# Computational Geometry
# Chapter 4: Voronoi Diagrams

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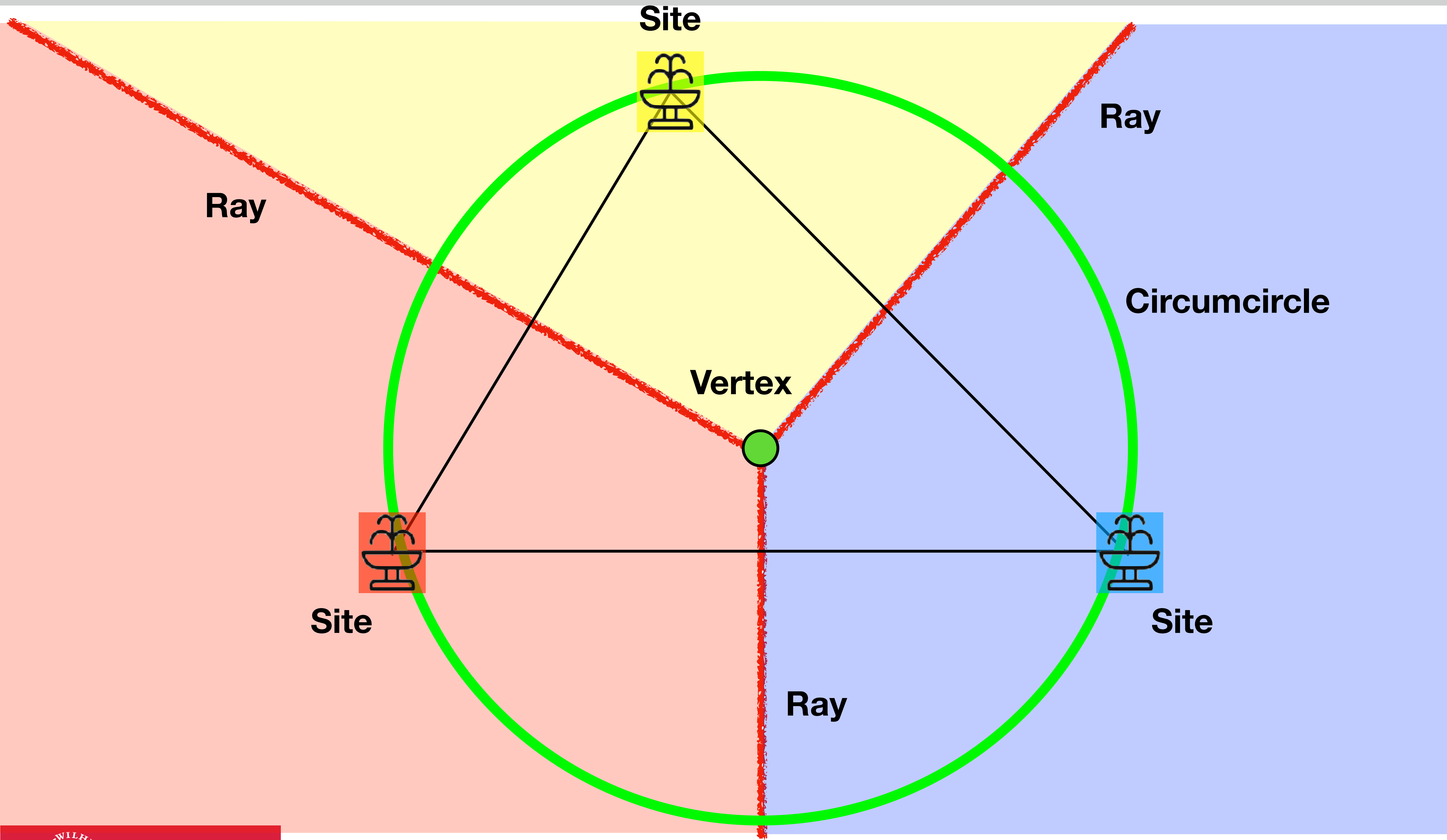


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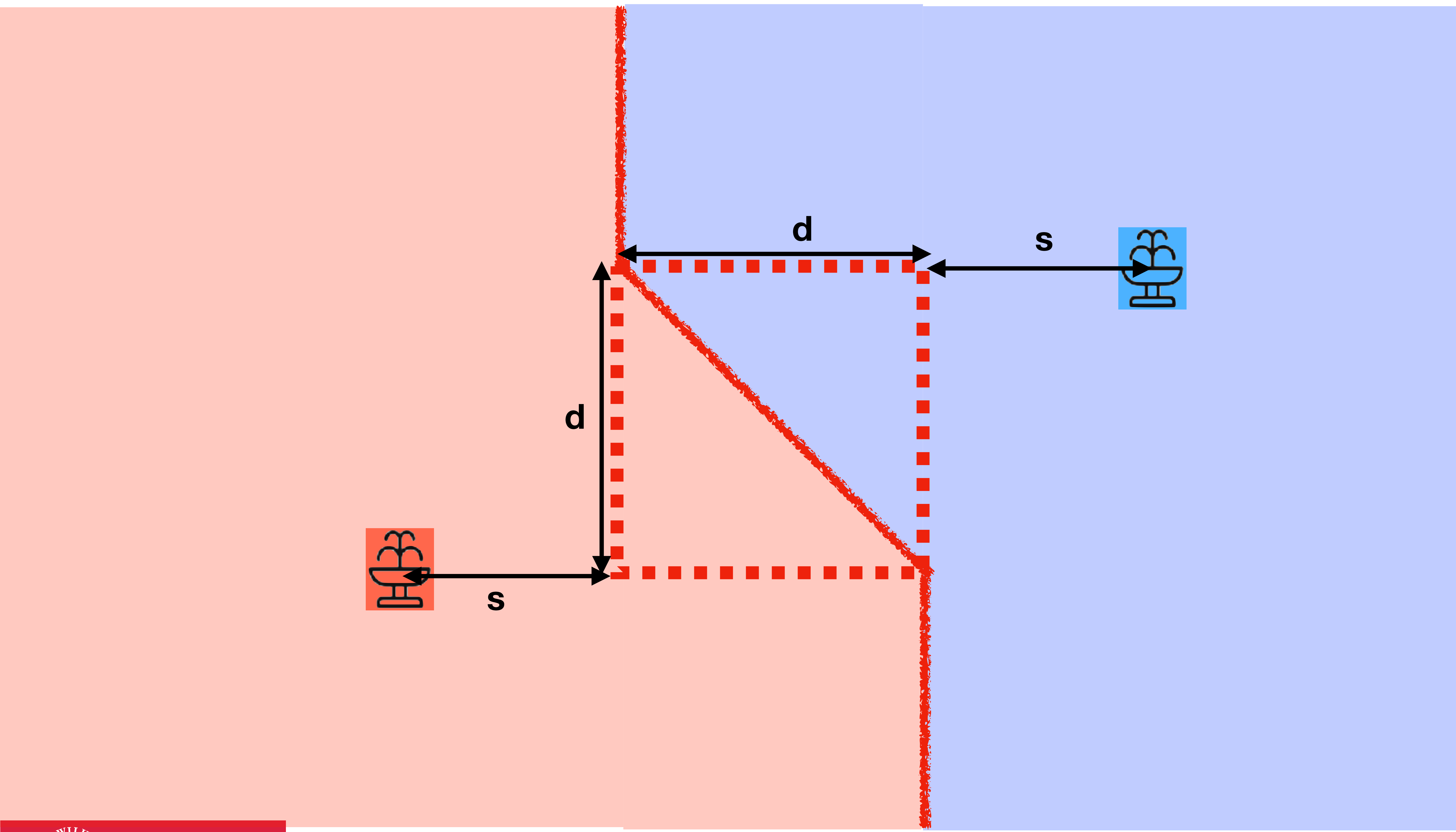


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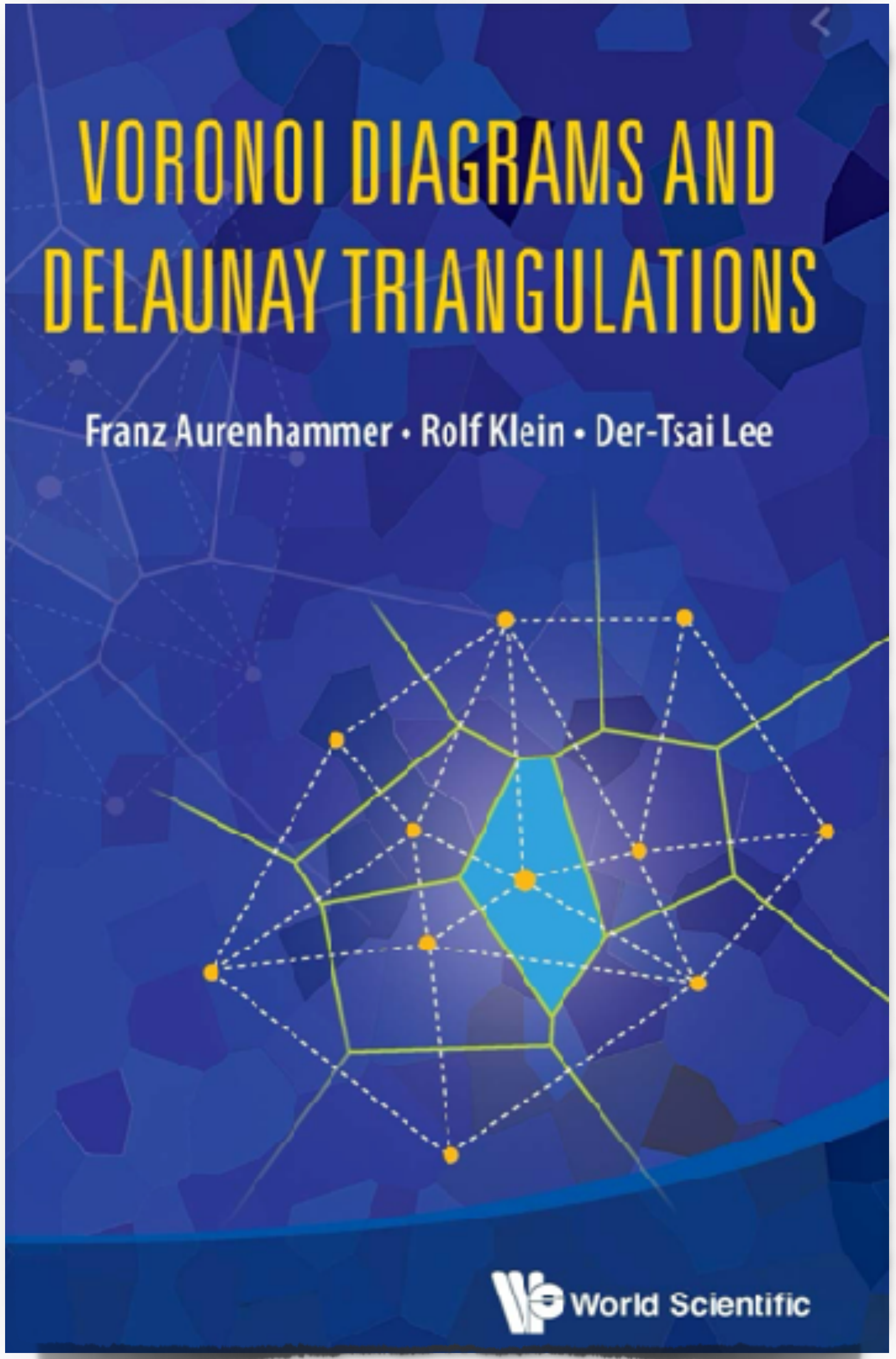
Balu Ertl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**In the following:** Distances and visualization in **Euclidean metric,** other metrics possible.

**Definition 4.1**

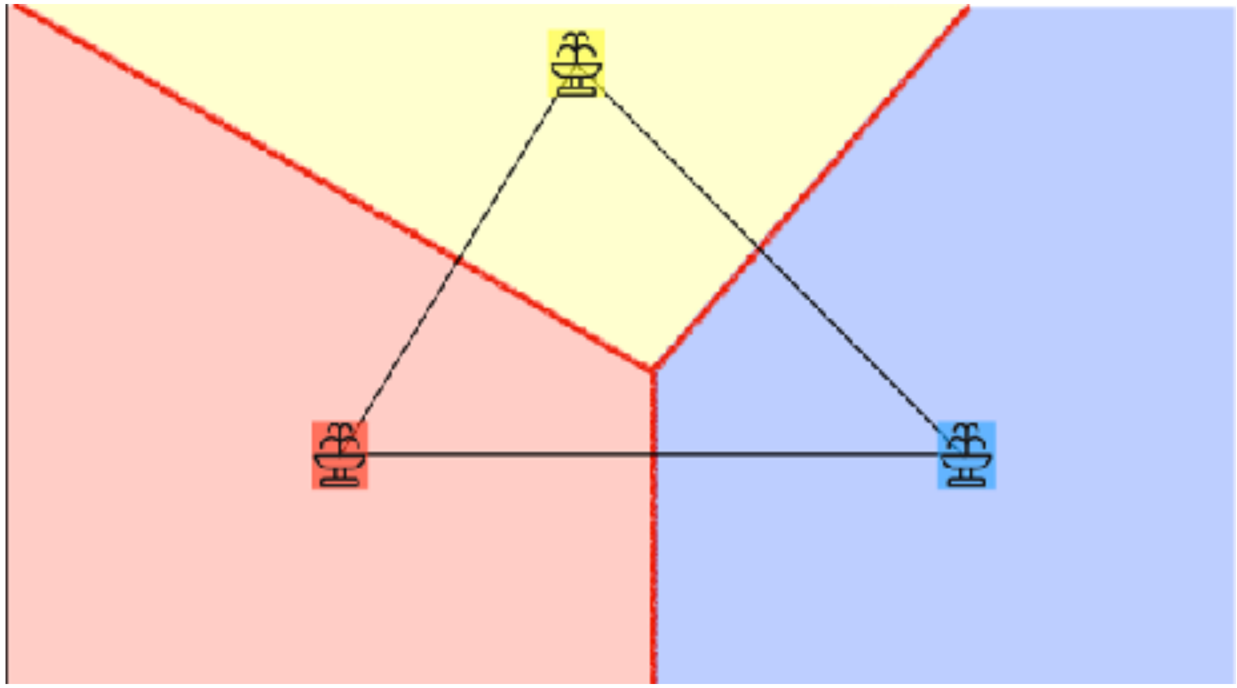
*Voronoi region* $V(p)$ of $p \in \mathcal{P}$:

$$V(p) := \{x \in \mathbb{R}^2 \mid \forall q \in \mathcal{P} : d(x,p) \leq d(x,q)\}$$

**Problem 4.2**

**Given:** Finite set of points $\mathcal{P}$ in $\mathbb{R}^2$

**Wanted**: For any $p \in \mathcal{P}$ find its Voronoi region



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**Definition 4.3**

For $p \neq q \in \mathcal{P}$ the halfspace of $p$ is $H(p,q) = \{x \in \mathbb{R}^2 \mid d(x,p) \leq d(x,q)\}$

For $p \neq q \in \mathcal{P}$ the bisector $B(p,q)$ with $p \in H(p,q), q \in H(q,p)$

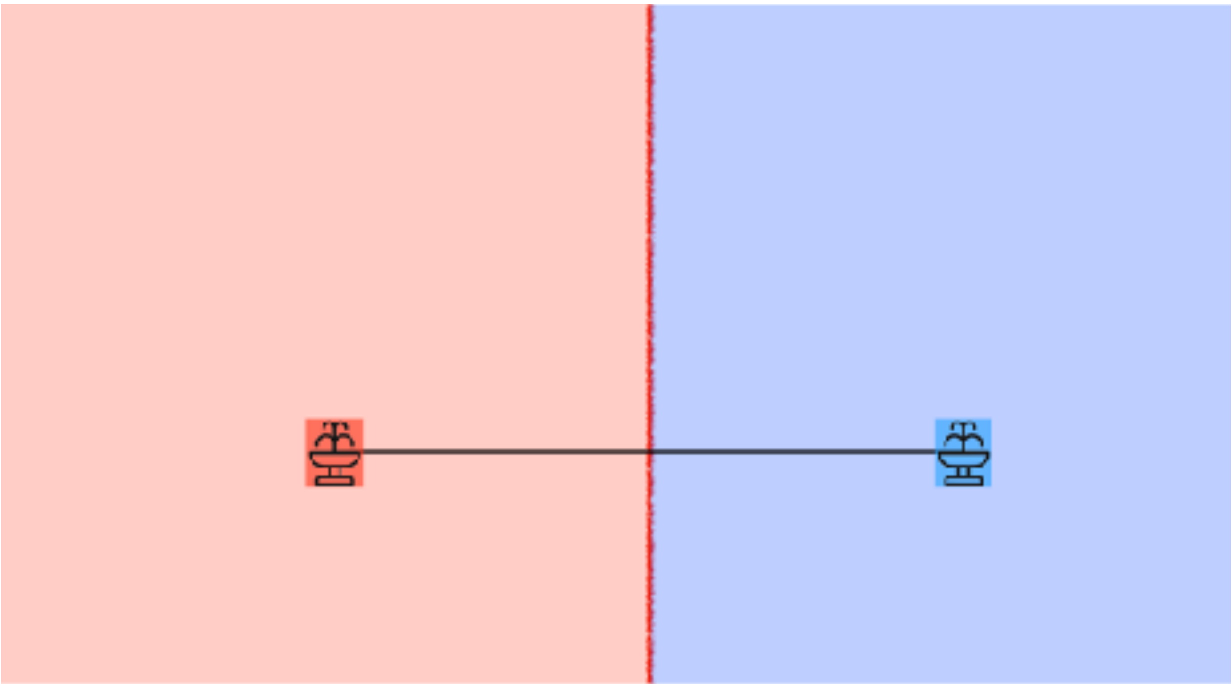
is $B(p,q) = B(q,p) = H(p,q) \cap H(q,p)$

- i.e., the set of all points with equal distance from $p$ and $q$.

**Corollary 4.4**

Voronoi region $V(p)$ of a point $p \in \mathcal{P}$:

$$V(p) = \bigcap_{q \in \mathcal{P} \setminus \{p\}} H(p,q)$$



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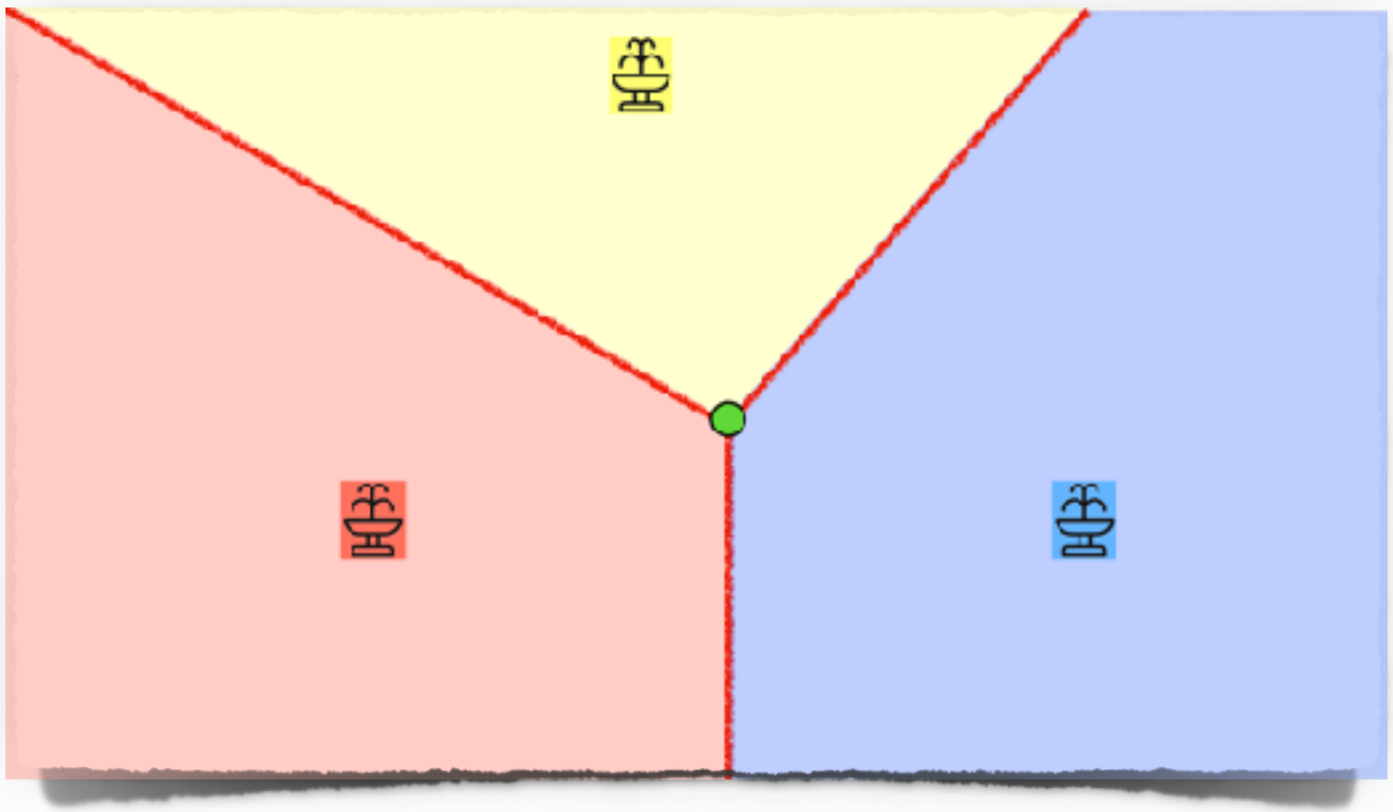
**Lemma 4.5**

$V(p_0), \ldots, V(p_{n-1})$ partition the plane into:

1. Convex set of points that are closest to precisely one site.
2. Sets of points (segments, rays or lines) that are closest to precisely two sites.
3. A finite number of points that are closest to at least three sites.



**Proof:**

Each $x \in \mathbb{R}^2$ has at least one closest site $\Rightarrow \mathbb{R}^2$ is completely partitioned.

Let $x \in \mathbb{R}^2$ closest to at least three sites $(q_1, q_2, q_3 \in \mathcal{P}$ )

$$\Rightarrow d(x, q_1) = d(x, q_2) = d(x, q_3) = \min_{q \in \mathcal{P}} d(x, q)$$

$\Rightarrow x$ center of circumcircle $\bigcirc$ with $q_1, q_2, q_3 \in \bigcirc$

$\Rightarrow x$ is uniquely defined for each triple, of which there is a finite number.

Let $x \in \mathbb{R}^2$ be closest to precisely two sites ( $q_1, q_2 \in \mathcal{P}$ )

$\Rightarrow x$ belongs to bisector $B(q_1, q_2)$

Let $x \in \mathbb{R}^2$ be closest to precisely one site ( $q_1 \in \mathcal{P}$ )

$\Rightarrow x \in V(q_1)$ (in the interior)

And: Voronoi regions are separated by bisectors.

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**Proof:**

Each $x \in \mathbb{R}^2$ has at least one closest site $\Rightarrow \mathbb{R}^2$ is completely partitioned.

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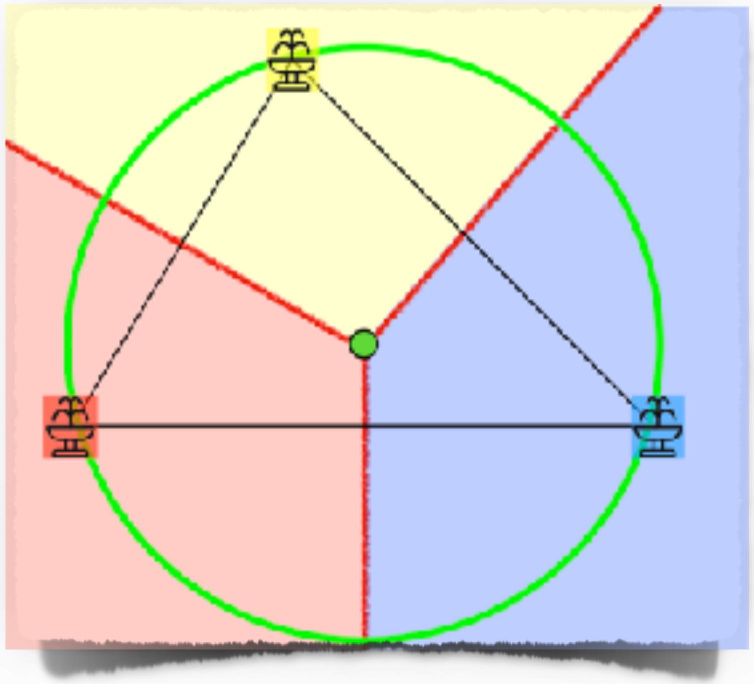
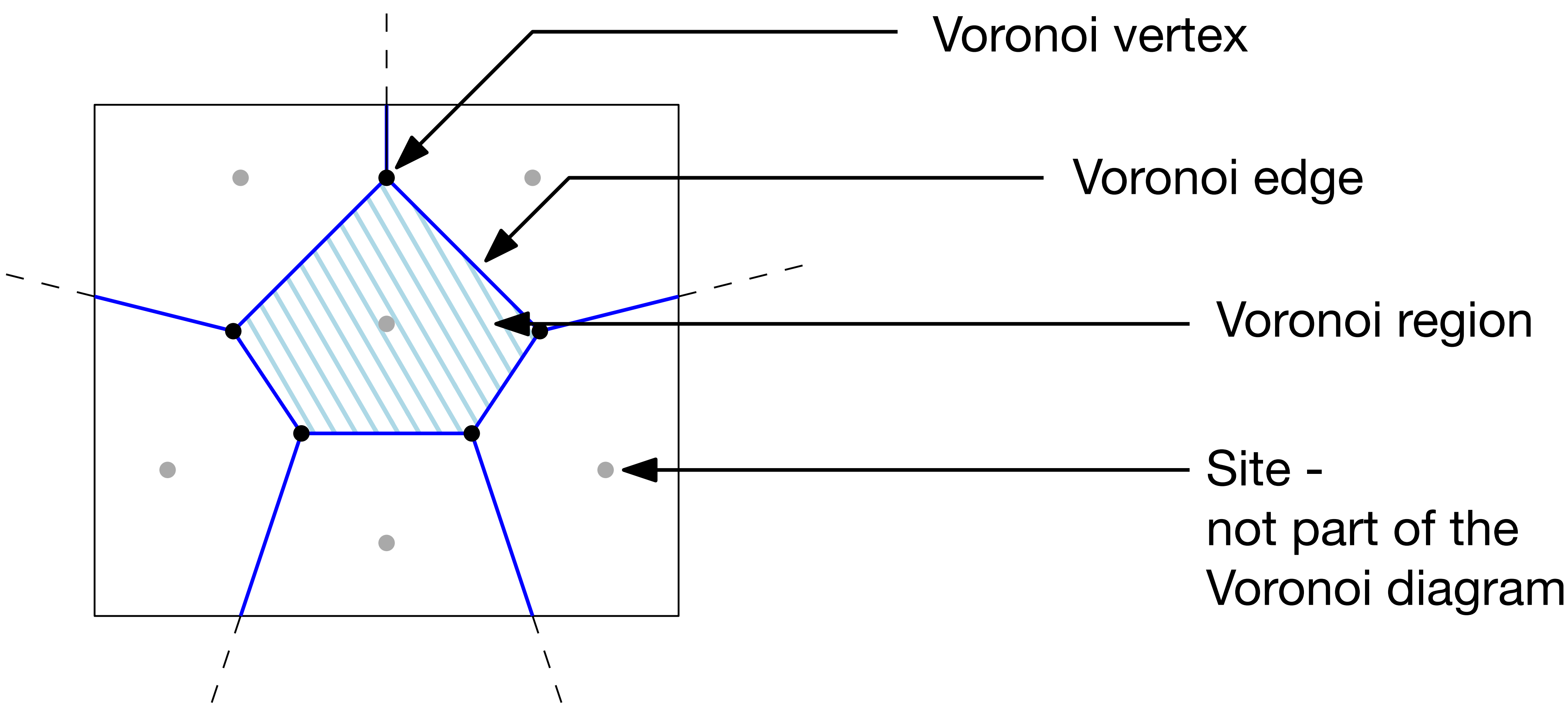
And: Voronoi regions are separated by bisectors. $\square$



**Definition 4.6**

The Voronoi diagram $Vor(\mathcal{P})$ is a partition of $\mathbb{R}^2$ into Voronoi regions with:

| | |
|---|---|
| **Voronoi vertices:** | Points closest to at least three sites |
| **Voronoi edges (or bisectors):** | Points closest to precisely two sites |
| **Voronoi regions:** | Points closest to precisely one site |

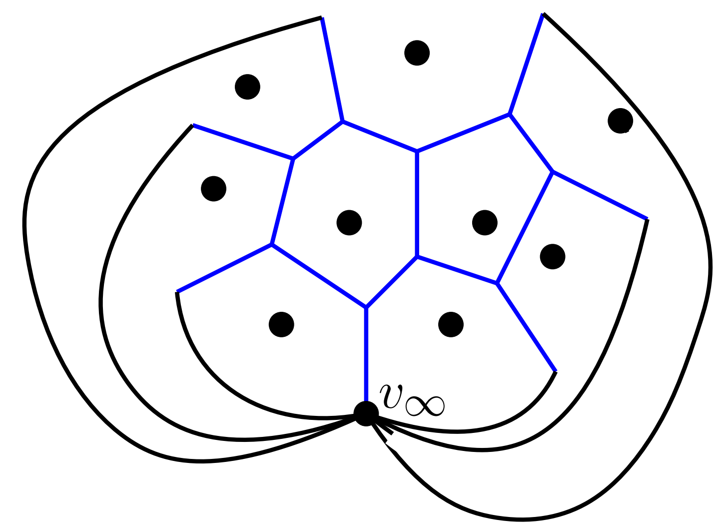


**Theorem 4.7**

$Vor(\mathcal{P})$ has precisely $n$ Voronoi regions, at most $2n-5$ Voronoi vertices and at most $3n-6$ Voronoi edges.

**Proof:**

- Each $p \in \mathcal{P}$ induces a region.
- Embedding as a planar graph
  $\rightarrow$ Consider extra vertex $v_\infty$




- Euler's formula: $v - e + f = 2$
- Number $f$ of faces: Number $n$ of Voronoi regions
- Number $e$ of edges: Number $n_e$ of Voronoi edges
- Number $v$ of vertices: Number $n_v$ of Voronoi vertices + 1
- Vertex degrees $\geq 3$
- Edge increases sum of degrees by $2$

$$2n_e \geq 3(n_v+1) \text{ \& } (n_v+1) - n_e + n \overset{(\dagger)}{=} 2 \quad \Leftrightarrow n_v \overset{(\star)}{=} n_e - n + 1$$

$\overset{(\star)}{\Rightarrow}$ 1.: $2n_e \geq 3(2+n_e-n) \Rightarrow 3n-6 \geq n_e$

$\overset{(\dagger)}{\Rightarrow}$ 2.: $2(n_v+1+n-2) \geq 3(n_v+1) \Rightarrow 2n-5 \geq n_v$

$\square$

1. Introduction and Motivation
2. Definitions
3. Representing planar partitions
4. Properties
5. Fortune's algorithm
6. The Voronoi game
7. Summary and conclusions

**Observation:**

$Vor(\mathcal{P})$ can be considered an embedded planar graph.

**Representing embedded graph:**

- Algorithm for constructing $Vor(\mathcal{P})$
  $\rightarrow$ Efficient representation of $Vor(\mathcal{P})$ required
- Objects:
  - Vertices with coordinates
  - Edges (Pointers to end points)
  - Faces (CCW sequence of boundary edges)

Theoretical Computer Science 7 (1978) 217–236.


**FINDING THE INTERSECTION OF T
POLYHEDRA***

D. E. MULLER[1] and F. I
*Coordinated Science Laboratory, U*

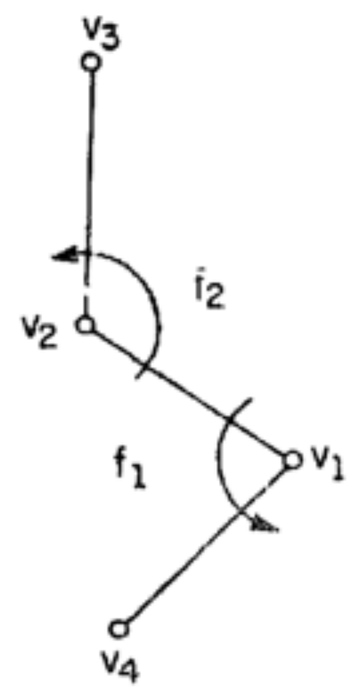
Communicated by M. Nivat
Received November 1977
Revised March 1978

**Abstract.** Given two convex polyhe
test whether their intersection is em
find a point in the intersection a
algorithm runs in time O ($n \log n$)
polyhedra. The part of the algorith
upon the fact that if a point in the i
from the convex hull of suitable ge
point.

**2. Derivation of a doubly connected edge list for a planar graph**

Let $V=\{v_1, \ldots, v_n\}$ and $E=\{e_1, \ldots, e_m\}$ be the sets of vertices and edges respectively, of a planar graph embedded in the plane without crossing edges. We assume that $(V, E)$ is represented as follows. To vertex $v_j \in V$ there corresponds cell $H[j]$ of an array $H[1:n]$, which contains a pointer to the first term of the cyclic list of the edges incident on $v_j$ arranged in the order in which they appear as one proceeds counterclockwise around $v_j$. The latter lists are realized by means of two ...ERTEX[$i$], NEXT[$i$]) is the ...h (V, E) is precisely the one ...which constructs the convex ... surface of a convex poly- ... this collection of lists the

... commonly used represen- ...he dual graph, i.e., the graph ...ph, is not readily available.




| | V1 | V2 | F1 | F2 | P1 | P2 |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 2 | | | | | | |
| ⋮ | | | | | | |
| a1 | 1 | 2 | 1 | 2 | a2 | a3 |
| | | | | | | |
| a2 | 4 | 1 | 1 | | | |
| | | | | | | |
| a3 | 2 | 3 | | 2 | | |
| | | | | | | |

Fig. 1. Illustration of the DCEL.

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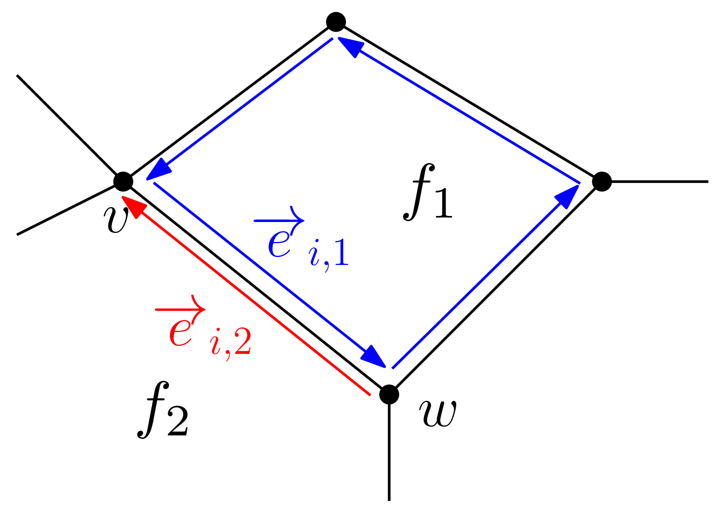
- Separate storage of vertices, edges and faces

- Subdividing edges into half-edges: $e_i = (v, w) \to \overrightarrow{e}_{i,1} = (v, w), \overrightarrow{e}_{i,2} = (w, v)$

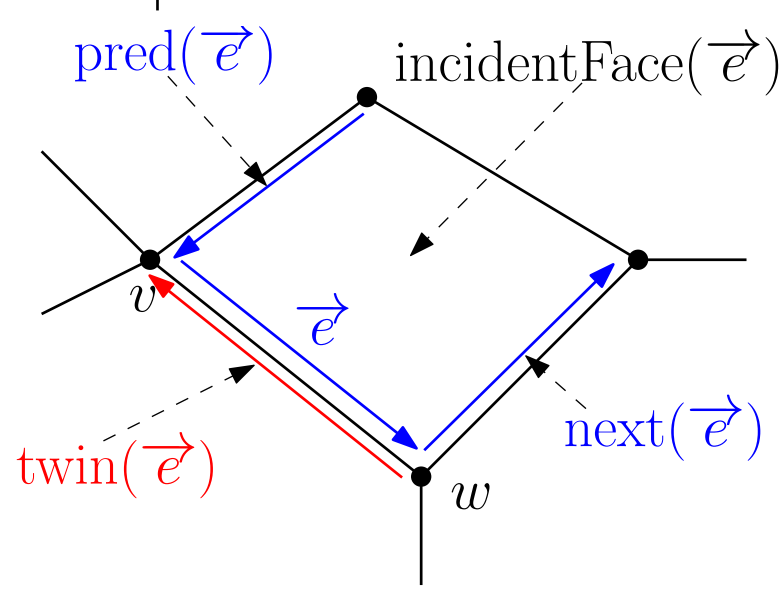
- $e = (v, w)$ separates two regions $f_1, f_2 \Rightarrow$

$$\left[ \begin{array}{c} \overrightarrow{e}_{i,1} = (v, w) \text{ is on boundary of } f_1 \\ \Leftrightarrow \\ w \text{ follows } v \text{ on boundary of } f_1 \text{(CCW)} \end{array} \right]$$



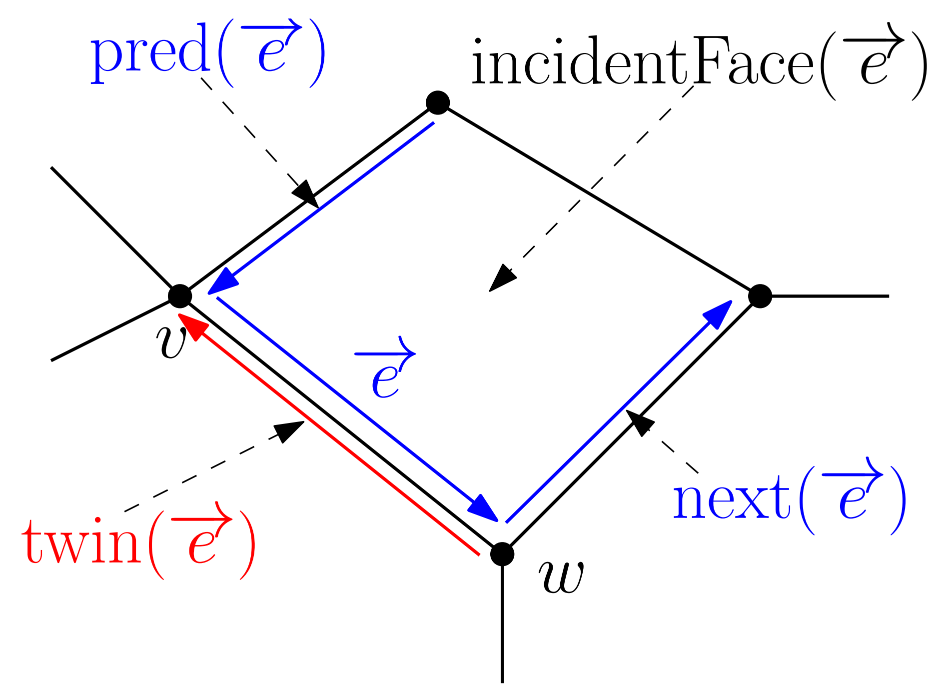

- Half-edge lies on boundary of unique face

$\Rightarrow$ Half-edges have unique predecessor and successor



**Representation:**

- Half-edge $\vec{e}$ stores:
  - Pointer $\text{incidentFace}(\vec{e})$ to the face $f$ bounded by edge $\vec{e}$
  - Pointer $\text{next}(\vec{e})$ to successor edge
  - Pointer $\text{pred}(\vec{e})$ to predecessor edge
  - Pointer $\text{origin}(\vec{e})$ to start vertex
  - Pointer $\text{twin}(\vec{e})$ to partner half-edge




- In essence: Storing boundary edges of a face $f$: Doubly linked list

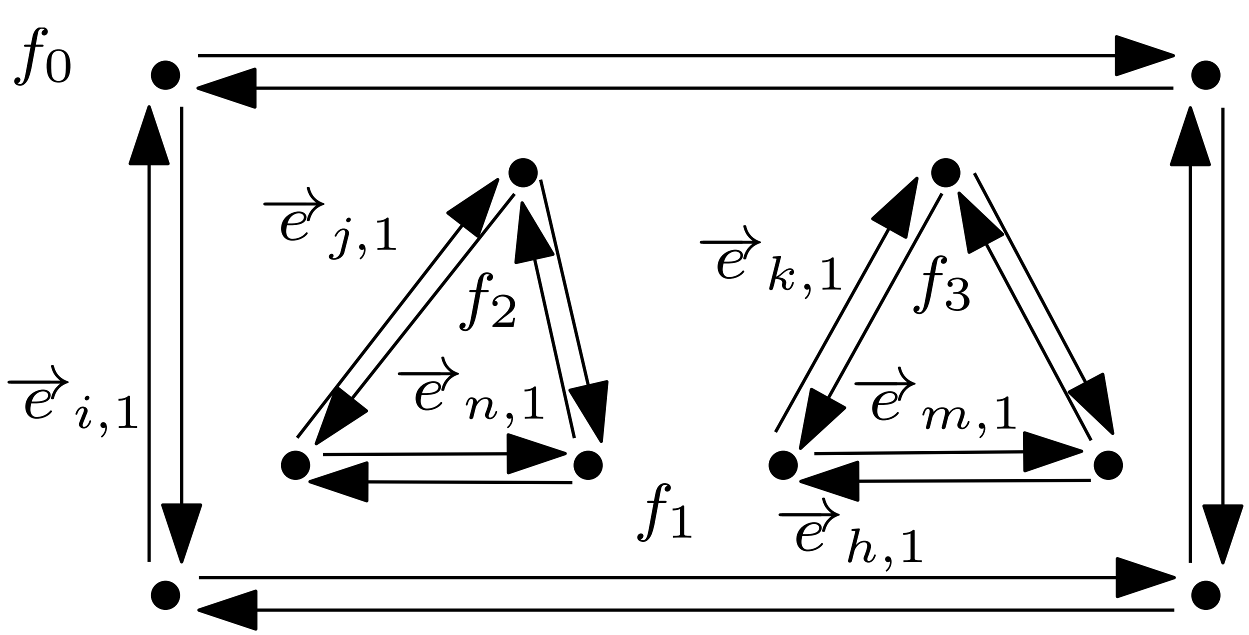
- Storing vertices. Each vertex $v$ stores:
  - Coordinates
  - Pointer to incident half-edge $\vec{e}$ with $\text{origin}(\vec{e}) = v$

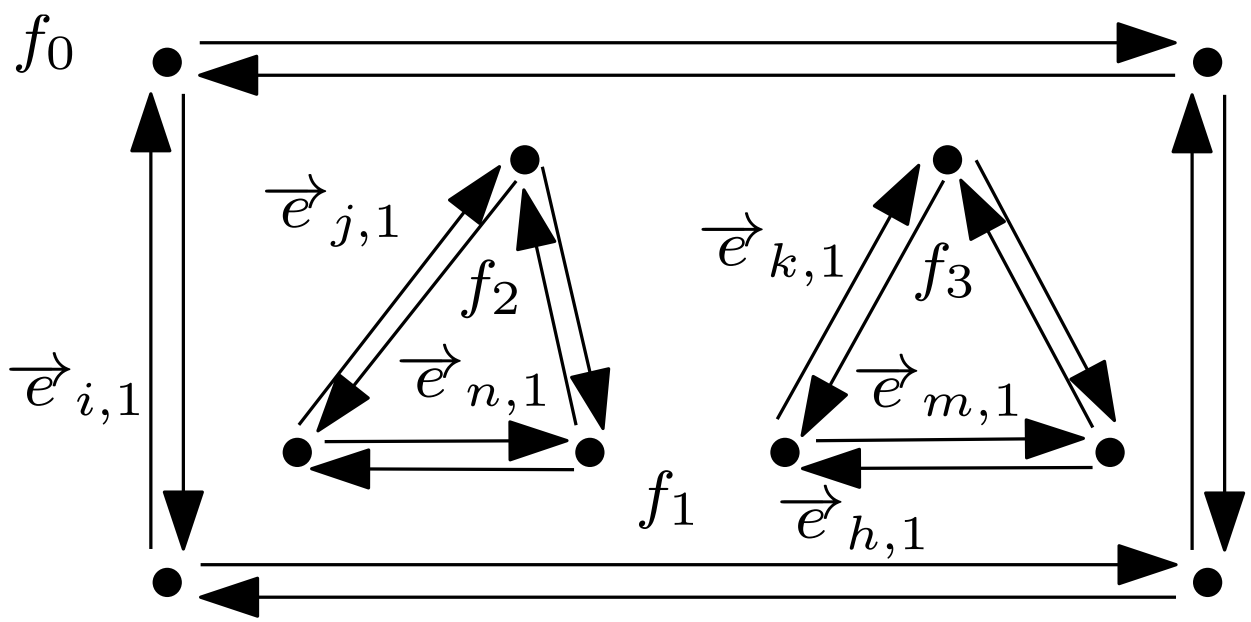
**Storing faces:**

- Exterior and interior boundaries (holes)
- Face $f$ stores pointer $\text{outerComponent}(f)$ to some edge on outer boundary

  Exterior face: $\text{outerComponent}(f) = \text{null}$
- Face $f$ stores list $\text{innerComponents}(f)$

  For each interior boundary one entry: pointer to some edge of component






## Storing faces:

| Face | outerComponent | innerComponents |
|---|---|---|
| $f_0$ | null | $\vec{e}_{i,1}$ |
| $f_1$ | $\vec{e}_{h,1}$ | $\{\vec{e}_{j,1}, \vec{e}_{k,1}\}$ |
| $f_2$ | $\vec{e}_{n,1}$ | null |
| $f_3$ | $\vec{e}_{m,1}$ | null |

**Doubly-Connected Edge List (DCEL):**

- Storing vertices, edges, faces in table
- Pointers to connect data; in particular: implicit storage of boundaries as doubly linked lists
- Constant memory per vertex and edge
- Total memory for faces: linear
- Total memory: linear
- Operations on DCEL $\rightarrow$ Exercise

**Assumption:** General position (no four points on same circle)

**Lemma 4.8**
Voronoi vertices $v$ have degree 3.

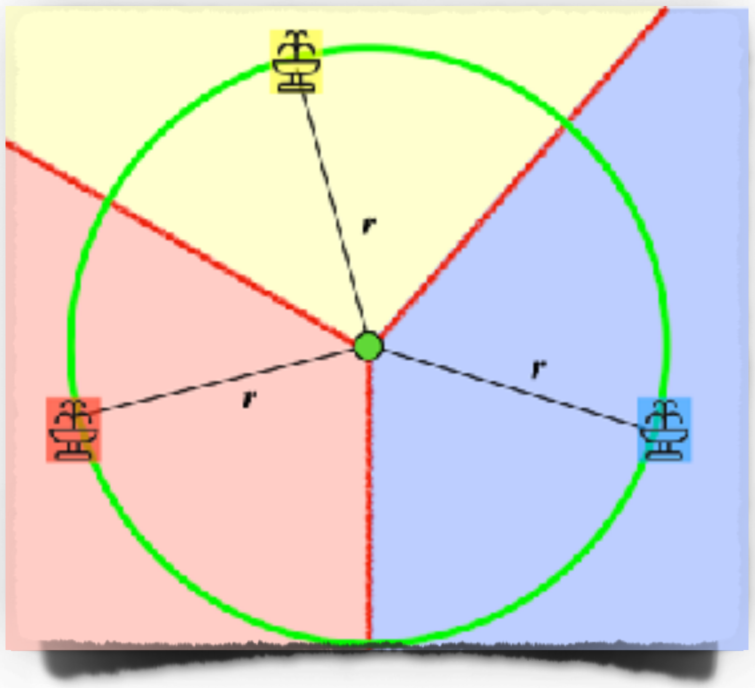
**Proof:**

A Voronoi vertex $v$ lies at the intersection of bisectors, so it is at identical distance to involved sites.

Therefore, it must be at the center of a circumcircle of all involved sites.

By assumption, the circle cannot contain more than three sites.

Because $v$ is a vertex, there must be more than two sites.

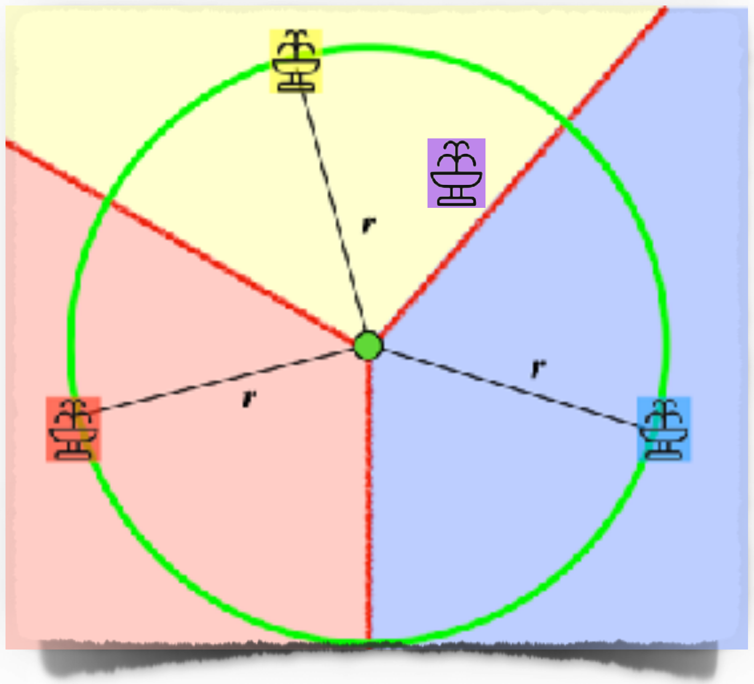


**Definition 4.9**

Let $p_{i_1}, p_{i_2}, p_{i_3}$ be three points that induce a Voronoi vertex $v$. Let $C(v)$ be the circle with center $v$ and $p_{i_1}, p_{i_2}, p_{i_3} \in C(v)$.

**Lemma 4.10**

$C(v)$ does not contain another site in $p \in \mathcal{P}$ in its interior.



**Proof:**

- Assumption: $C(v)$ contains $p \in \mathcal{P}$ in its interior.
- Then $p$ is closer to $v$ than the three sites.
- Therefore, $v$ cannot lie on the boundaries of $V(p_{i_1}), V(p_{i_2}), V(p_{i_2})$ ⚡

$\square$

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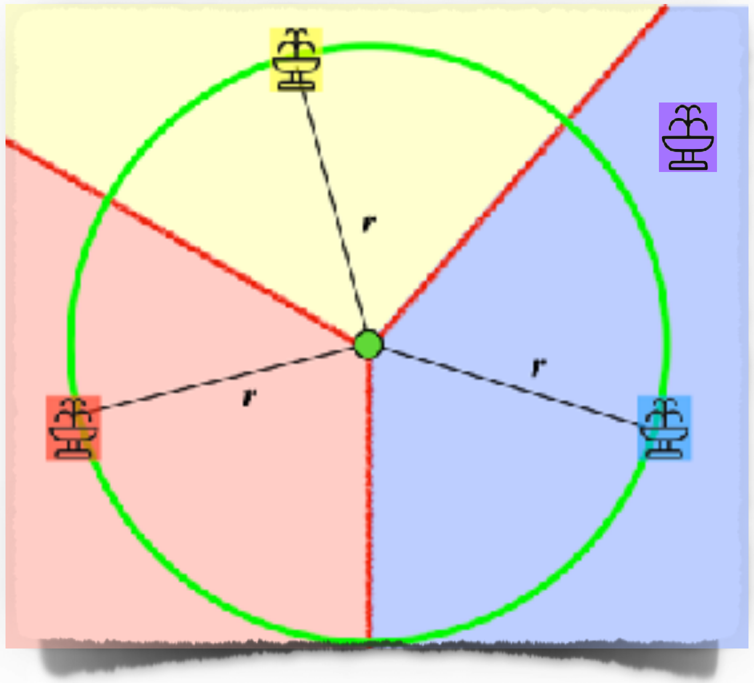
**Lemma 4.11**

Let $p_{i_1}, p_{i_2}, p_{i_3}$ be three sites with empty circumcircle $C$.
Then they induce a Voronoi vertex.

**Proof:**

- Assumption: $C$ contains no $p \in \mathcal{P}$ in its interior.
- Then the center $v$ is closest to all three sites, thus on their pairwise bisectors.
- Therefore, $v$ is a Voronoi vertex.




□

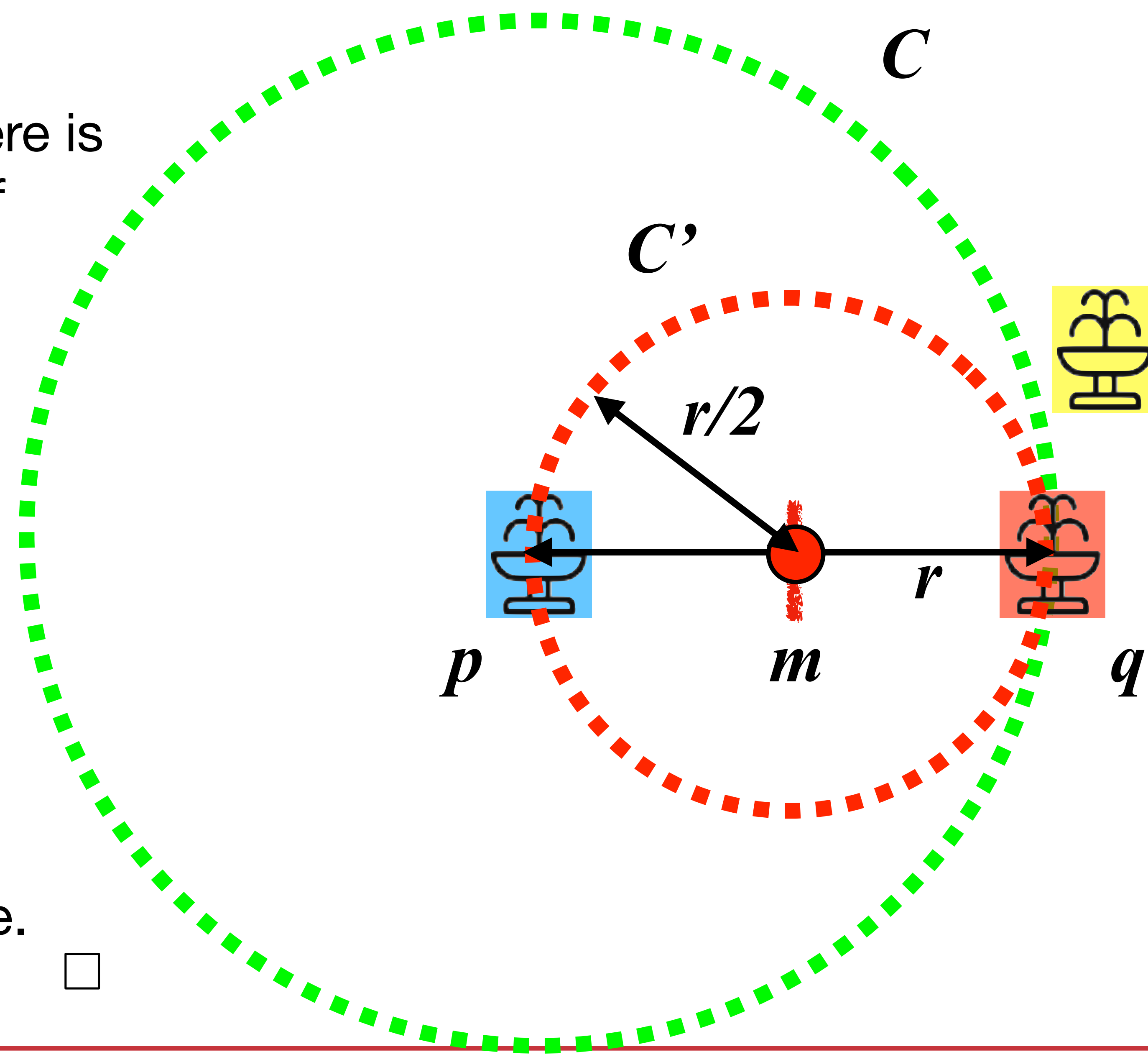
**Lemma 4.12**

A nearest neighbor $q \in \mathcal{P}$ of $p \in \mathcal{P}$ induces a Voronoi edge of $V(p)$.

**Proof:**

Because $q$ is a nearest neighbor of $p$, there is no other site strictly inside the circle $C$ of radius $r=d(p,q)$ around $p$.

The circle $C'$ of radius $r/2$ around the midpoint $m$ between $p$ and $q$ $m$ lies completely inside $C$, and its only point on the boundary of $C$ is $q$. Therefore, $m$ is only closest to $p$ and q. Because all other sites have at least distance $r/2+\delta$ from m, at least some $\varepsilon$-portion of the bisector of $p$ and $q$ must belong to a Voronoi edge. □



**Lemma 4.13**

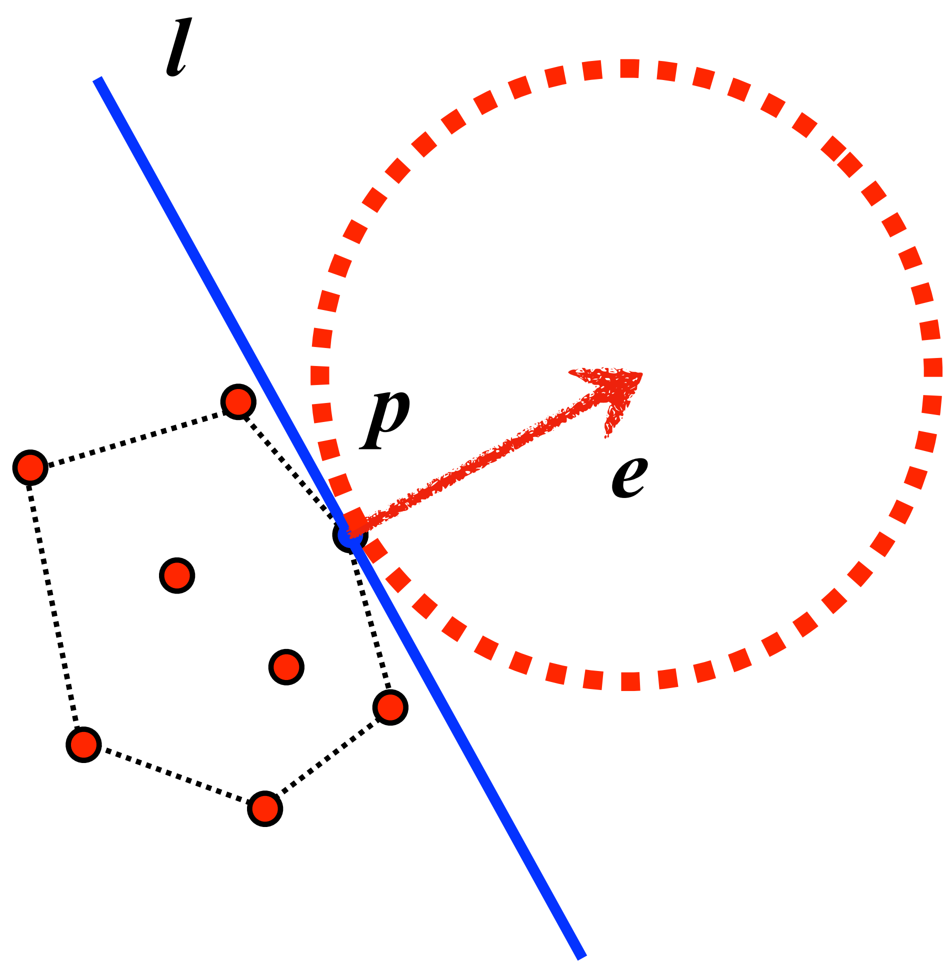
$p \in \mathcal{P}$ lies on boundary of $\text{conv}(\mathcal{P}) \Leftrightarrow V(p)$ unbounded.

**Proof:**

$\Rightarrow$:

Because $p$ lies on the boundary of the convex hull, there must be a line $l$ through $p$, such that all sites lie in the same half-plane.

Then the ray $e$ from $p$ orthogonal to $l$ consists of points that all have $p$ as their unique closest site.



**Lemma 4.13**

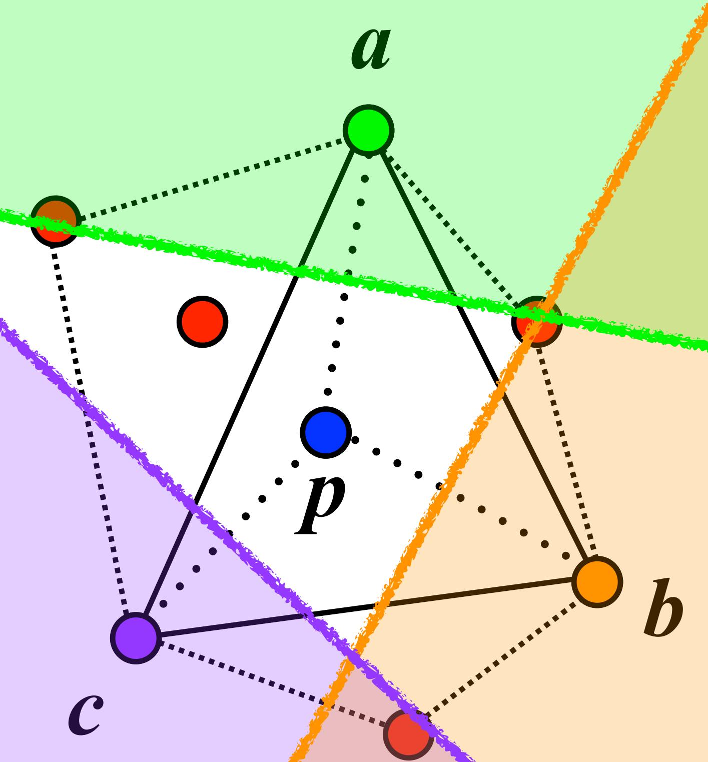
$p \in \mathcal{P}$ lies on boundary of $\mathrm{conv}(\mathcal{P}) \Leftrightarrow V(p)$ unbounded.

**Proof:**

$\Leftarrow$

Suppose $p$ lies strictly inside the convex hull.
Then $p$ must lie inside a triangle of three other sites.

Then the union of half-planes consisting
of points that are closer to $a$, $b$, or $c$ than to $p$
leaves only a bounded triangle
as set of points that can be contained in $V(p)$.




□

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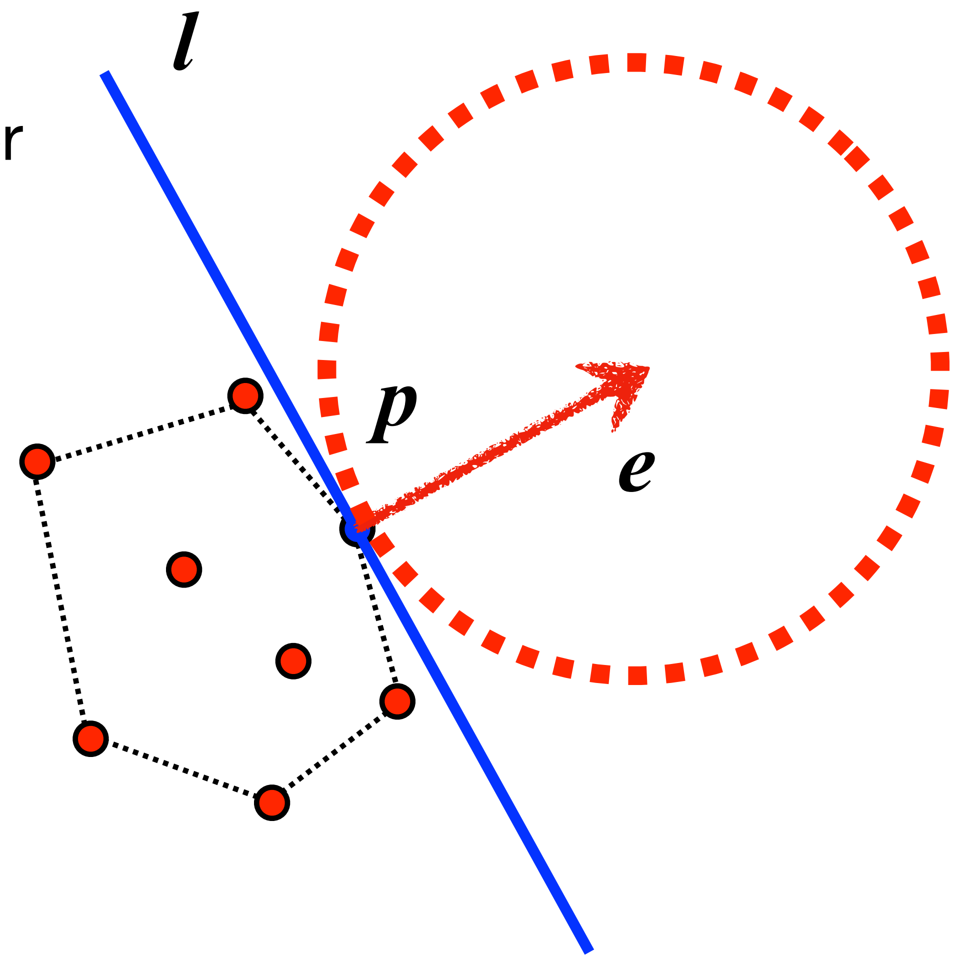
**Lemma 4.13**

$p \in \mathcal{P}$ lies on boundary of $\mathrm{conv}(\mathcal{P}) \Leftrightarrow V(p)$ unbounded.

**Corollary 4.14:**

Computing the Voronoi diagram for $n$ points has a lower bound of $\Omega(n \log n)$.

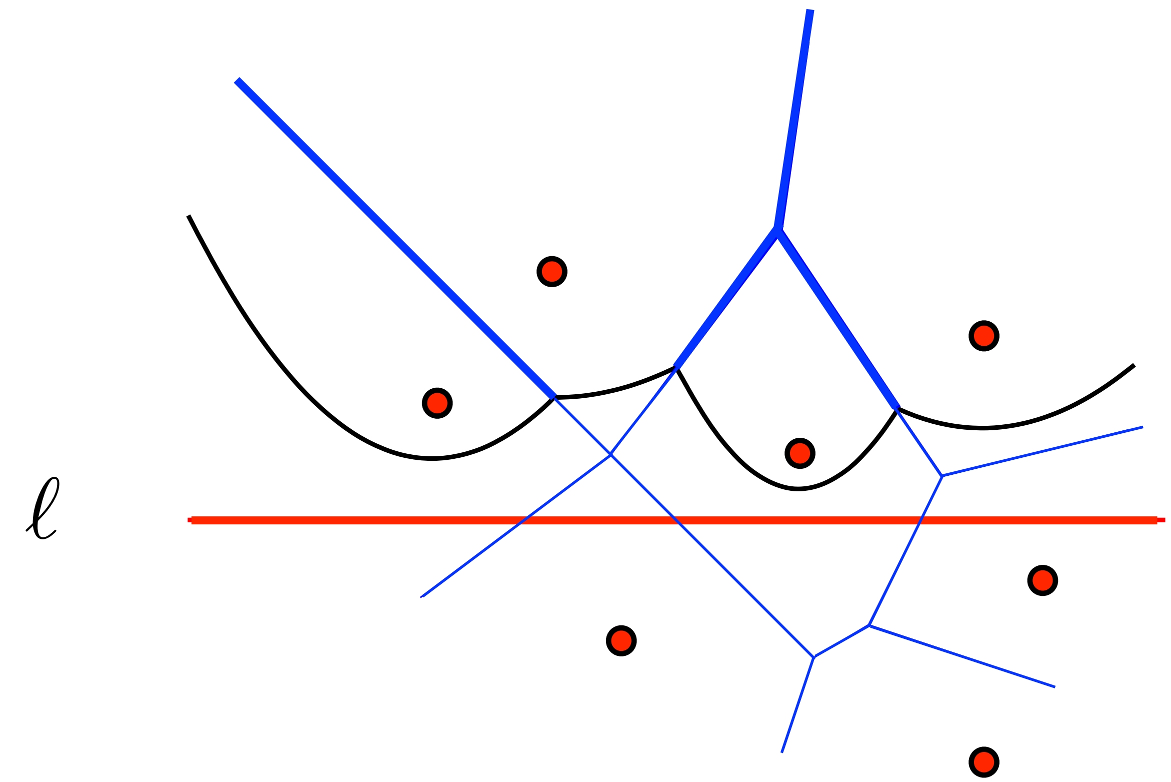


Vironoi Man

Vironoi Man

**Approach:**

- Consider a moving „frontier“ between resolved and unresolved part.

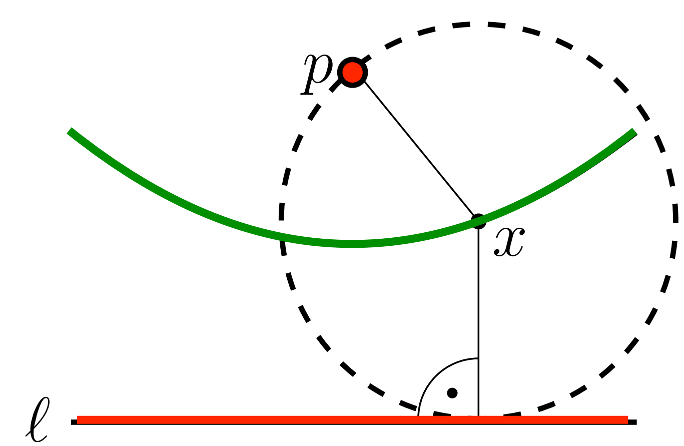



**Crucial issue:**

- $p \in \mathcal{P}$ below $\ell$ can influence $Vor(p)$ above $\ell$.

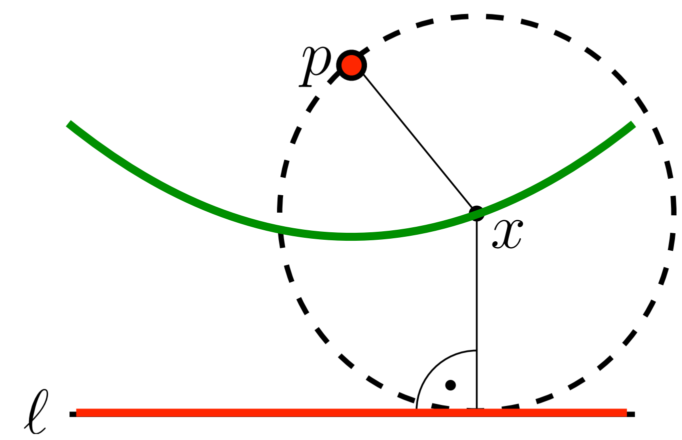
**Observation:**

- The separation between resolved and unresolved part for a point $p$ and line $\ell$ is a curve consisting of points that have equal distance from $p$ and $\ell$.



**Consider:**

$$\{x \in \mathbb{R}^2 \mid d(x,p) = d(x,\ell)\}$$




**Theorem 4.15:**

The curve is a parabola (with *focus* $p$ and *directrix* $\ell$ ).

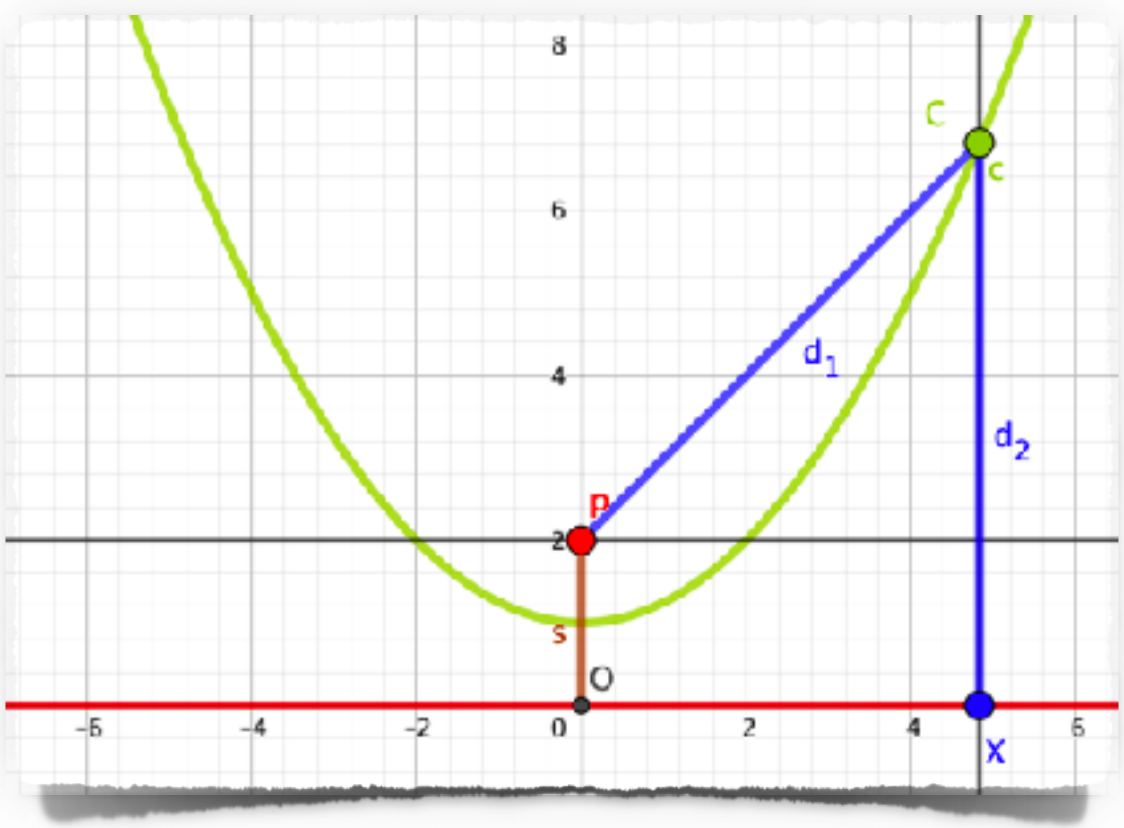
**Proof:**

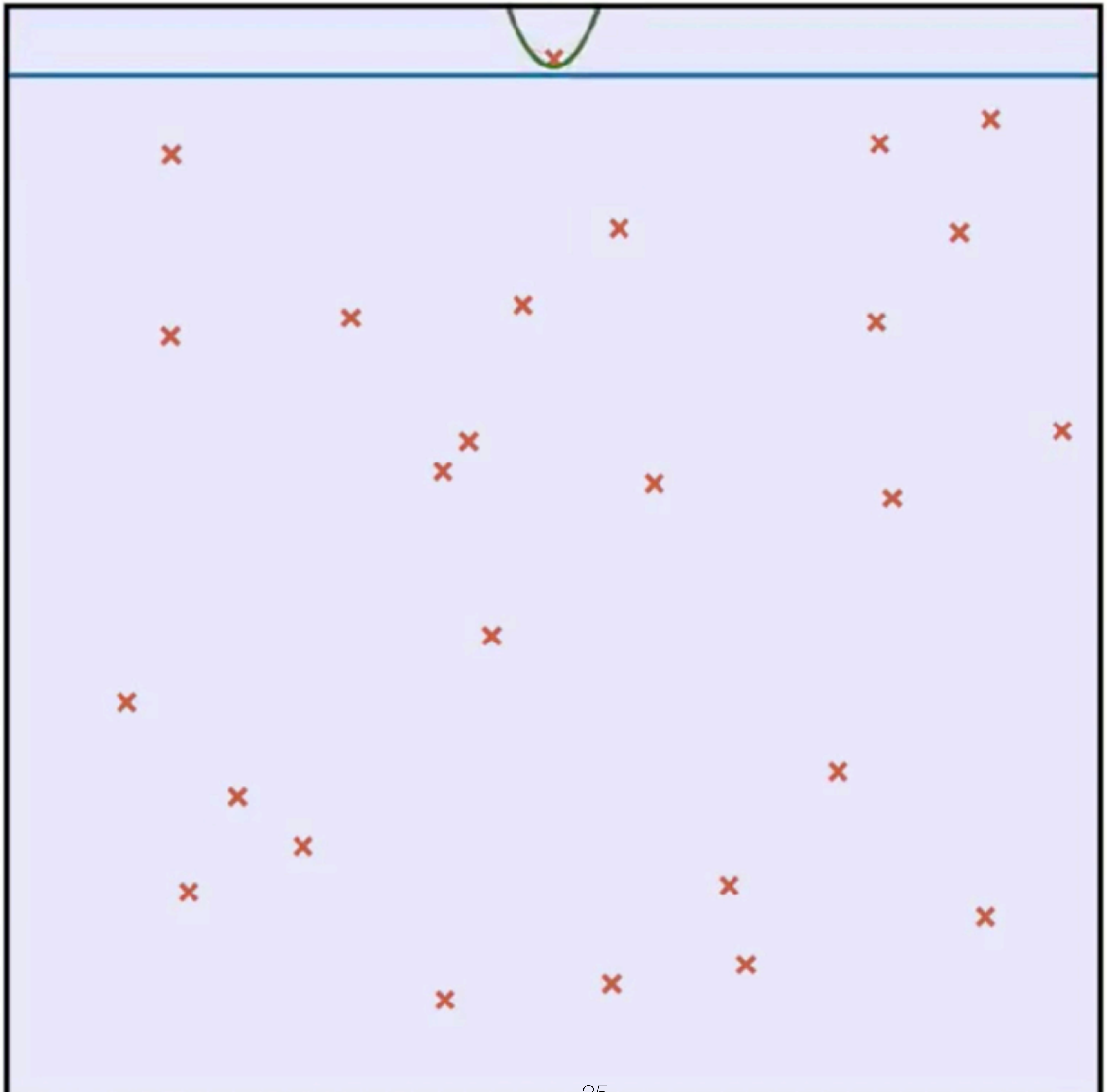
Consider $p=(0,s)$ and $X=(x,0)$.
Then $C=(x,y)$ with

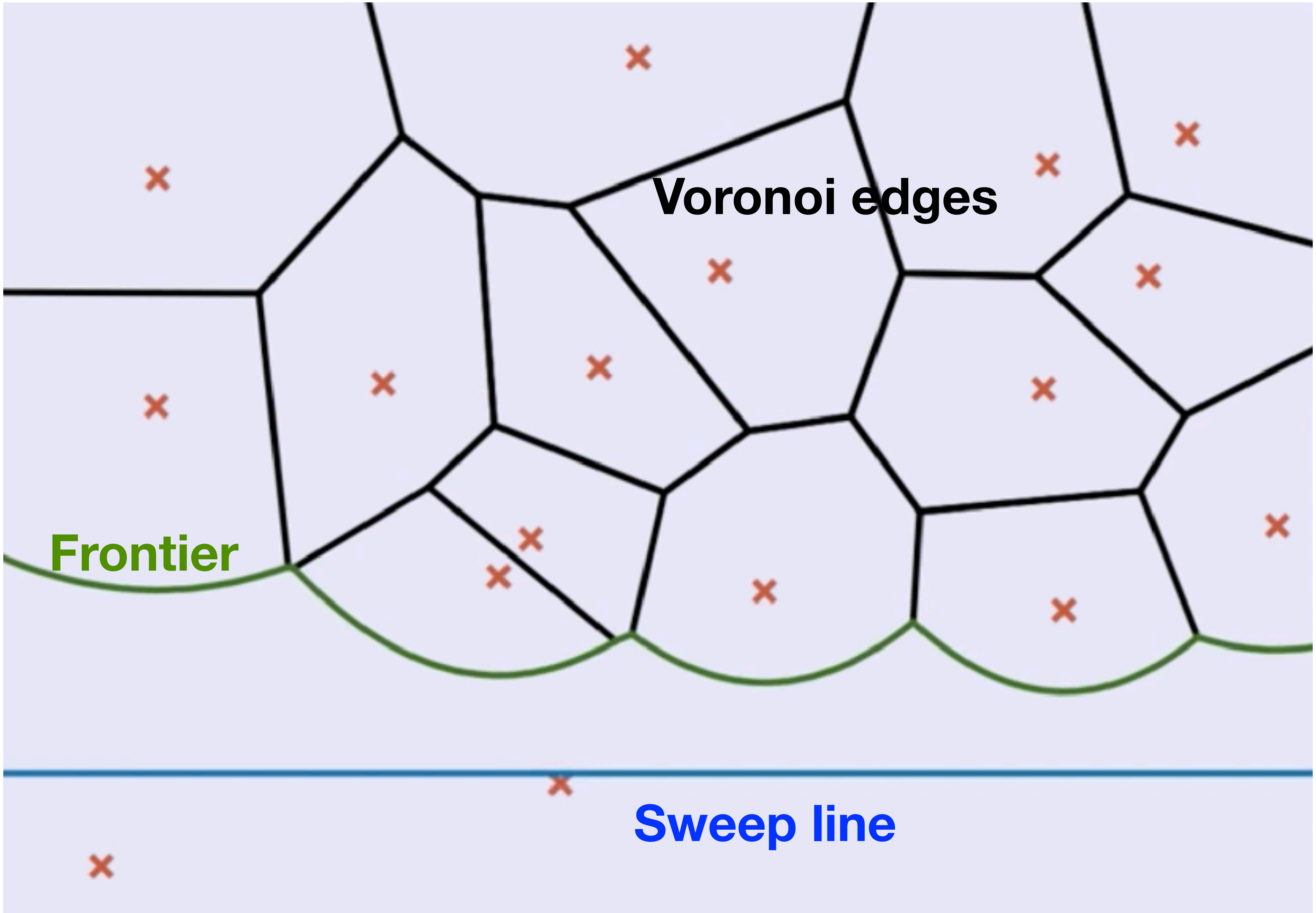
$$d_1^2 = x^2 + (y-s)^2 = x^2 + y^2 - 2ys + s^2$$
$$d_2^2 = y^2$$

So

$$y = \frac{1}{2s}x^2 + \frac{s}{2}$$






Voronoi edges
Frontier
Sweep line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Voronoi edges
Frontier
Sweep line


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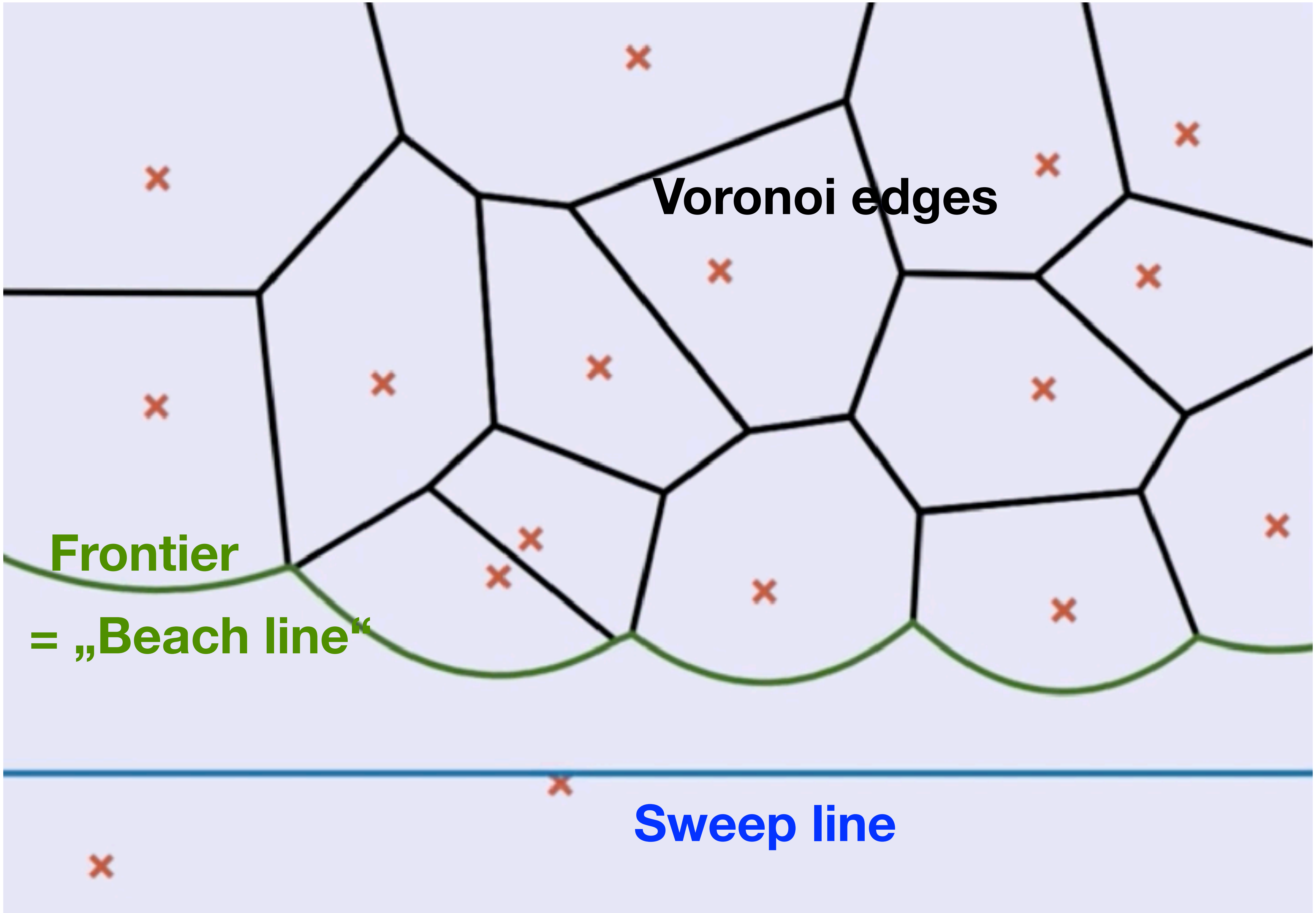


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# Thank you for today!