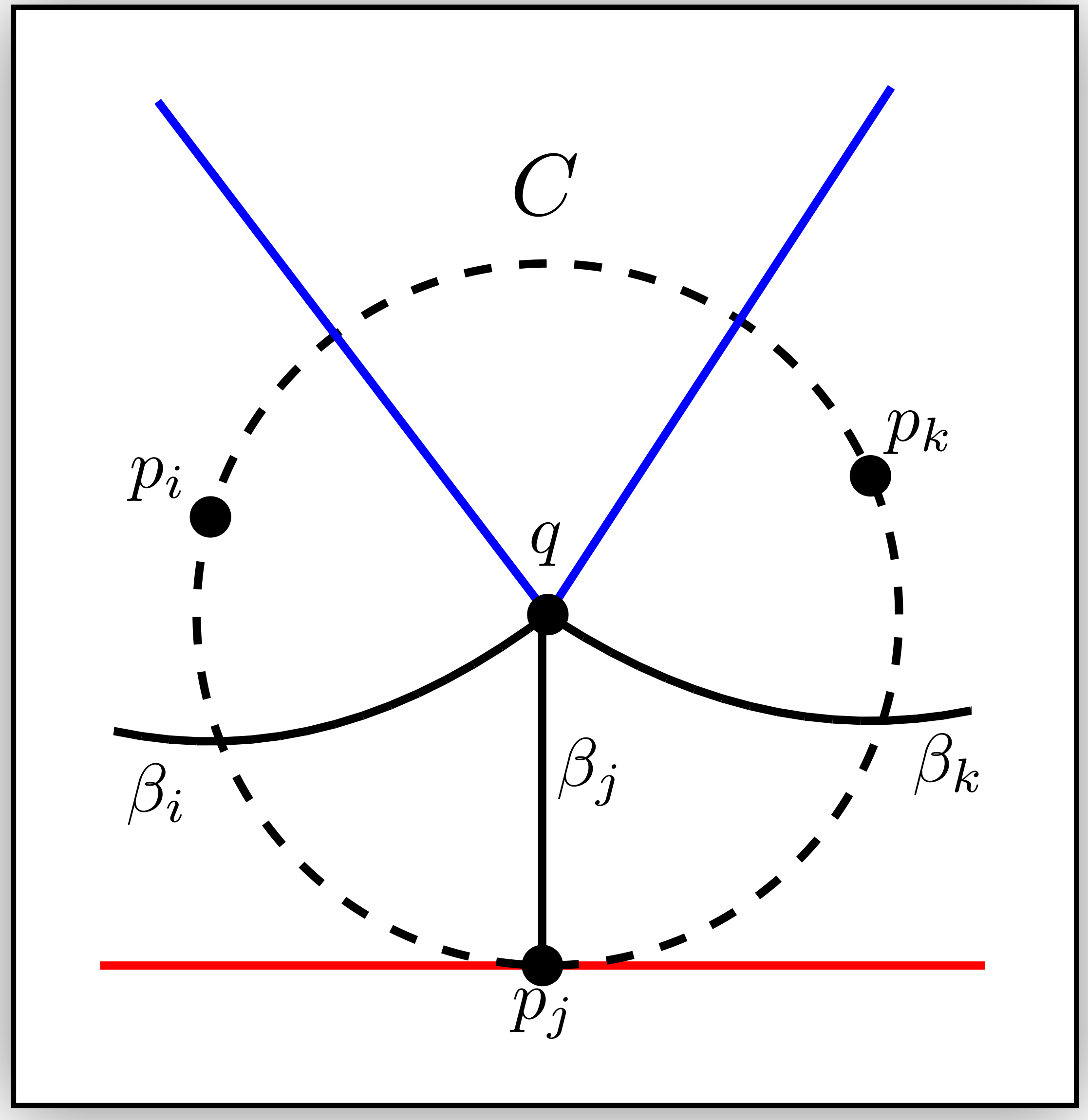
- Generate Voronoi vertex q.



Algorithm 1: Computation of $V^*(S)$.
Input: S is a set of $n \geq 1$ points with unique bottommost point.
Output: The bisectors and vertices of V^*.
Data structures: Q: a priority queue of points in the plane, ordered lexicographically. Each point is labeled as a site, or labeled as the intersection of a pair of boundaries of a single region. Q may contain duplicate instances of the same point with distinct labels; the ordering of duplicates is irrelevant.
L: a sequence $(r_1, c_1, r_2, \ldots, r_k)$ of regions (labeled by site) and boundaries (labeled by a pair of sites). Note that a region can appear many times on L.

1. initialize Q with all sites
2. $p \leftarrow$ extract_min(Q)
3. $L \leftarrow$ the list containing R_p.
4. **while** Q is not empty **begin**
5. $p \leftarrow$ extract_min(Q)
6. **case**
7. p is a site:
8. Find an occurrence of a region R_q^* on L containing p.
9. Create bisector B_{pq}^*.
10. Update list L so that it contains $\ldots, R_q^*, C_{pq}^-, R_p^*, C_{pq}^+, R_q^*, \ldots$ in place of R_q^*.
11. Delete from Q the intersection between the left and right boundary of R_q^*, if any.
12. Insert into Q the intersection between C_{pq}^- and its neighbor to the left on L, if any, and the intersection between C_{pq}^+ and its neighbor to the right, if any.
13. p is an intersection:
14. Let p be the intersection of boundaries C_{qr} and C_{rs}.
15. Create the bisector B_{qs}^*.
16. Update list L so it contains $C_{qs} = C_{qs}^-$ or C_{qs}^+, as appropriate, instead of C_{qr}, R_r^*, C_{rs}.
17. Delete from Q any intersection between C_{qr} and its neighbor to the left and between C_{rs} and its neighbor to the right.
18. Insert any intersections between C_{qs} and its neighbors to the left or right into Q.
19. Mark p as a vertex and as an endpoint of B_{qr}^*, B_{rs}^*, and B_{qs}^*.
20. **end**

Fig. 2.4. Algorithm 1: computation of $V^*(S)$.

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Fortune's algorithm computes the Voronoi diagram of n points in time $\Theta(n \log n)$.

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THEOREM 2.8. *Algorithm 1 can be implemented to run in time $O(n \log n)$ and space $O(n)$.*

Steve's Questions

Steven Fortune 4. Dezember 2020 um 14:48

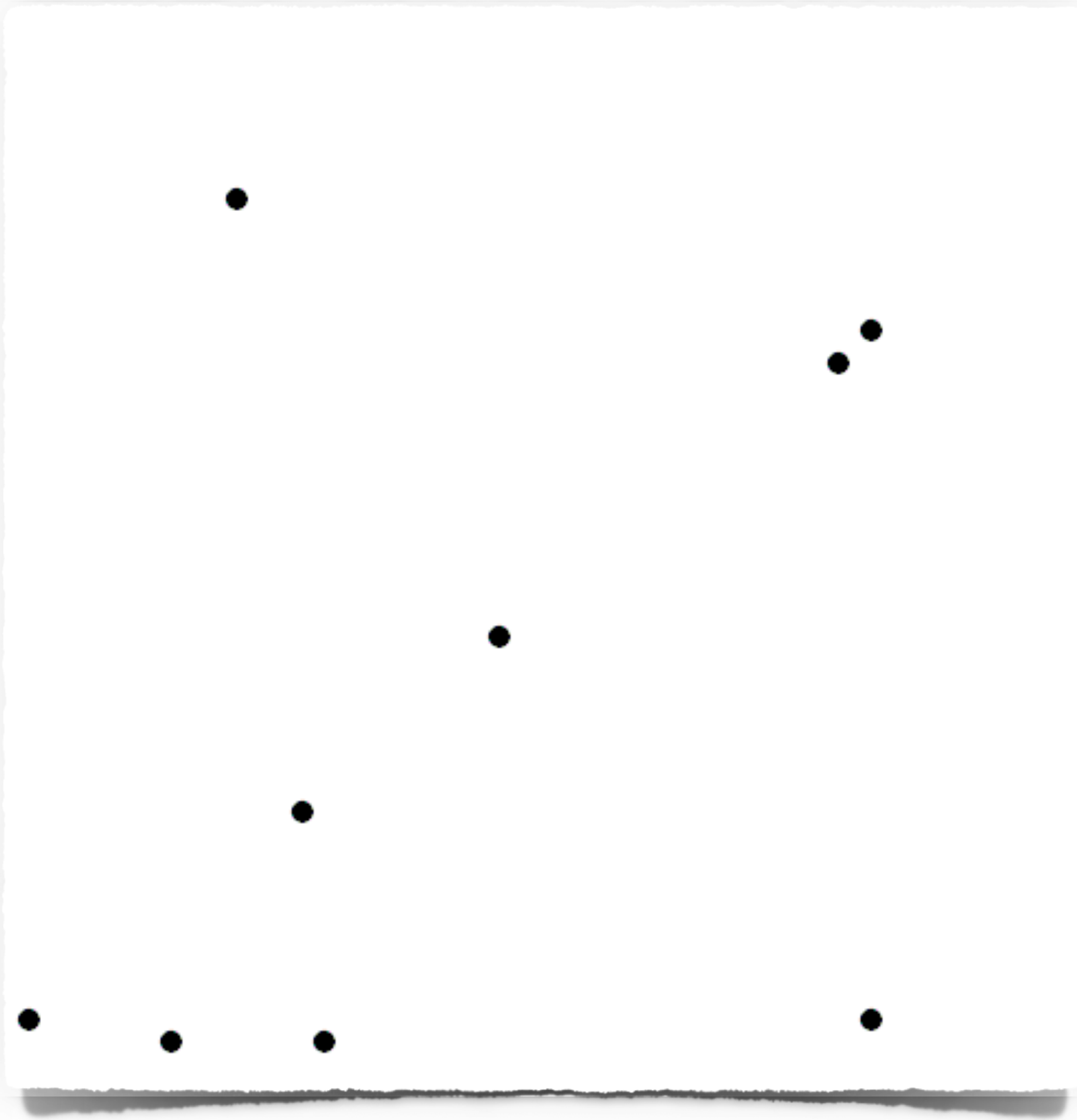
Aw: Computational Geometry - video message?

An: Sándor Fekete,

Umgeleitet von: fekete@tu-braunschweig.de

The ways I know:

1. original paper, transforming the plane
2. beach line (maintaining the “known” part of the VD)
3. sweep line maintains the top of circles
4. in 3d, sweep a plane at 45 degrees to xy, watch it intersect 45 degree cones centered at siteas



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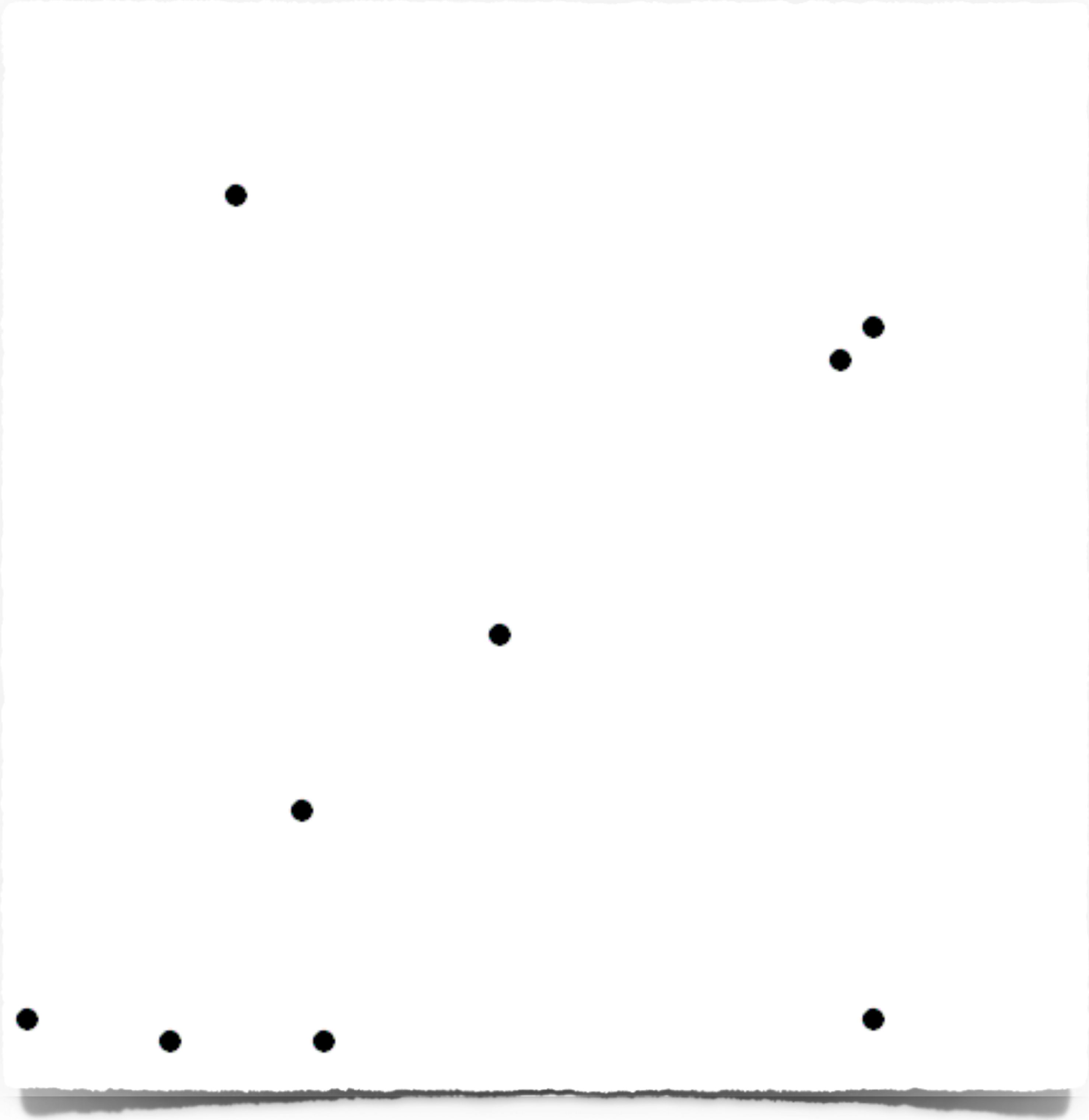

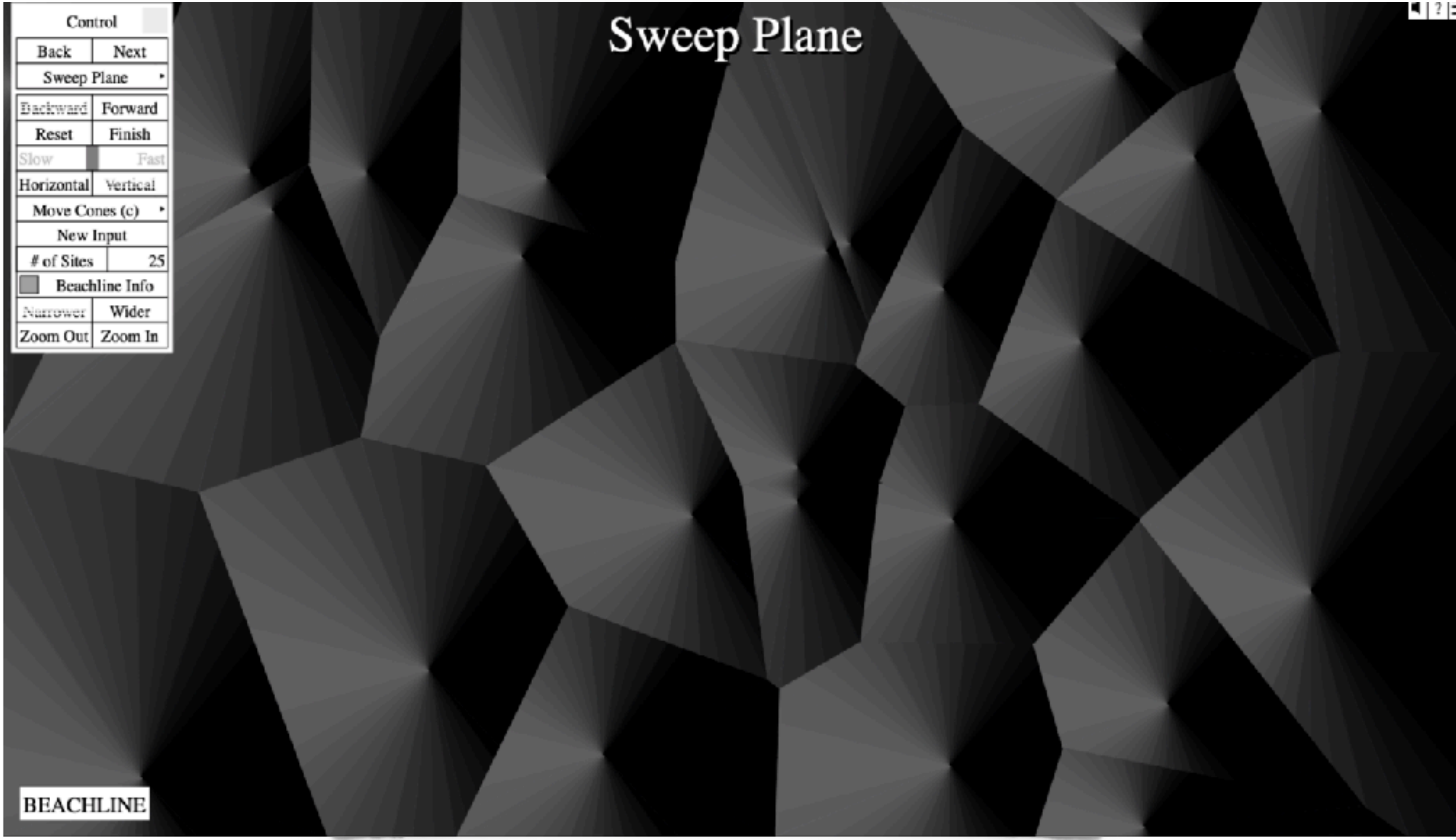

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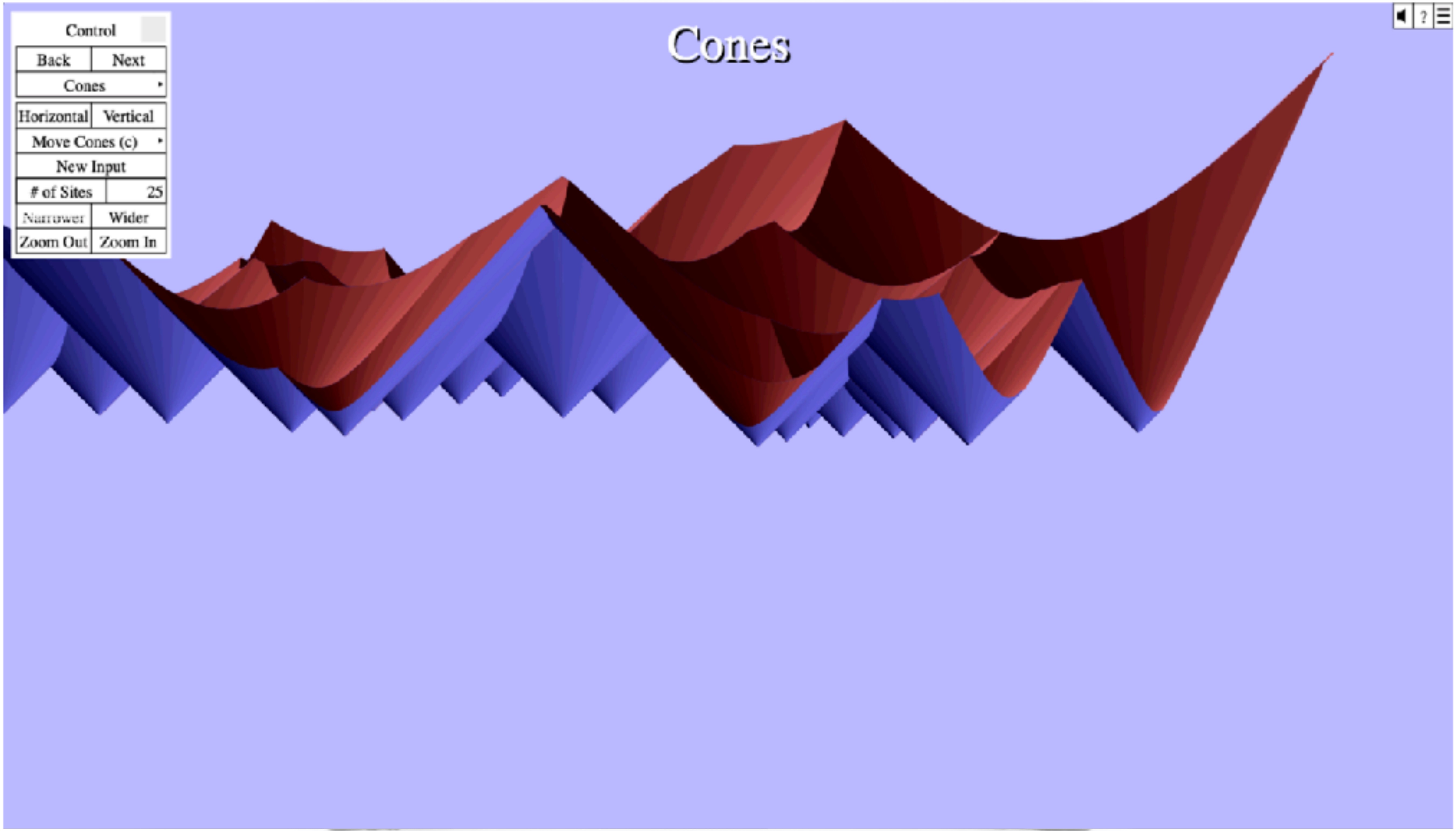
Steve's Questions



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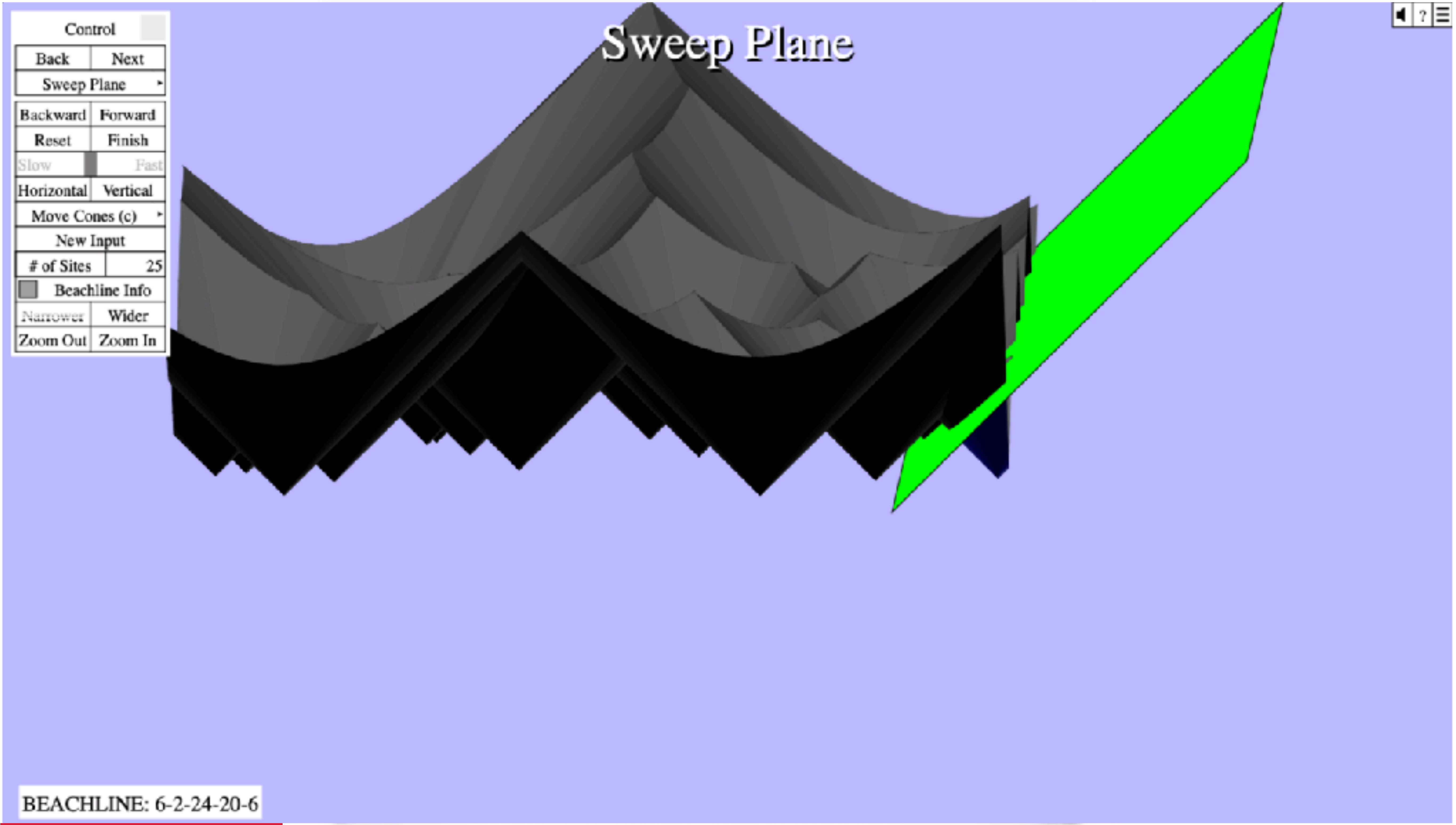
Sweep Plane
Control
Back
Next
Sweep Plane
Backward
Forward
Reset
Finish
Slow
Fast
Horizontal
Vertical
Move Cones (c)
New Input
# of Sites
25
Beachline Info
Narrower
Wider
Zoom Out
Zoom In
BEACHLINE: 6-2-24-20-6


Control
Back
Next
Sweep Plane
Halt (h)
Reset
Finish
Slow
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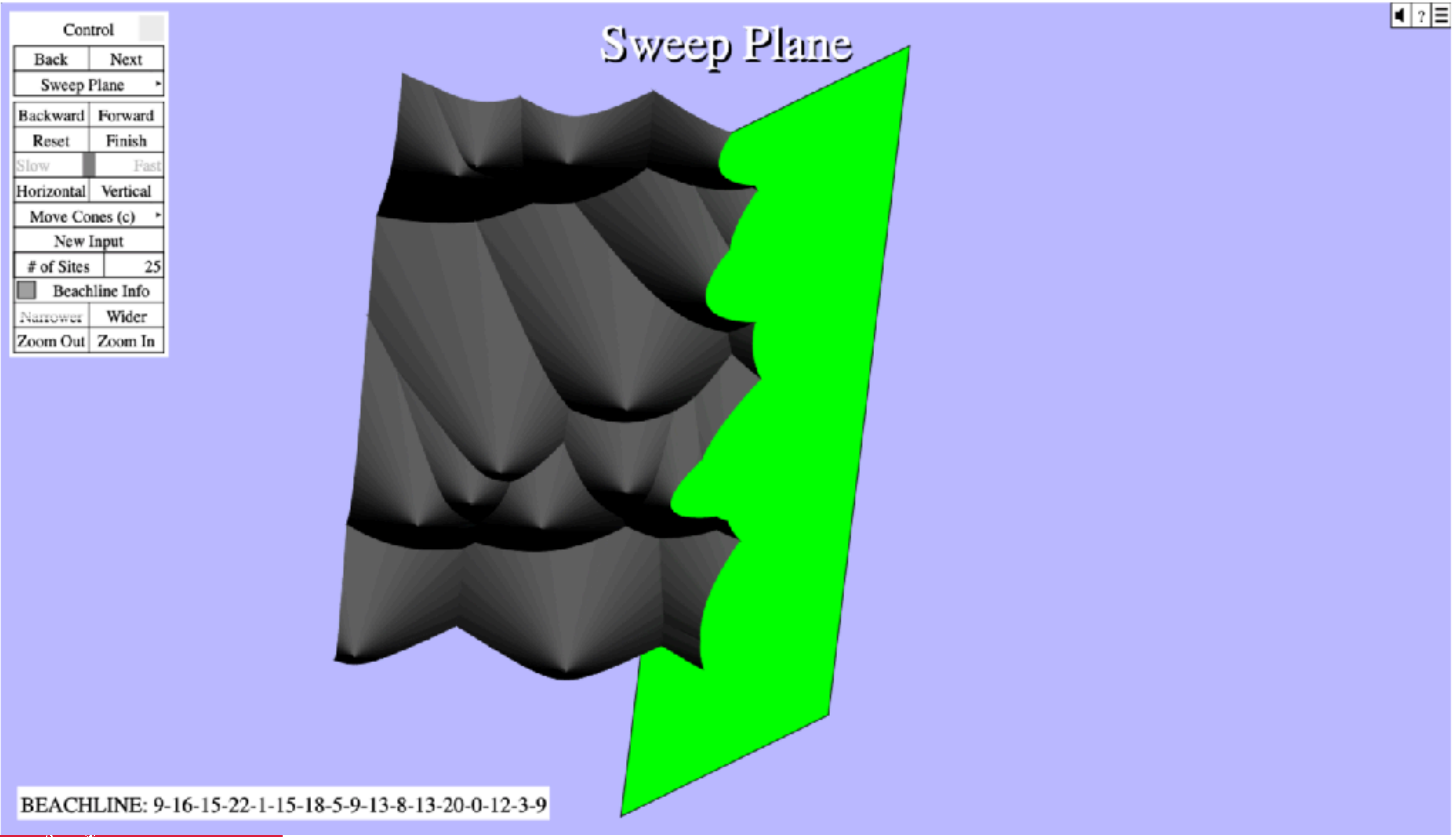
Sweep Plane
BEACHLINE: 6-2-13-5-9-7-8-7-21-12-6


Sweep Plane
Control
Back
Next
Sweep Plane
Backward
Forward
Reset
Finish
Slow
Fast
Horizontal
Vertical
Move Cones (c)
New Input
# of Sites
25
Beachline Info
Narrower
Wider
Zoom Out
Zoom In
BEACHLINE: 9-16-15-22-1-15-18-5-9-13-8-13-20-0-12-3-9

Technische Universität Braunschweig

Thank you for today!