Computational Geometry
Chapter 4: Voronoi Diagrams

Prof. Dr. Sándor Fekete

Algorithms Division
Department of Computer Science
TU Braunschweig

Technische
Universität
Braunschweig

1. **Introduction and Motivation**
2. **Definitions**
3. **Representing planar partitions**
4. **Properties**
5. **Fortune's algorithm**
6. **Variations**
7. **The Voronoi Game**
8. **Summary and conclusions**

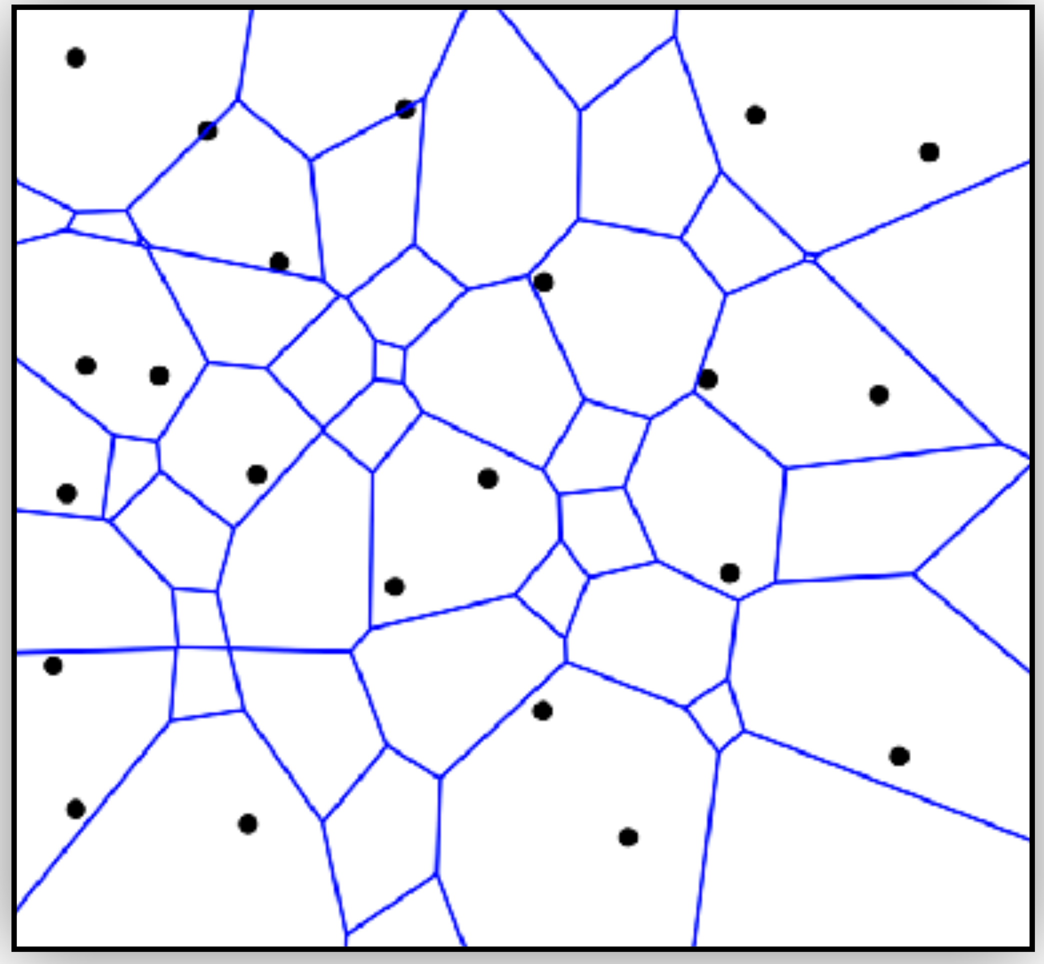
Higher-order Voronoi diagrams:

Higher-order Voronoi diagrams:

- Voronoi region := point set with same set of k nearest neighbors

Higher-order Voronoi diagrams:

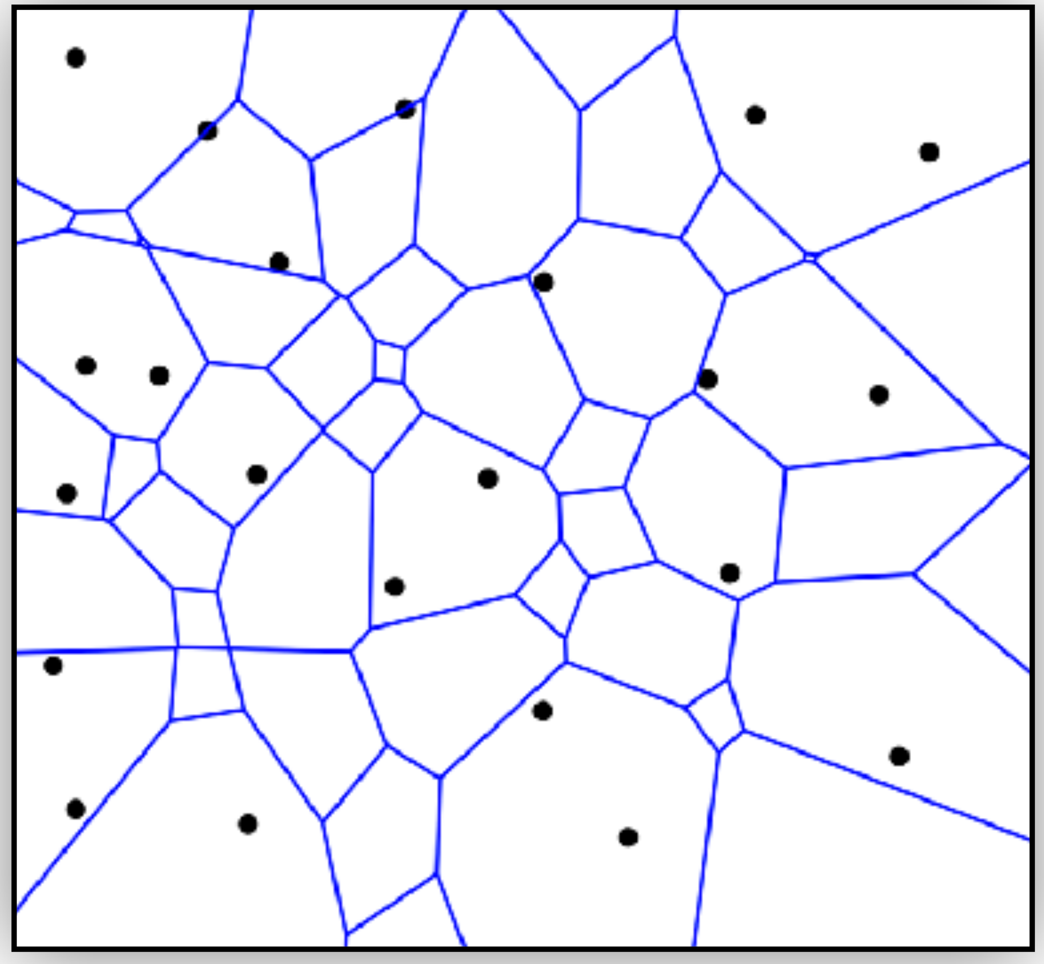
- Voronoi region := point set with same set of k nearest neighbors



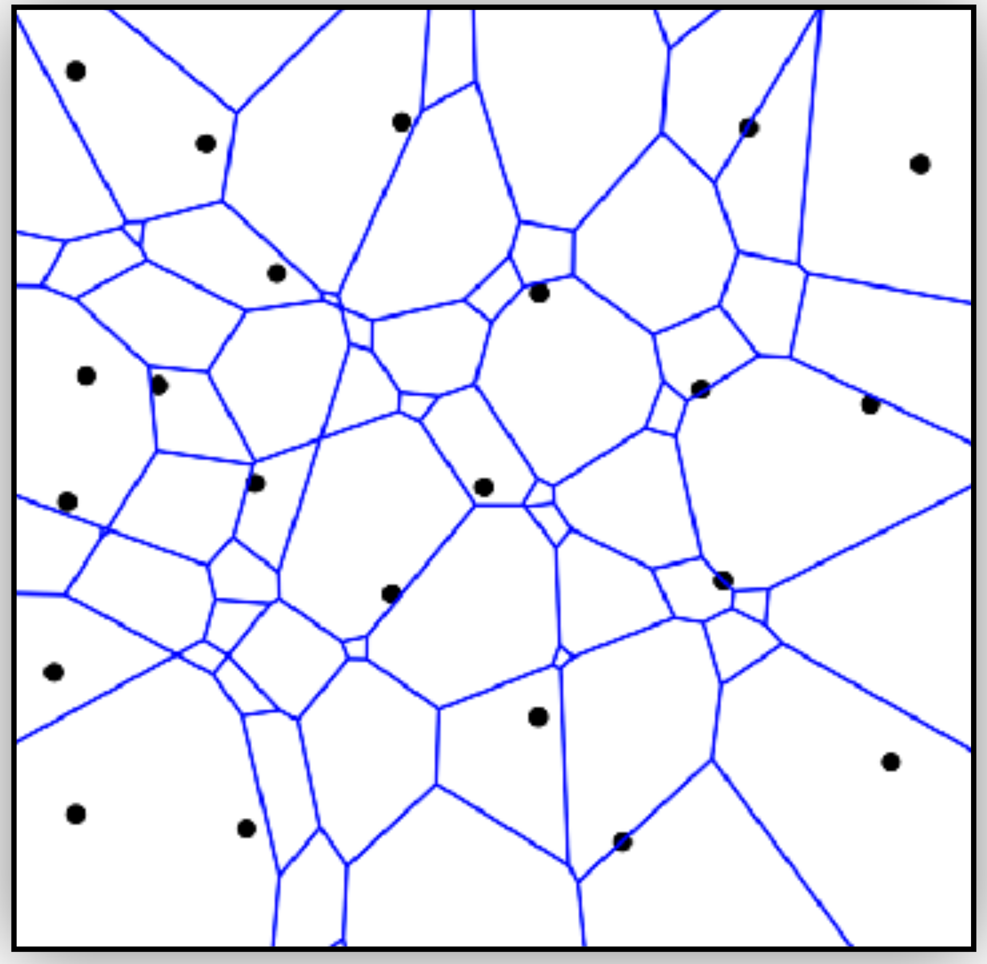
$k = 2$

Higher-order Voronoi diagrams:

- Voronoi region := point set with same set of k nearest neighbors



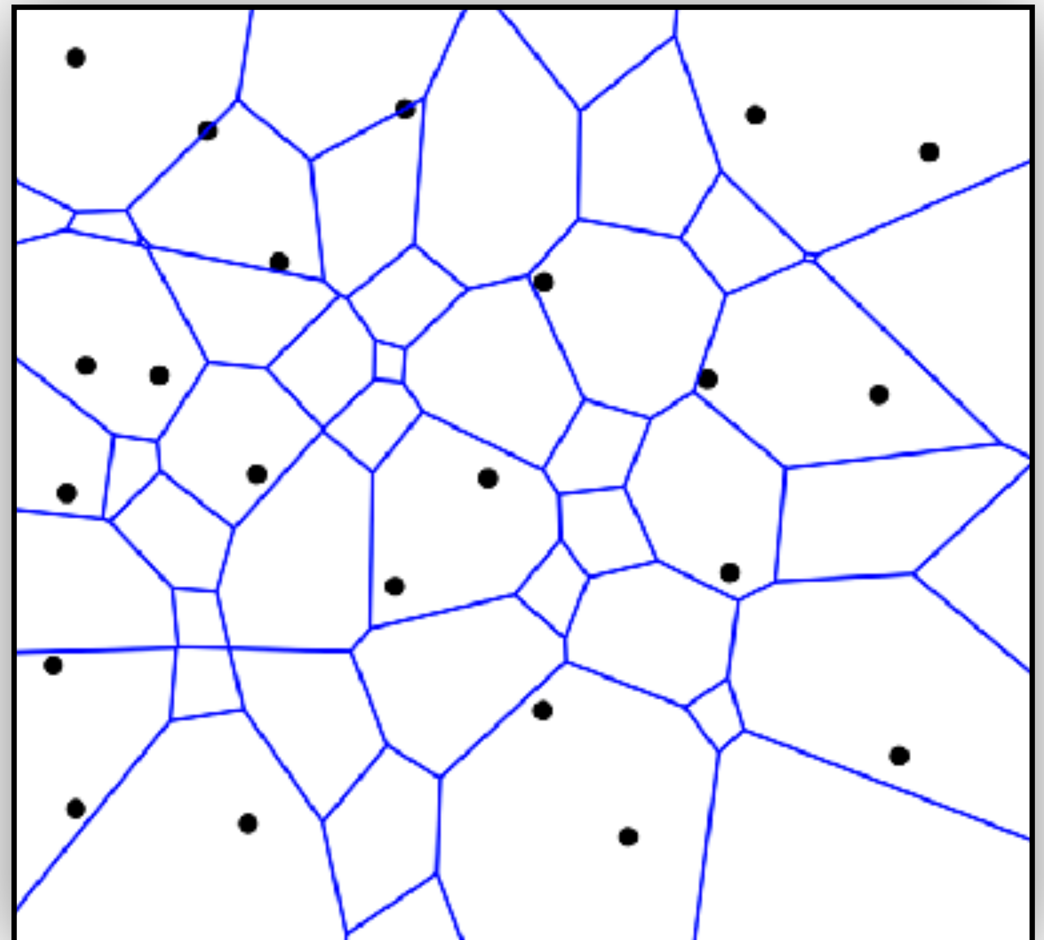
$k = 2$



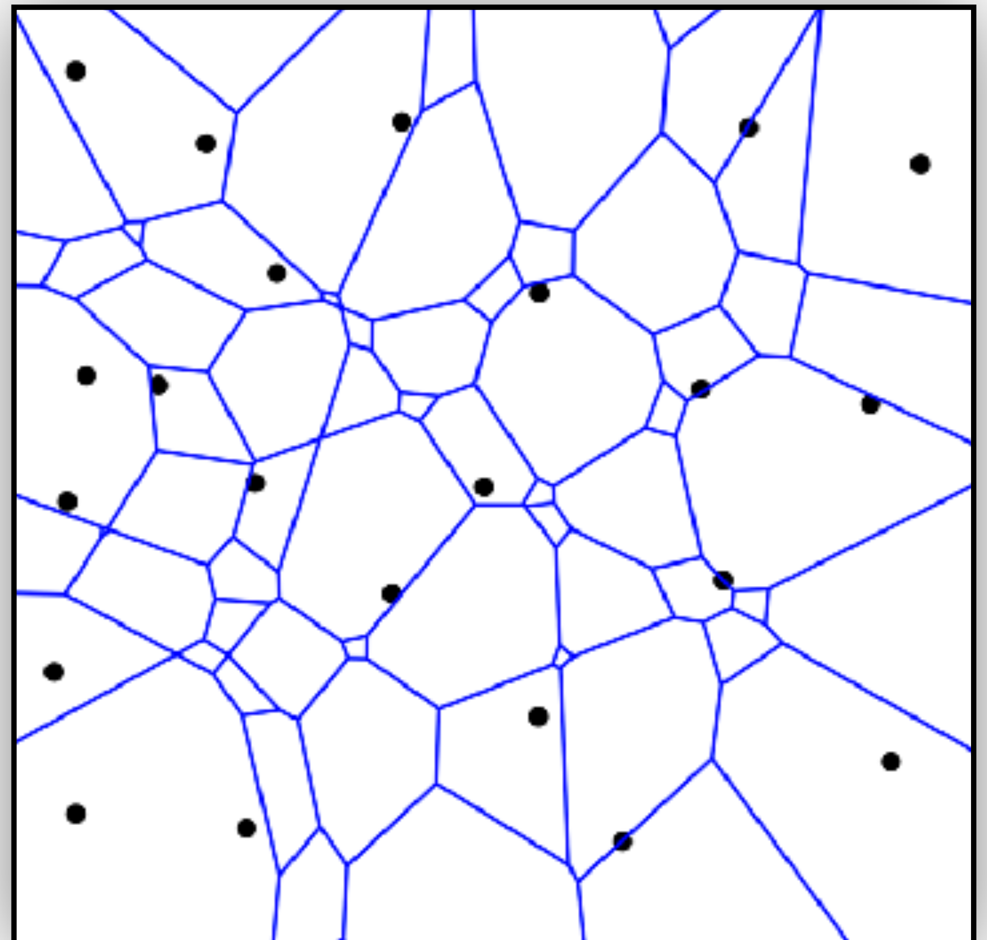
$k = 3$

Higher-order Voronoi diagrams:

- Voronoi region := point set with same set of k nearest neighbors

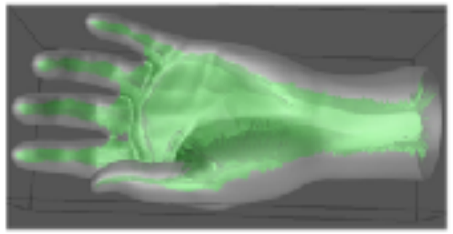


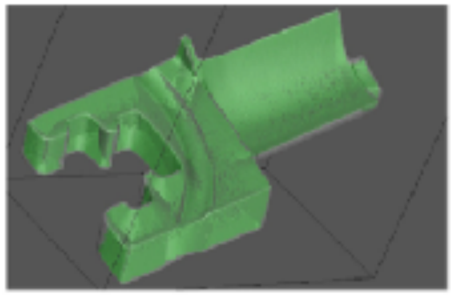
$k = 2$

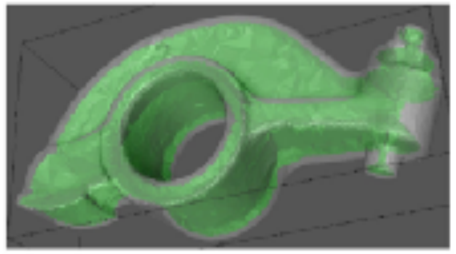


$k = 3$

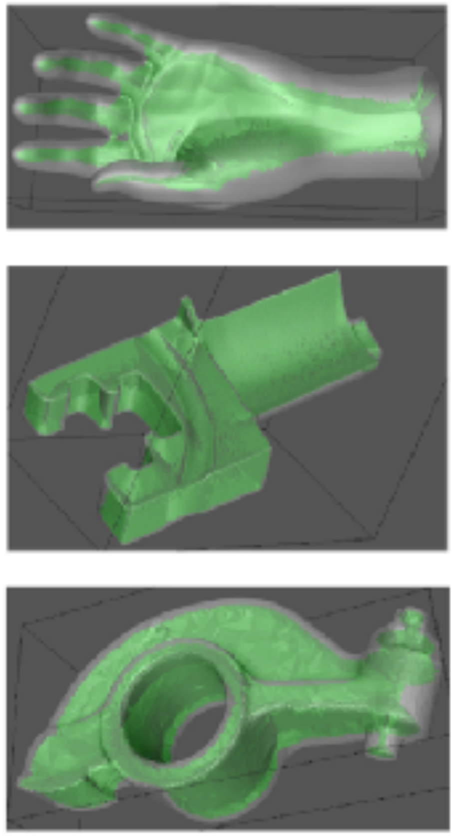
- *Worst-Case*-optimal: $\mathcal{O}(n^3)$ for arbitrary (but fixed) $k \geq 2$ [Edelsbrunner and Seidel, 1986].





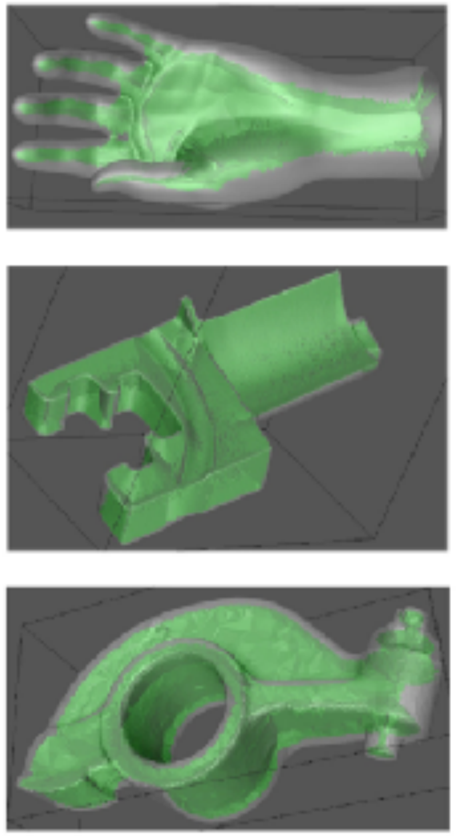


Medial axis of a simple polygon



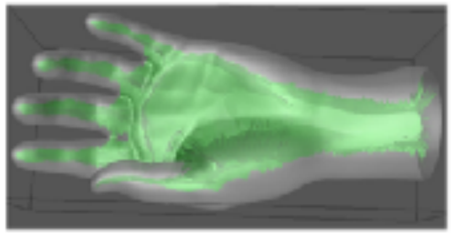
Medial axis of a simple polygon

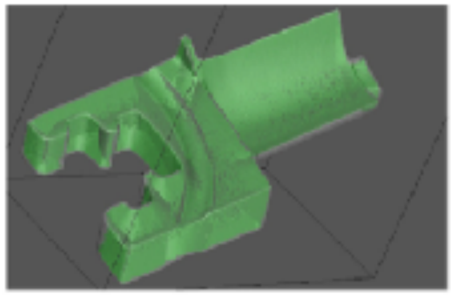
- Analogously: polygon instead of $\mathcal{P}$

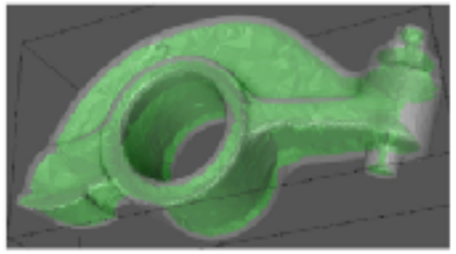


Medial axis of a simple polygon

- Analogously: polygon instead of $\mathcal{P}$
- Voronoi edges and vertices: two or three nearest neighbors on polygon boundary

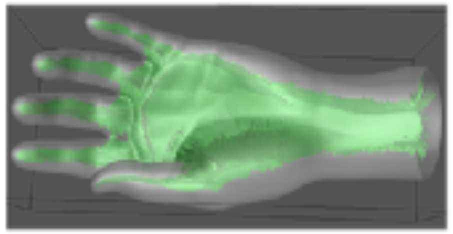


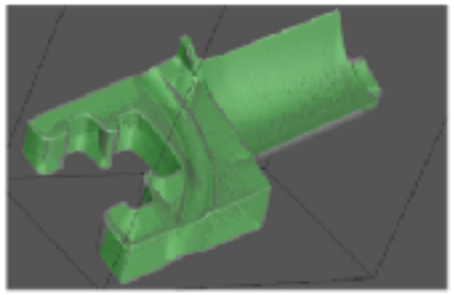


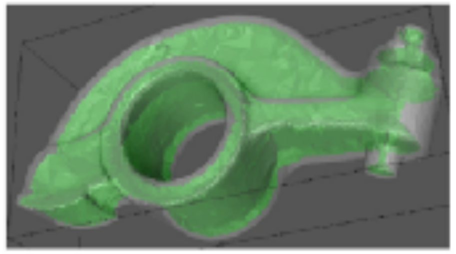


Medial axis of a simple polygon

- Analogously: polygon instead of $\mathcal{P}$
- Voronoi edges and vertices: two or three nearest neighbors on polygon boundary
- Computation for simple polygons in linear(!) time [Chin *et al.*, 1995].

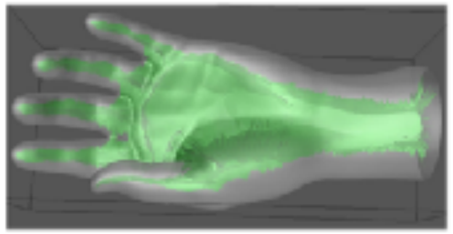


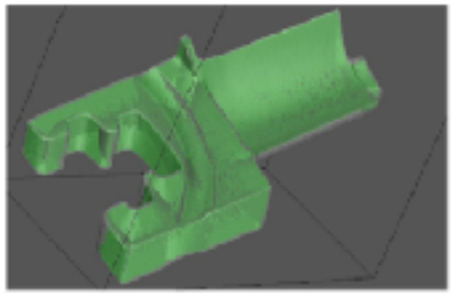


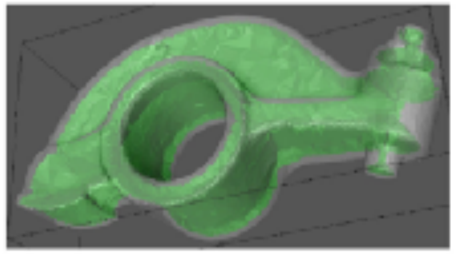


Medial axis of a simple polygon

- Analogously: polygon instead of $\mathcal{P}$
- Voronoi edges and vertices: two or three nearest neighbors on polygon boundary
- Computation for simple polygons in linear(!) time [Chin *et al.*, 1995].
- Application: Surface reconstruction



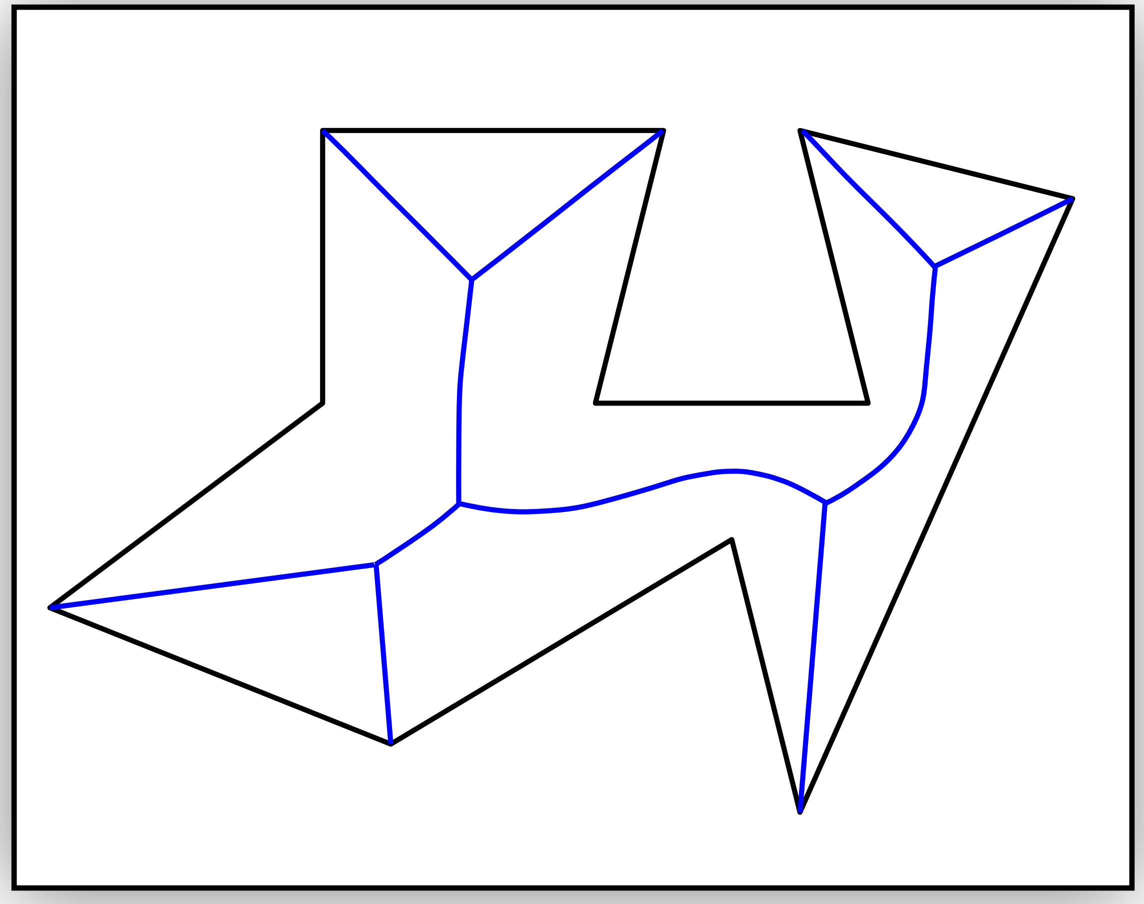


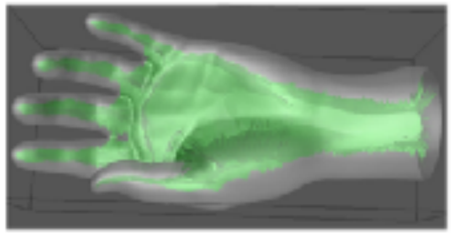


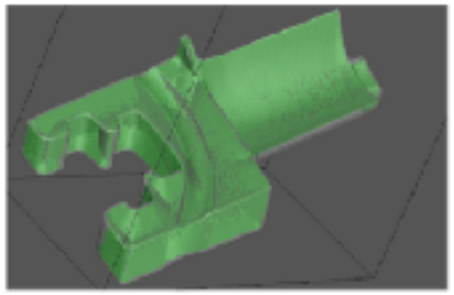
Technische Universität Braunschweig

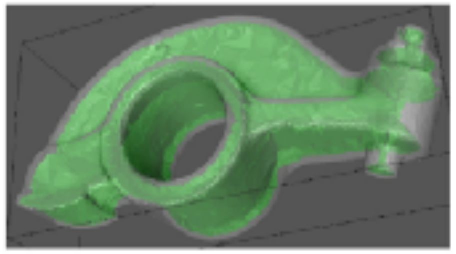
Medial axis of a simple polygon

- Analogously: polygon instead of $\mathcal{P}$
- Voronoi edges and vertices: two or three nearest neighbors on polygon boundary
- Computation for simple polygons in linear(!) time [Chin *et al.*, 1995].
- Application: Surface reconstruction



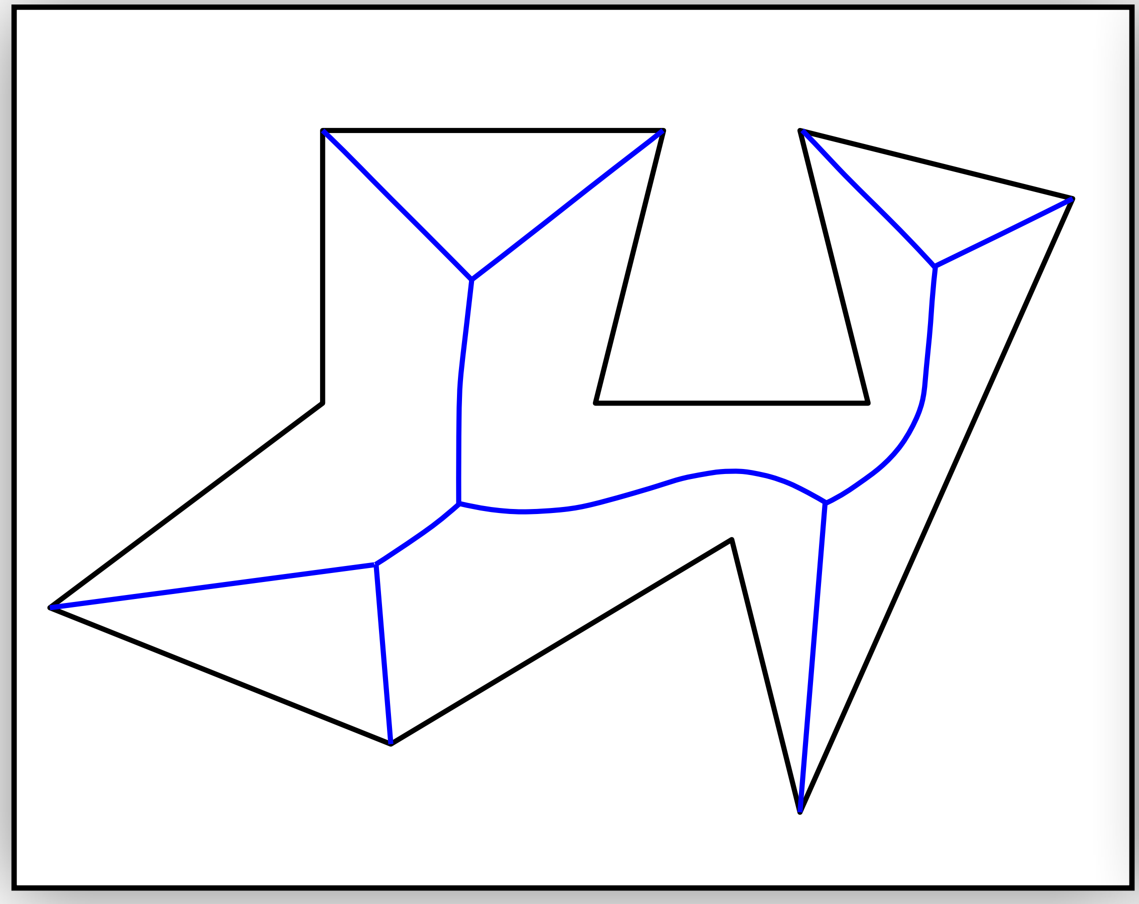


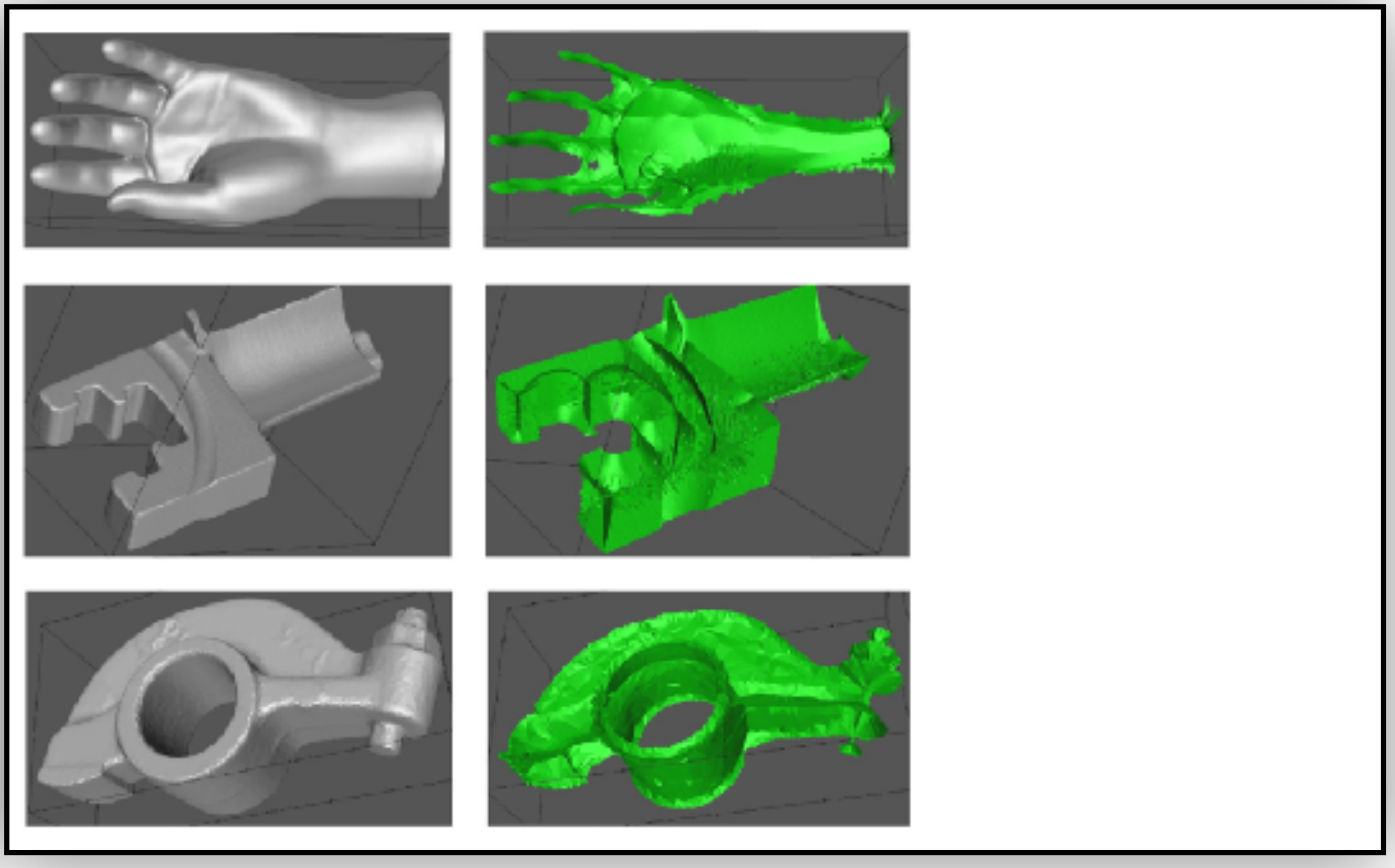




Medial axis of a simple polygon

- Analogously: polygon instead of $\mathcal{P}$
- Voronoi edges and vertices: two or three nearest neighbors on polygon boundary
- Computation for simple polygons in linear(!) time [Chin *et al.*, 1995].
- Application: Surface reconstruction





Technische Universität Braunschweig

Straight skeleton of a simple polygon:

Straight skeleton of a simple polygon:

- Voronoi edges and vertices: Intersection points of parallel wave fronts [Aichholzer *et al.*, 1995].

Straight skeleton of a simple polygon:

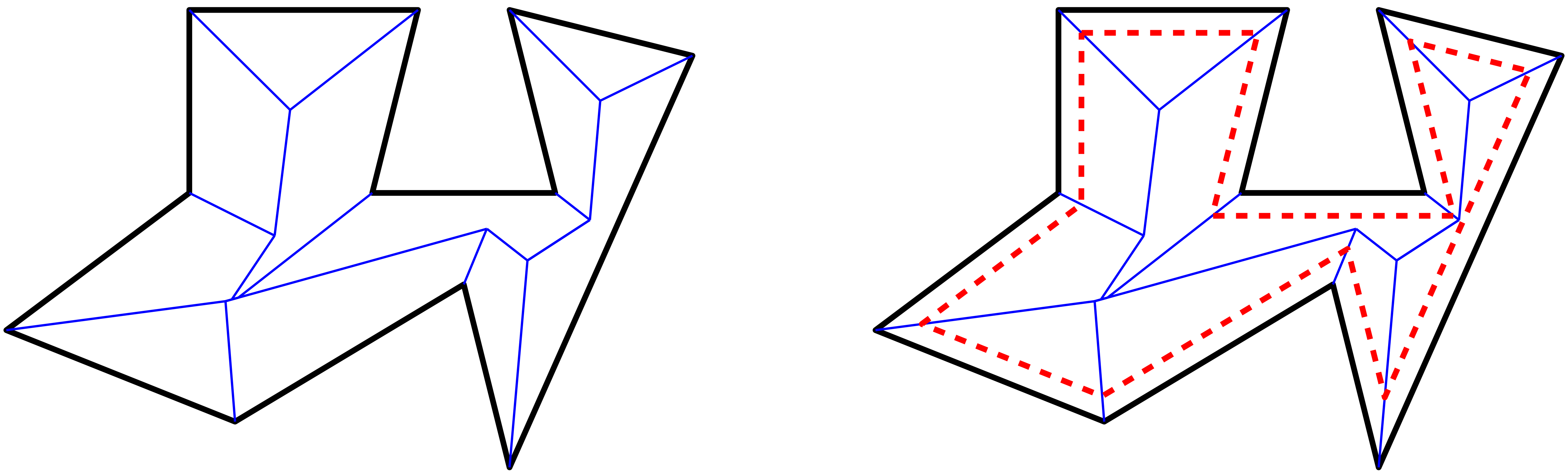
- Voronoi edges and vertices: Intersection points of parallel wave fronts [Aichholzer *et al.*, 1995].

- Computation in $\mathcal{O}(mn + n \log n)$ with $m =:$ # reflex vertices [Felkel and Obdrzalek, 1998].

Technische Universität Braunschweig

Straight skeleton of a simple polygon:

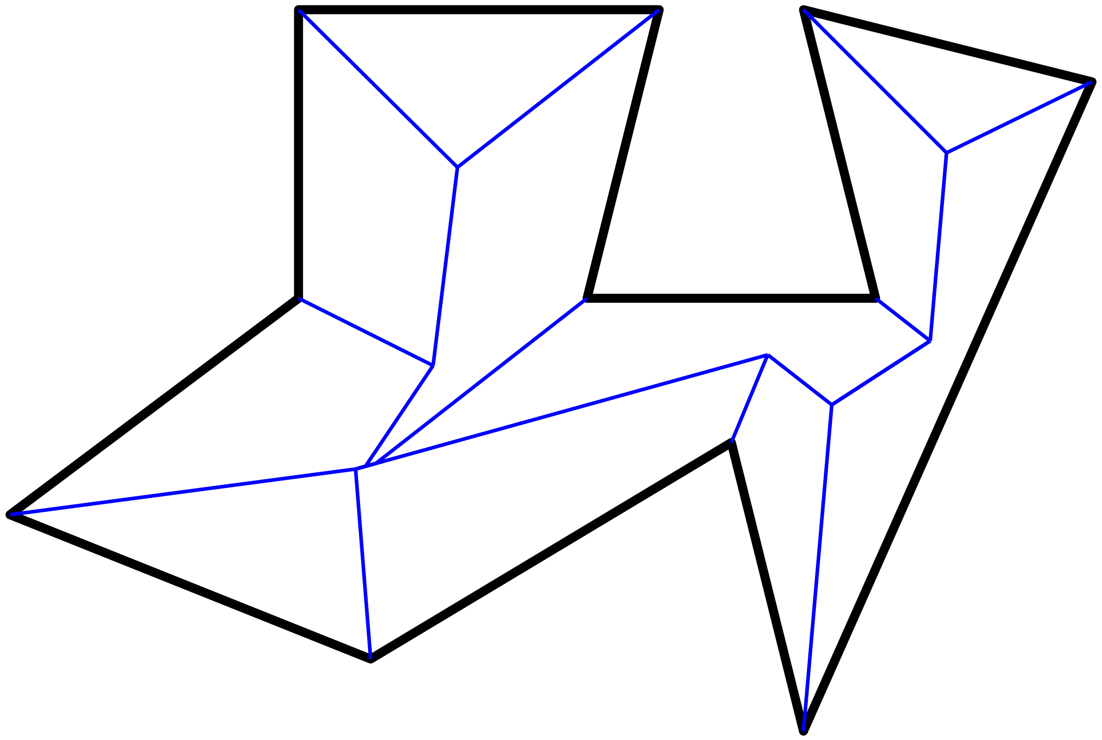
- Voronoi edges and vertices: Intersection points of parallel wave fronts [Aichholzer *et al.*, 1995].



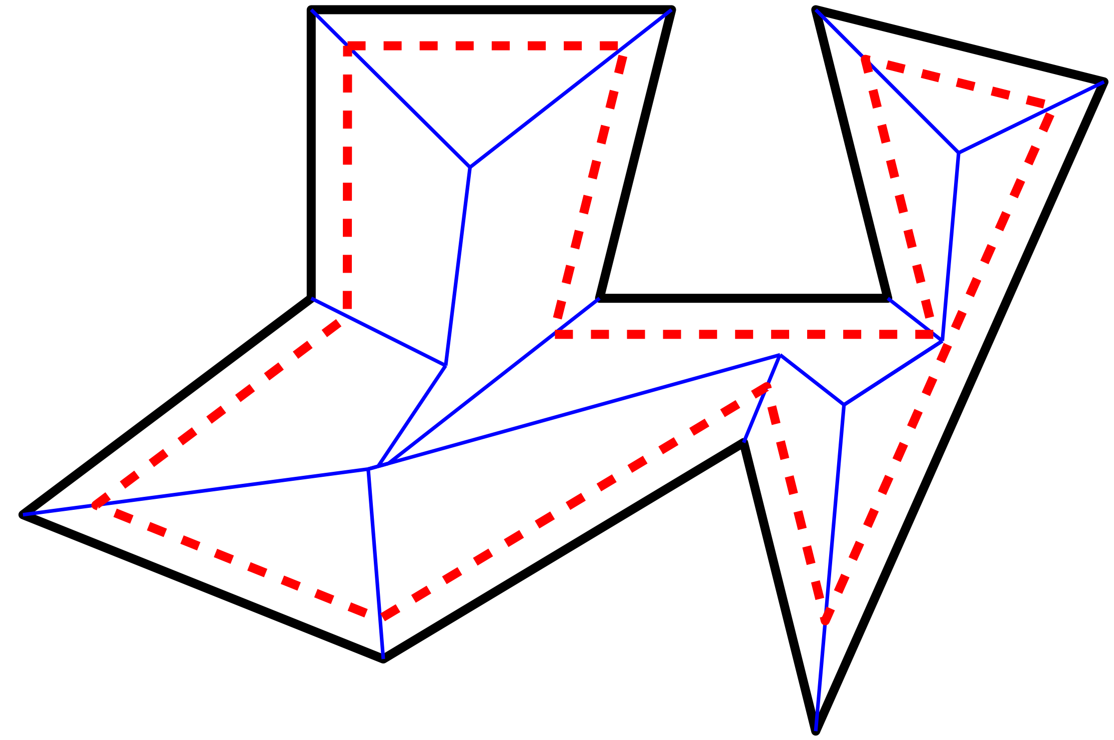
- Computation in $\mathcal{O}(mn + n \log n)$ with $m =$: # reflex vertices [Felkel and Obdrzalek, 1998].

Straight skeleton of a simple polygon:

- Voronoi edges and vertices: Intersection points of parallel wave fronts [Aichholzer *et al.*, 1995].



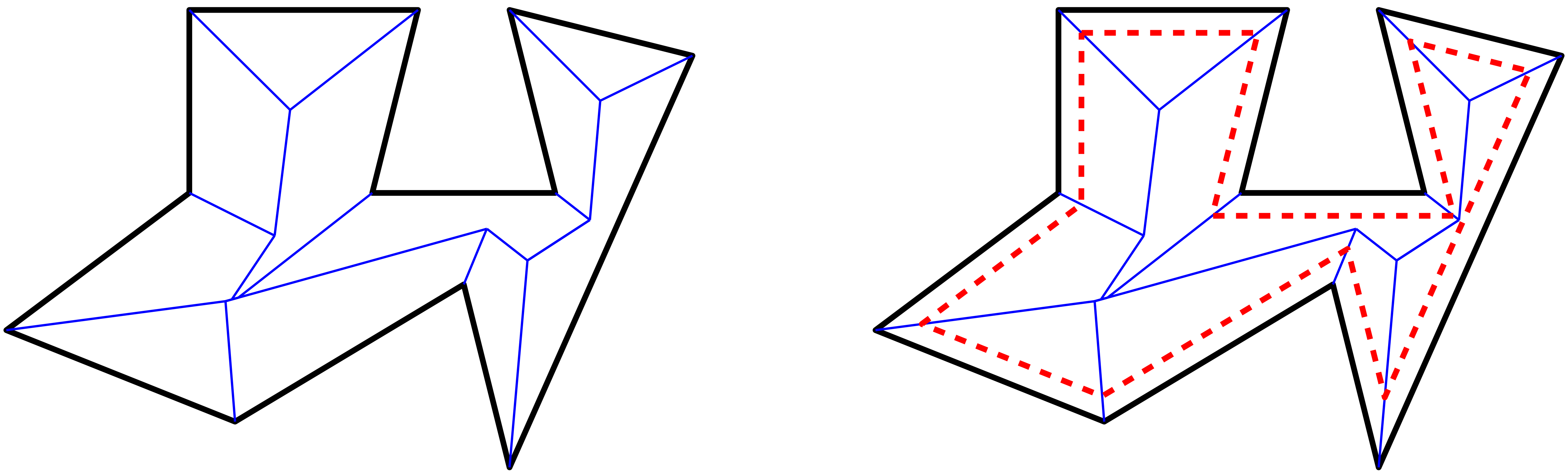
Straight Skeleton



- Computation in $\mathcal{O}(mn + n \log n)$ with $m =:$ # reflex vertices [Felkel and Obdrzalek, 1998].

Straight skeleton of a simple polygon:

- Voronoi edges and vertices: Intersection points of parallel wave fronts [Aichholzer *et al.*, 1995].



Straight Skeleton Parallel wave front

- Computation in $\mathcal{O}(mn + n \log n)$ with $m =:$ # reflex vertices [Felkel and Obdrzalek, 1998].

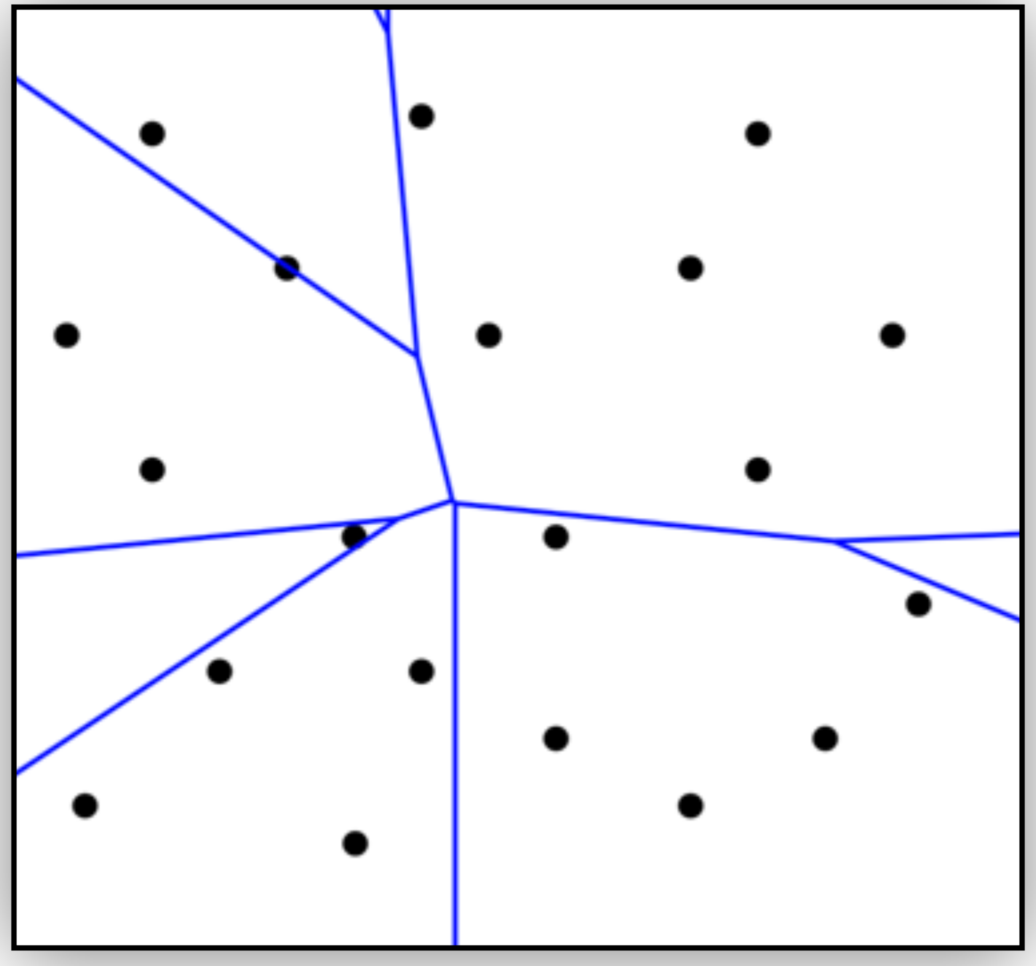
Furthest-point Voronoi diagram:

Furthest-point Voronoi diagram:

- Voronoi region: Set of points with same furthest site $p \in \mathcal{P}$.

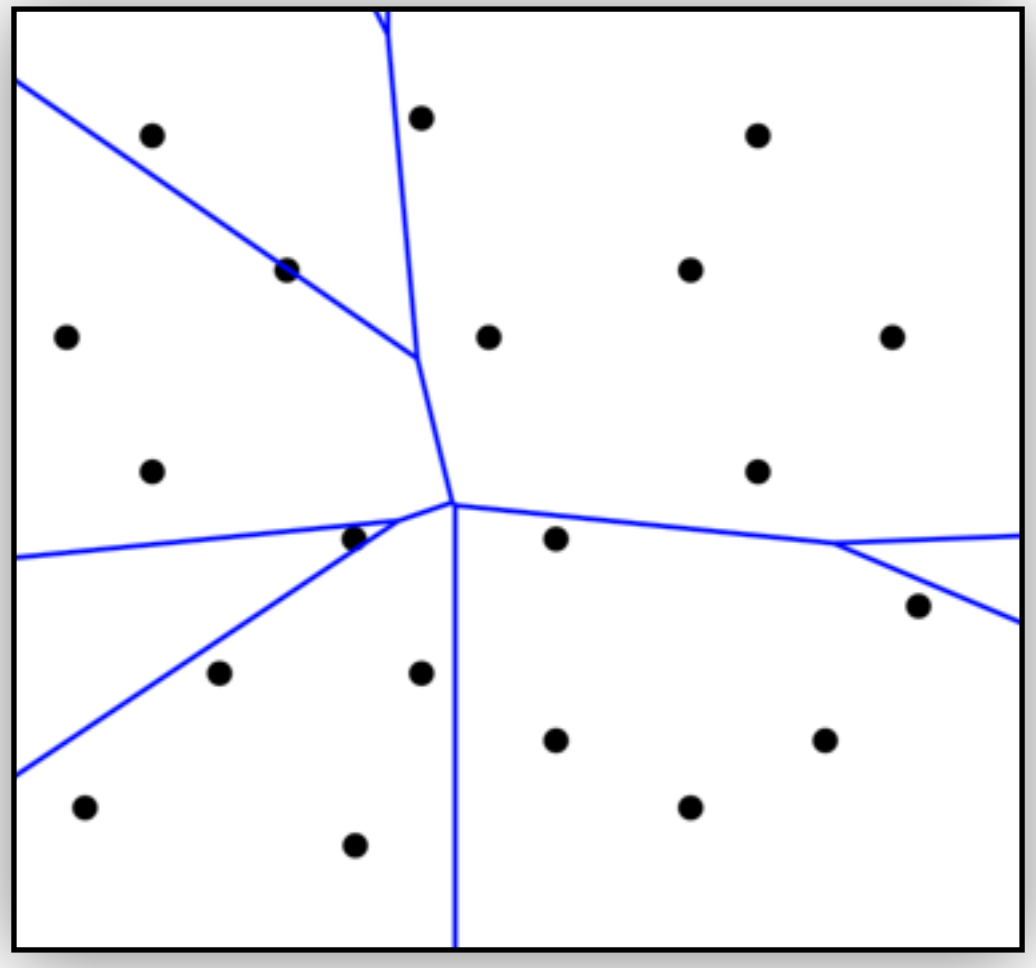
Furthest-point Voronoi diagram:

- Voronoi region: Set of points with same furthest site $p \in \mathcal{P}$.



Furthest-point Voronoi diagram:

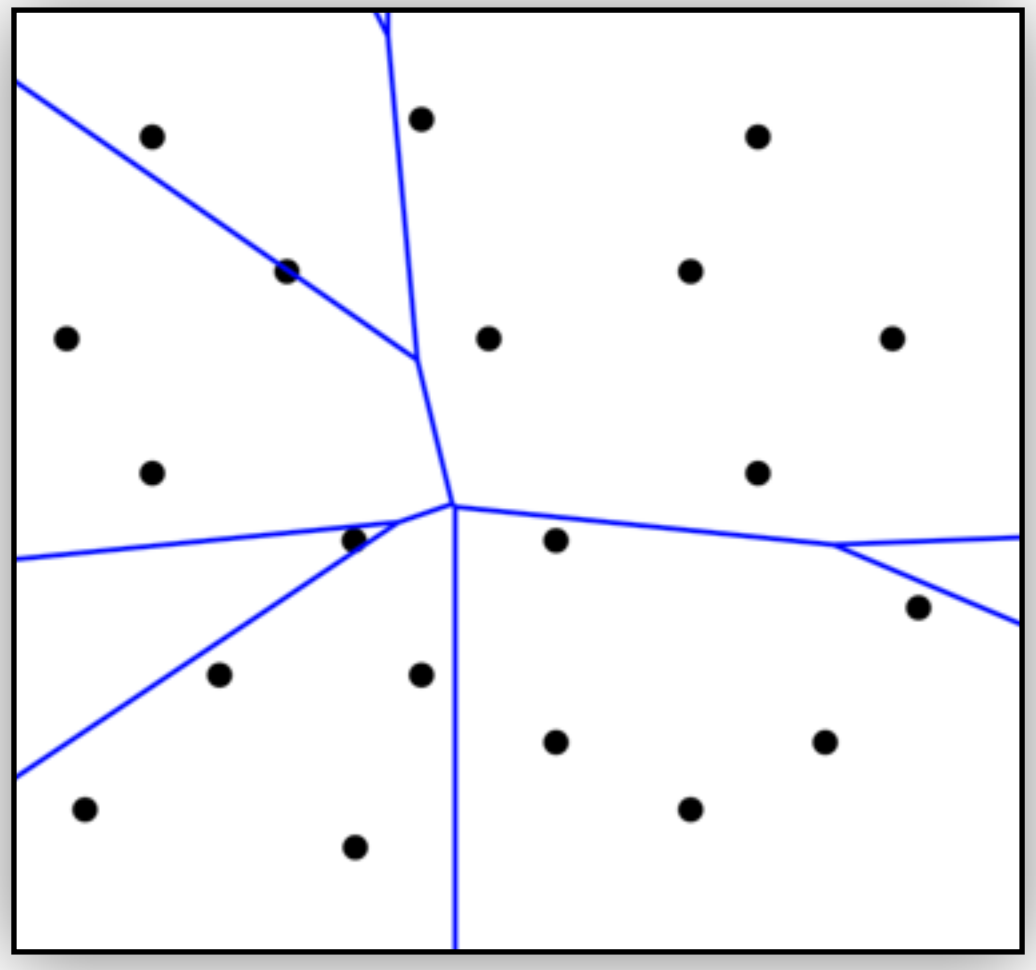
- Voronoi region: Set of points with same furthest site $p \in \mathcal{P}$.

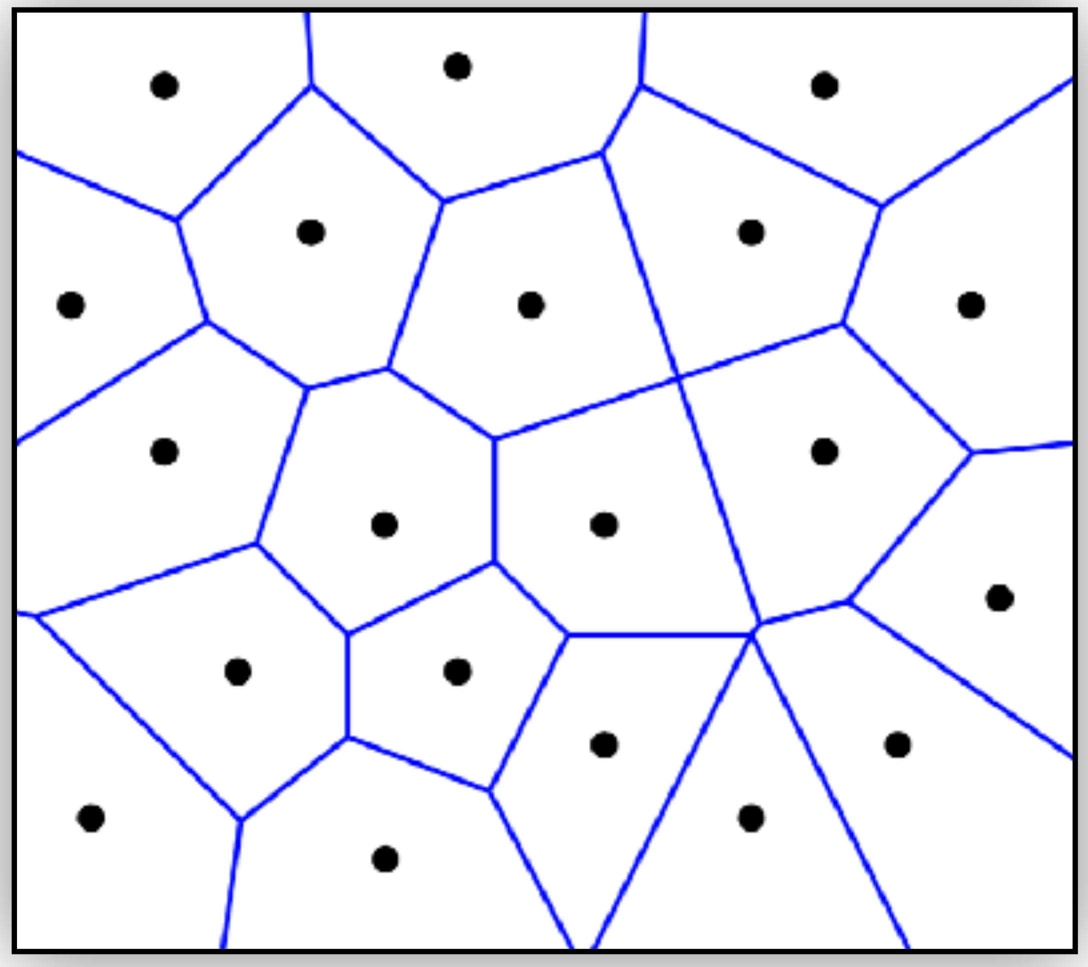


Furthtest-point Voronoi diagram

Furthest-point Voronoi diagram:

- Voronoi region: Set of points with same furthest site $p \in \mathcal{P}$.



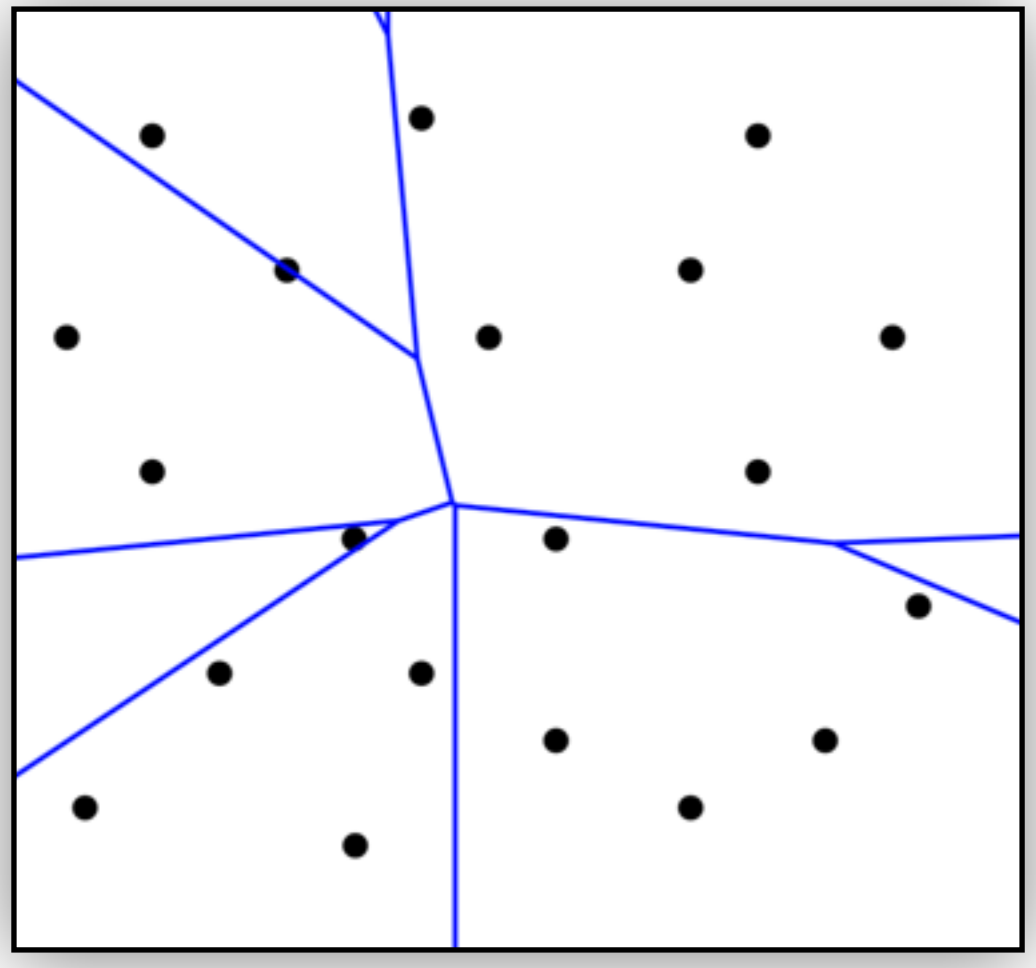


Furthtest-point Voronoi diagram

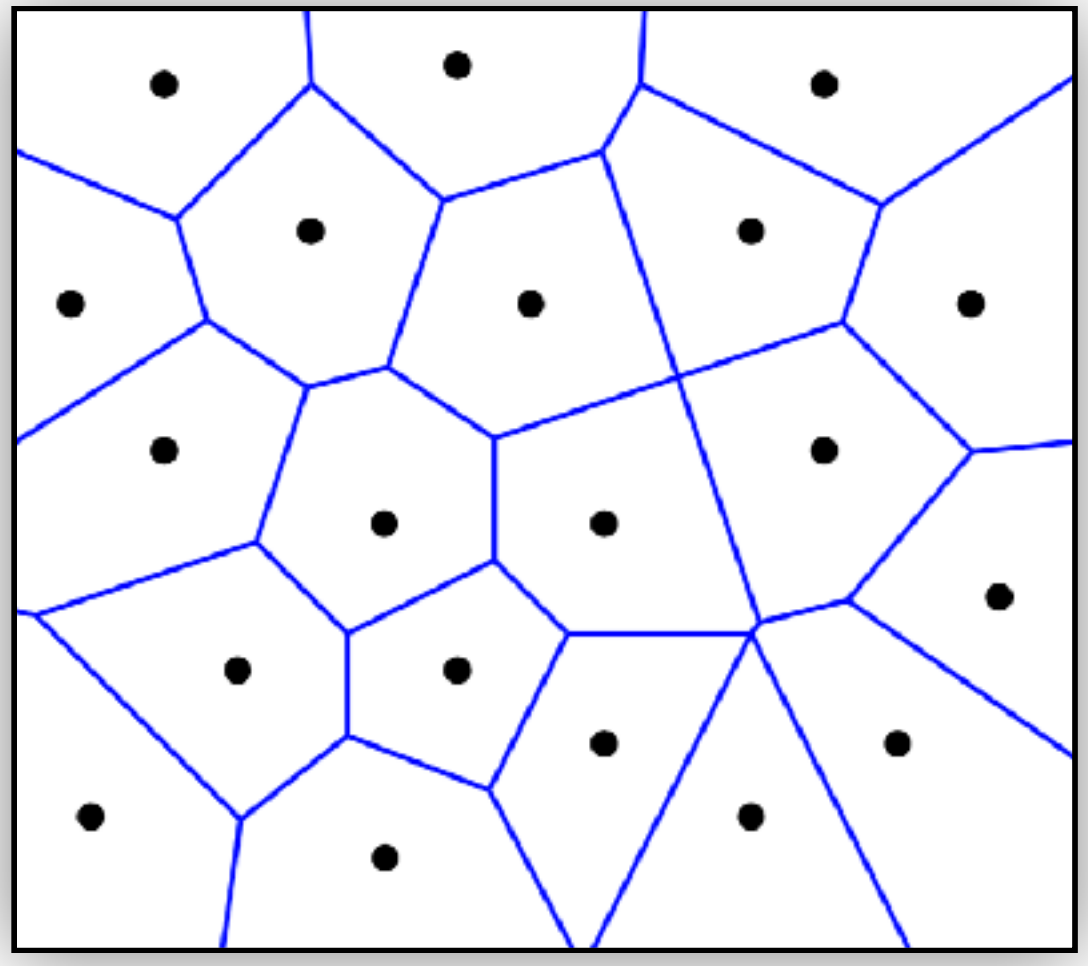
Technische Universität Braunschweig

Furthest-point Voronoi diagram:

- Voronoi region: Set of points with same furthest site $p \in \mathcal{P}$.



Furthtest-point Voronoi diagram

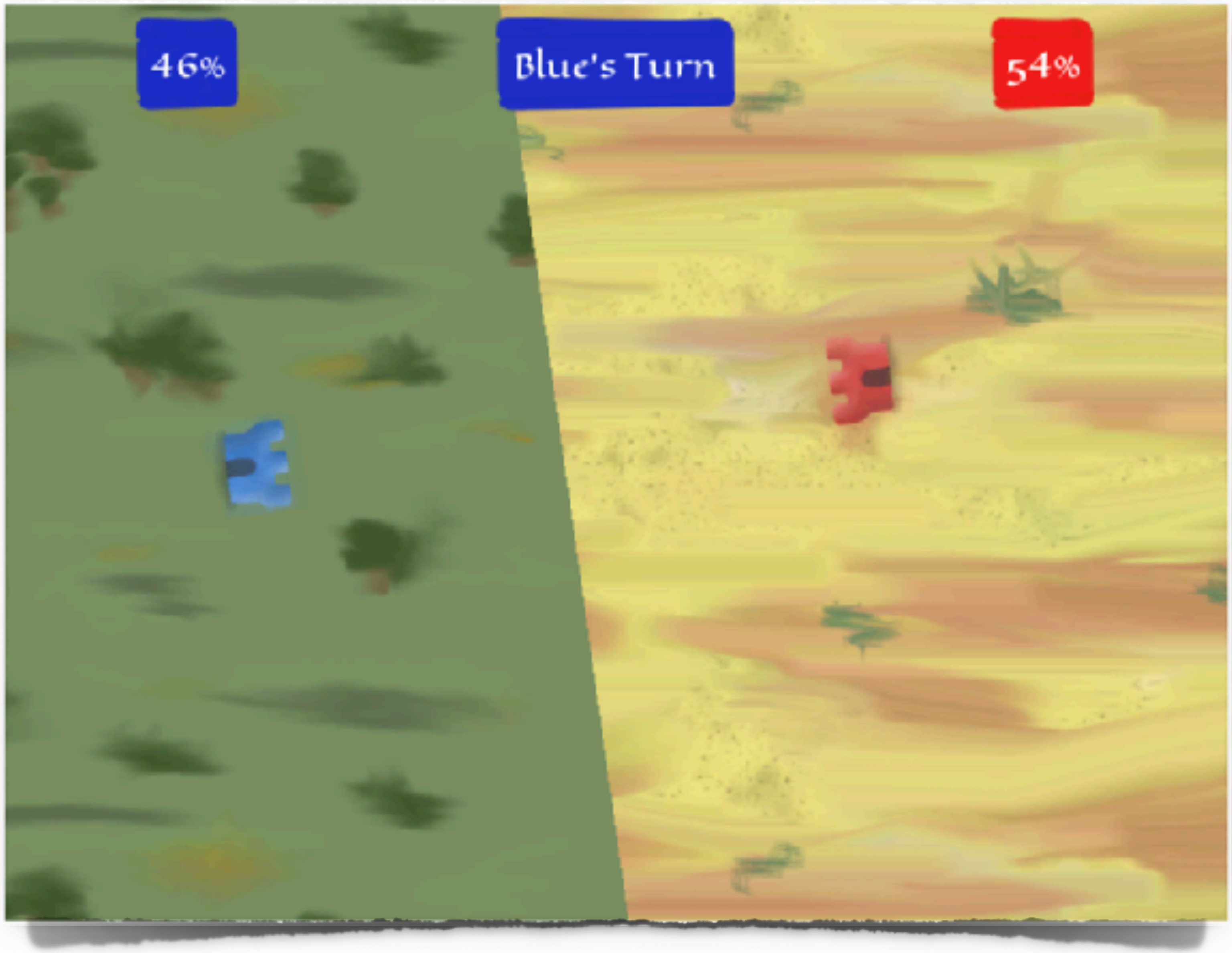


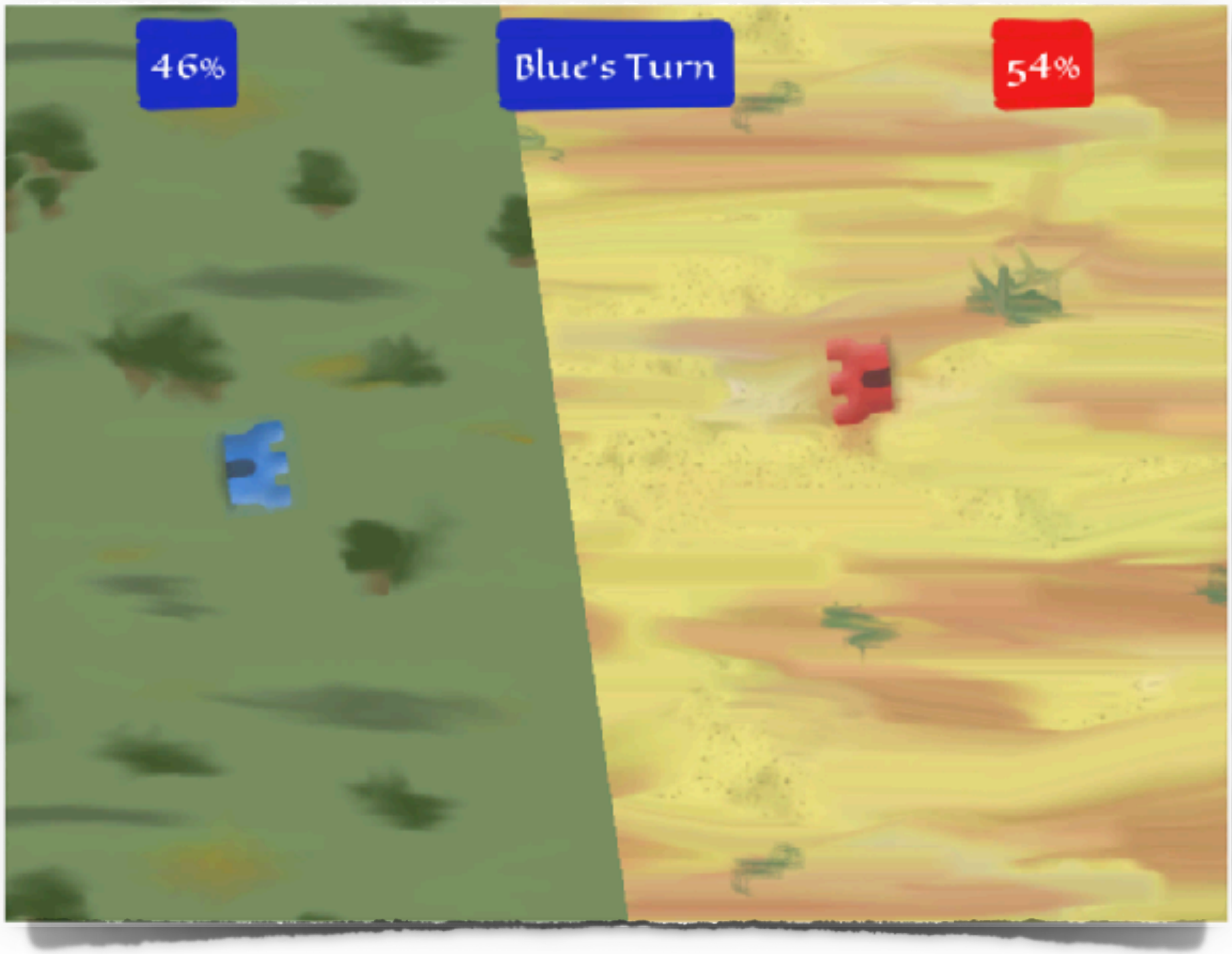
Voronoi diagram

1. Introduction and Motivation
2. Definitions
3. Representing planar partitions
4. Properties
5. Fortune's algorithm
6. Variations
7. The Voronoi game
8. Summary and conclusions

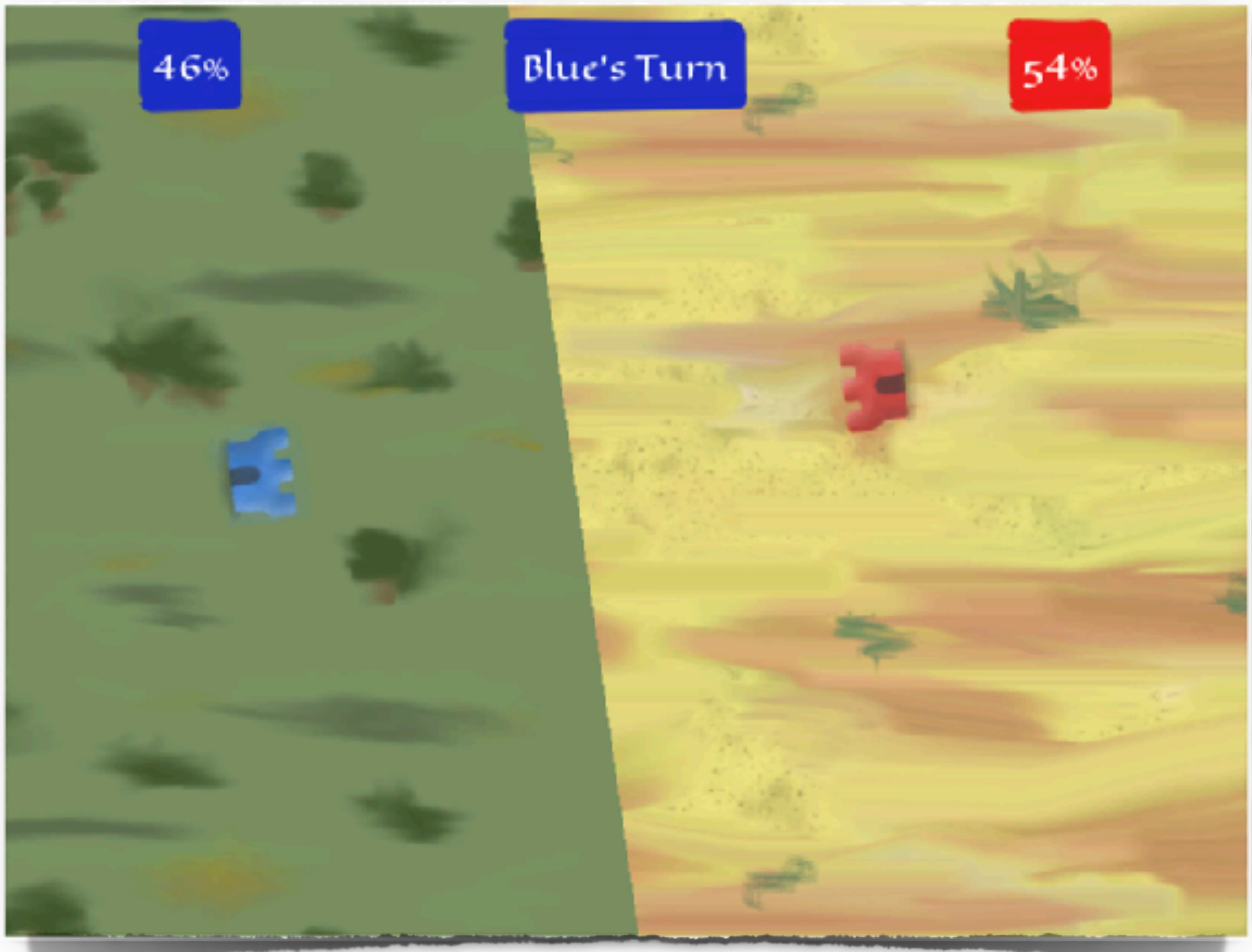
Technische Universität Braunschweig

Ruler of the Plane

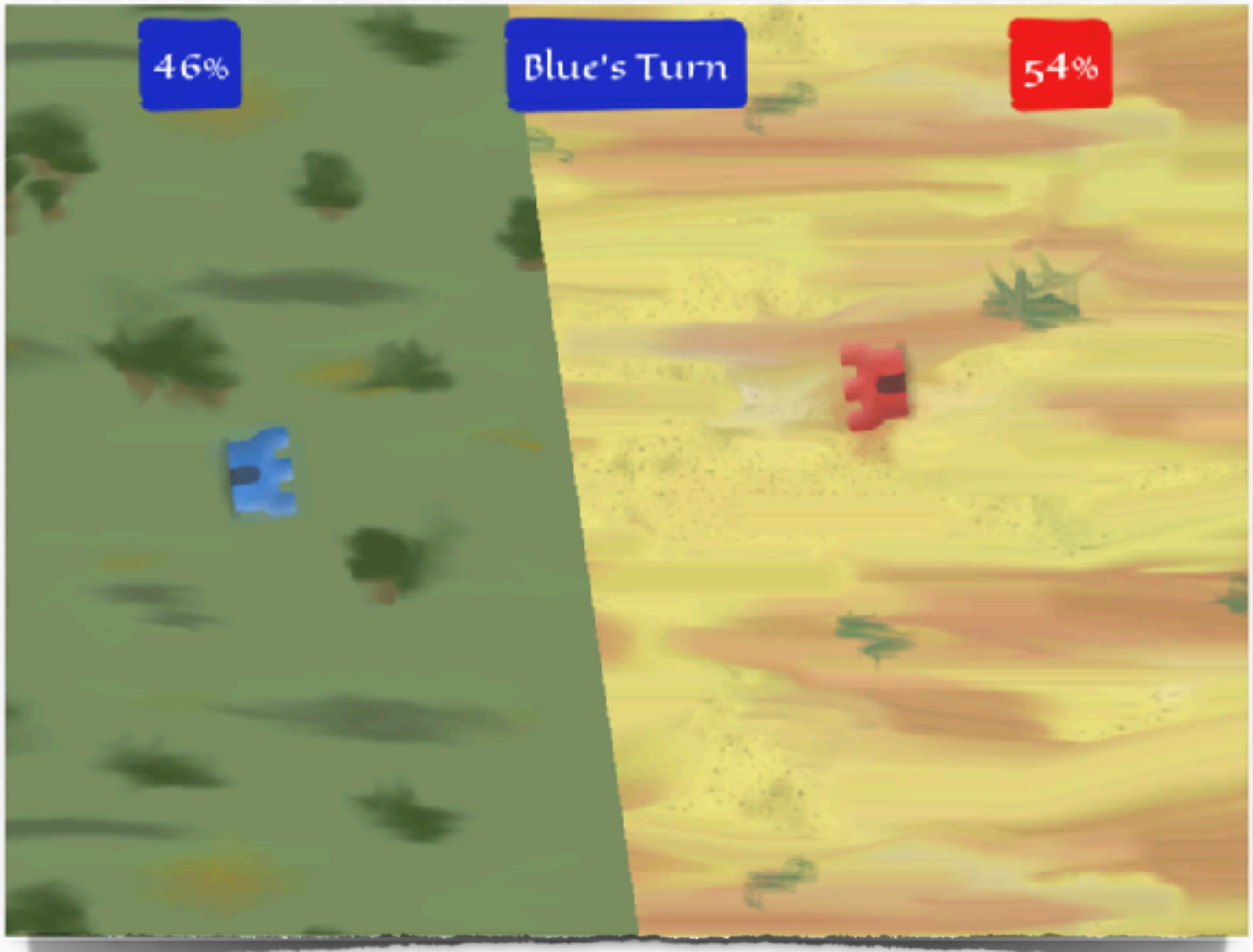

46%
Blue's Turn
54%



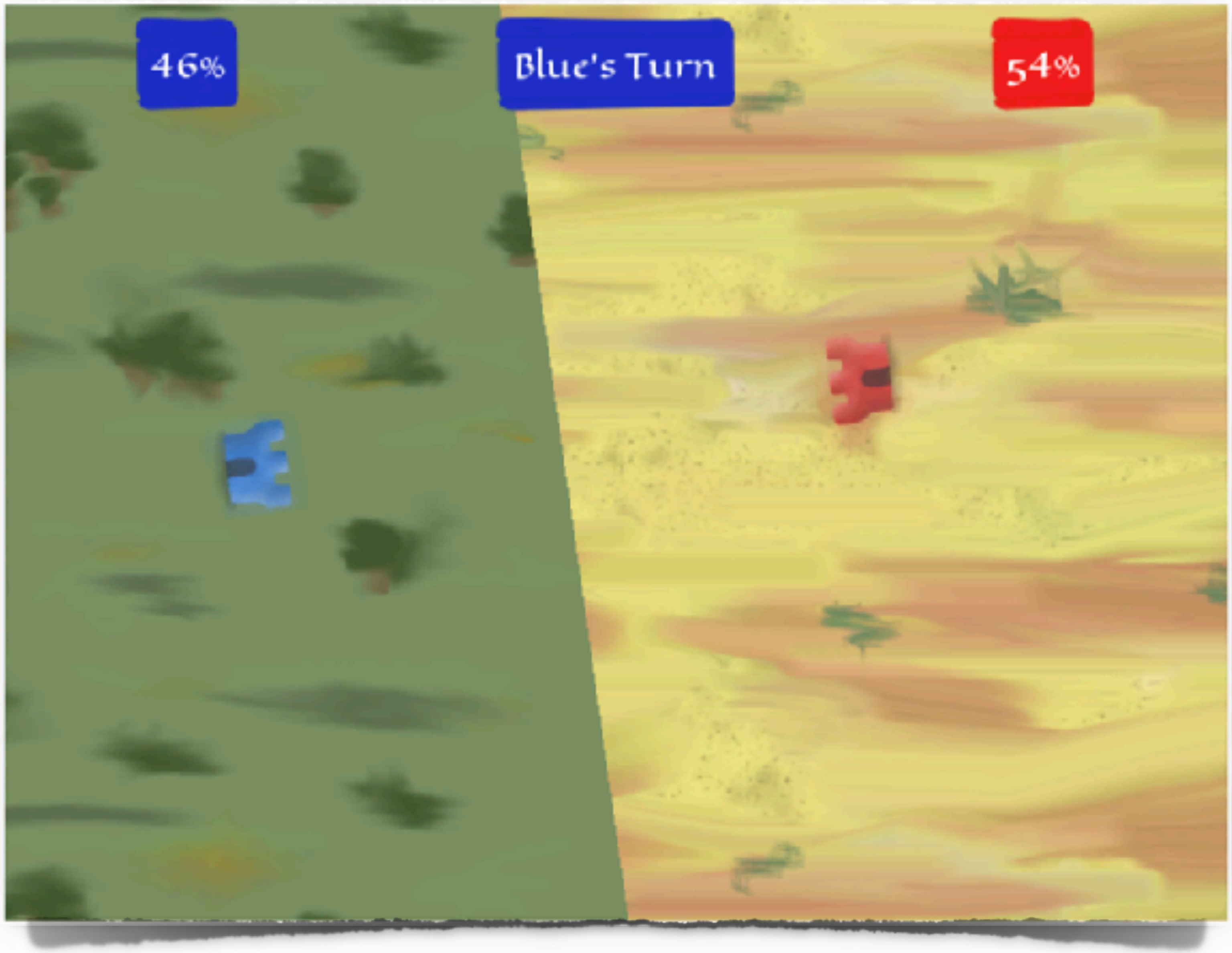
- **A domain**



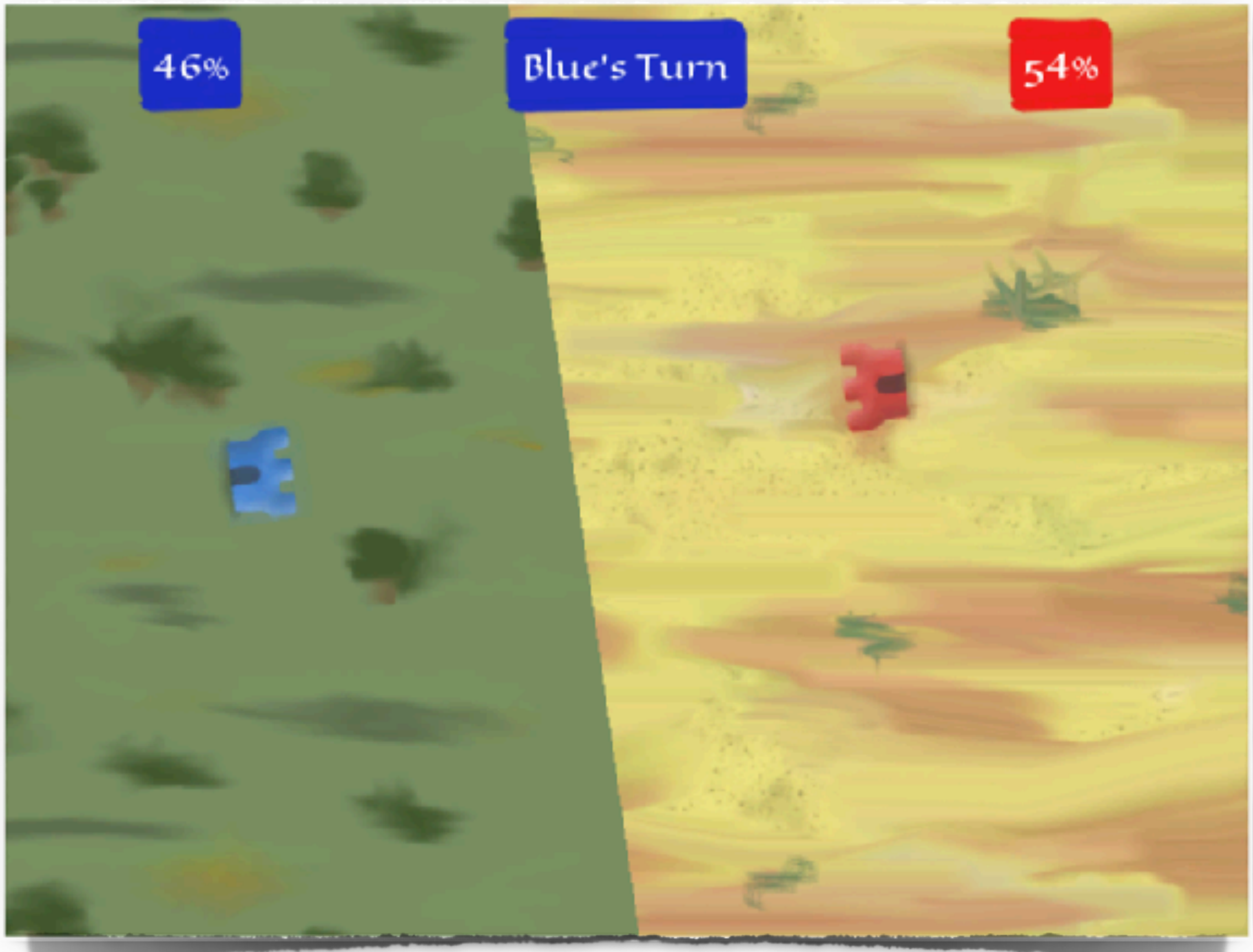
- A domain
- Two players



- A domain
- Two players
- Players take turns



- A domain
- Two players
- Players take turns
- Voronoi diagram is computed



- A domain
- Two players
- Players take turns
- Voronoi diagram is computed
- Player with larger area wins

1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]

Competitive Facility Location along a Highway*

Hee-Kap Ahn[1], Siu-Wing Cheng[2], Otfried Cheong[1],
Mordecai Golin[2], and René van Oostrum[1]

[1] Department of Computer Science, Utrecht University, Netherlands,
{heekap,otfried,rene}@cs.uu.nl
[2] Department of Computer Science, HKUST, Hong Kong,
{scheng,golin}@cs.ust.hk

Abstract. We consider a competitive facility location problem with two players. Players alternate placing points, one at a time, into the playing arena, until each of them has placed n points. The arena is then subdivided according to the nearest-neighbor rule, and the player whose points control the larger area wins. We present a winning strategy for the second player, where the arena is a circle or a line segment.

1 Introduction

The classical facility location problem [5] asks for the optimum location of a new facility (police station, super market, transmitter, etc.) with respect to a given set of customers. Typically, the function to be optimized is the maximum distance from customers to the facility — this results in the minimum enclosing disk problem studied by Megiddo [8], Welzl [12] and Aronov et al. [2].

Competitive facility location deals with the placement of sites by competing market players. Geometric arguments are combined with arguments from *game theory* to see how the behavior of these decision makers affect each other. Competitive location models have been studied in many different fields, such as spatial economics and industrial organization [1,9], mathematics [6] and operations research [3,7,11]. Comprehensive overviews of competitive facility locations models are the surveys by Friesz et al. [11], Eiselt and Laporte [3] and Eiselt et al. [4].

We consider a model where the behavior of the customers is deterministic in the sense that a facility can determine the set of customers more attracted to it than to any other facility. This set is called the *market area* of the facility. The collection of market areas forms a tessellation of the underlying space. If customers choose the facility on the basis of distance in some metric, the tessellation is the Voronoi Diagram of the set of facilities [10].

We address a competitive facility location problem that we call the *Voronoi Game*. It is played by two players, Blue and Red, who place a specified number, n,

* Part of the work was done while the first, third, and fifth authors were at the Dept. of Computer Science, HKUST, Hong Kong. The work described in this paper has been supported by the Research Grants Council of Hong Kong, China (HKUST6074/97E, HKUST6088/99E, HKUST6094/99E, HKUST6162/00E, and HKUST6137/98E).

J. Wang (Ed.): COCOON 2001, LNCS 2108, pp. 237–246, 2001.

Technische Universität Braunschweig

1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]

Competitive Facility Location along a Highway*

Hee-Kap Ahn[1], Siu-Wing Cheng[2], Otfried Cheong[1], Mordecai Golin[2], and René van Oostrum[1]

[1] Department of Computer Science, Utrecht University, Netherlands, {heekap,otfried,rene}@cs.uu.nl
[2] Department of Computer Science, HKUST, Hong Kong, {scheng,golin}@cs.ust.hk

Abstract. We consider a competitive facility location problem with two players. Players alternate placing points, one at a time, into the playing arena, until each of them has placed n points. The arena is then subdivided according to the nearest-neighbor rule, and the player whose points control the larger area wins. We present a winning strategy for the second player, where the arena is a circle or a line segment.

1 Introduction

The classical facility location problem [5] asks for the optimum location of a new facility (police station, super market, transmitter, etc.) with respect to a given set of customers. Typically, the function to be optimized is the maximum distance from customers to the facility — this results in the minimum enclosing disk problem studied by Megiddo [8], Welzl [12] and Aronov et al. [2].

Competitive facility location deals with the placement of sites by competing market players. Geometric arguments are combined with arguments from *game theory* to see how the behavior of these decision makers affect each other. Competitive location models have been studied in many different fields, such as spatial economics and industrial organization [1,9], mathematics [6] and operations research [3,7,11]. Comprehensive overviews of competitive facility locations models are the surveys by Friesz et al. [11], Eiselt and Laporte [3] and Eiselt et al. [4].

We consider a model where the behavior of the customers is deterministic in the sense that a facility can determine the set of customers more attracted to it than to any other facility. This set is called the *market area* of the facility. The collection of market areas forms a tessellation of the underlying space. If customers choose the facility on the basis of distance in some metric, the tessellation is the Voronoi Diagram of the set of facilities [10].

We address a competitive facility location problem that we call the *Voronoi Game*. It is played by two players, Blue and Red, who place a specified number, n,

* Part of the work was done while the first, third, and fifth authors were at the Dept. of Computer Science, HKUST, Hong Kong. The work described in this paper has been supported by the Research Grants Council of Hong Kong, China (HKUST6074/97E, HKUST6088/99E, HKUST6094/99E, HKUST6162/00E, and HKUST6137/98E).

J. Wang (Ed.): COCOON 2001, LNCS 2108, pp. 237–246, 2001.

The most natural Voronoi Game is played in a two-dimensional arena U using the Euclidean metric. Unfortunately nobody knows how to win this game, even for very restricted regions U. In this note we present strategies for winning one-dimensional versions of the game, where the arena is a circle or a line segment, and variations. In other words, we consider competitive facility location along an Australian highway.

1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]

Competitive Facility Location along a Highway*

Hee-Kap Ahn[1], Siu-Wing Cheng[2], Otfried Cheong[1], Mordecai Golin[2], and René van Oostrum[1]

[1] Department of Computer Science, Utrecht University, Netherlands, {heekap,otfried,rene}@cs.uu.nl
[2] Department of Computer Science, HKUST, Hong Kong, {scheng,golin}@cs.ust.hk

Abstract. We consider a competitive facility location problem with two players. Players alternate placing points, one at a time, into the playing arena, until each of them has placed n points. The arena is then subdivided according to the nearest-neighbor rule, and the player whose points control the larger area wins. We present a winning strategy for the second player, where the arena is a circle or a line segment.

1 Introduction

The classical facility location problem [5] asks for the optimum location of a new facility (police station, super market, transmitter, etc.) with respect to a given set of customers. Typically, the function to be optimized is the maximum distance from customers to the facility — this results in the minimum enclosing disk problem studied by Megiddo [8], Welzl [12] and Aronov et al. [2].

Competitive facility location deals with the placement of sites by competing market players. Geometric arguments are combined with arguments from *game theory* to see how the behavior of these decision makers affect each other. Competitive location models have been studied in many different fields, such as spatial economics and industrial organization [1,9], mathematics [6] and operations research [3,7,11]. Comprehensive overviews of competitive facility locations models are the surveys by Friesz et al. [11], Eiselt and Laporte [3] and Eiselt et al. [4].

We consider a model where the behavior of the customers is deterministic in the sense that a facility can determine the set of customers more attracted to it than to any other facility. This set is called the *market area* of the facility. The collection of market areas forms a tessellation of the underlying space. If customers choose the facility on the basis of distance in some metric, the tessellation is the Voronoi Diagram of the set of facilities [10].

We address a competitive facility location problem that we call the *Voronoi Game*. It is played by two players, Blue and Red, who place a specified number, n,

* Part of the work was done while the first, third, and fifth authors were at the Dept. of Computer Science, HKUST, Hong Kong. The work described in this paper has been supported by the Research Grants Council of Hong Kong, China (HKUST6074/97E, HKUST6088/99E, HKUST6094/99E, HKUST6162/00E, and HKUST6137/98E).

J. Wang (Ed.): COCOON 2001, LNCS 2108, pp. 237–246, 2001.

The most natural Voronoi Game is played in a two-dimensional arena U using the Euclidean metric. Unfortunately nobody knows how to win this game, even for very restricted regions U. In this note we present strategies for winning one-dimensional versions of the game, where the arena is a circle or a line segment, and variations. In other words, we consider competitive facility location along an Australian highway.

- **Blue and Red**

1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]

Competitive Facility Location along a Highway*

Hee-Kap Ahn[1], Siu-Wing Cheng[2], Otfried Cheong[1], Mordecai Golin[2], and René van Oostrum[1]

[1] Department of Computer Science, Utrecht University, Netherlands, {heekap,otfried,rene}@cs.uu.nl
[2] Department of Computer Science, HKUST, Hong Kong, {scheng,golin}@cs.ust.hk

Abstract. We consider a competitive facility location problem with two players. Players alternate placing points, one at a time, into the playing arena, until each of them has placed n points. The arena is then subdivided according to the nearest-neighbor rule, and the player whose points control the larger area wins. We present a winning strategy for the second player, where the arena is a circle or a line segment.

1 Introduction

The classical facility location problem [5] asks for the optimum location of a new facility (police station, super market, transmitter, etc.) with respect to a given set of customers. Typically, the function to be optimized is the maximum distance from customers to the facility — this results in the minimum enclosing disk problem studied by Megiddo [8], Welzl [12] and Aronov et al. [2].

Competitive facility location deals with the placement of sites by competing market players. Geometric arguments are combined with arguments from *game theory* to see how the behavior of these decision makers affect each other. Competitive location models have been studied in many different fields, such as spatial economics and industrial organization [1,9], mathematics [6] and operations research [3,7,11]. Comprehensive overviews of competitive facility locations models are the surveys by Friesz et al. [11], Eiselt and Laporte [3] and Eiselt et al. [4].

We consider a model where the behavior of the customers is deterministic in the sense that a facility can determine the set of customers more attracted to it than to any other facility. This set is called the *market area* of the facility. The collection of market areas forms a tessellation of the underlying space. If customers choose the facility on the basis of distance in some metric, the tessellation is the Voronoi Diagram of the set of facilities [10].

We address a competitive facility location problem that we call the *Voronoi Game*. It is played by two players, Blue and Red, who place a specified number, n,

* Part of the work was done while the first, third, and fifth authors were at the Dept. of Computer Science, HKUST, Hong Kong. The work described in this paper has been supported by the Research Grants Council of Hong Kong, China (HKUST6074/97E, HKUST6088/99E, HKUST6094/99E, HKUST6162/00E, and HKUST6137/98E).

J. Wang (Ed.): COCOON 2001, LNCS 2108, pp. 237–246, 2001.

The most natural Voronoi Game is played in a two-dimensional arena U using the Euclidean metric. Unfortunately nobody knows how to win this game, even for very restricted regions U. In this note we present strategies for winning one-dimensional versions of the game, where the arena is a circle or a line segment, and variations. In other words, we consider competitive facility location along an Australian highway.

- **Blue and Red**
- $n > 1$

Technische Universität Braunschweig

Competitive Facility Location along a Highway*

Hee-Kap Ahn[1], Siu-Wing Cheng[2], Otfried Cheong[1], Mordecai Golin[2], and René van Oostrum[1]

[1] Department of Computer Science, Utrecht University, Netherlands, {heekap,otfried,rene}@cs.uu.nl
[2] Department of Computer Science, HKUST, Hong Kong, {scheng,golin}@cs.ust.hk

Abstract. We consider a competitive facility location problem with two players. Players alternate placing points, one at a time, into the playing arena, until each of them has placed n points. The arena is then subdivided according to the nearest-neighbor rule, and the player whose points control the larger area wins. We present a winning strategy for the second player, where the arena is a circle or a line segment.

1 Introduction

The classical facility location problem [5] asks for the optimum location of a new facility (police station, super market, transmitter, etc.) with respect to a given set of customers. Typically, the function to be optimized is the maximum distance from customers to the facility — this results in the minimum enclosing disk problem studied by Megiddo [8], Welzl [12] and Aronov et al. [2].

Competitive facility location deals with the placement of sites by competing market players. Geometric arguments are combined with arguments from *game theory* to see how the behavior of these decision makers affect each other. Competitive location models have been studied in many different fields, such as spatial economics and industrial organization [1,9], mathematics [6] and operations research [3,7,11]. Comprehensive overviews of competitive facility locations models are the surveys by Friesz et al. [11], Eiselt and Laporte [3] and Eiselt et al. [4].

We consider a model where the behavior of the customers is deterministic in the sense that a facility can determine the set of customers more attracted to it than to any other facility. This set is called the *market area* of the facility. The collection of market areas forms a tessellation of the underlying space. If customers choose the facility on the basis of distance in some metric, the tessellation is the Voronoi Diagram of the set of facilities [10].

We address a competitive facility location problem that we call the *Voronoi Game*. It is played by two players, Blue and Red, who place a specified number, n,

* Part of the work was done while the first, third, and fifth authors were at the Dept. of Computer Science, HKUST, Hong Kong. The work described in this paper has been supported by the Research Grants Council of Hong Kong, China (HKUST6074/97E, HKUST6088/99E, HKUST6094/99E, HKUST6162/00E, and HKUST6137/98E).

J. Wang (Ed.): COCOON 2001, LNCS 2108, pp. 237–246, 2001.

The most natural Voronoi Game is played in a two-dimensional arena U using the Euclidean metric. Unfortunately nobody knows how to win this game, even for very restricted regions U. In this note we present strategies for winning one-dimensional versions of the game, where the arena is a circle or a line segment, and variations. In other words, we consider competitive facility location along an Australian highway.

- **Blue and Red**
- $n > 1$
- **Circle or line segment**

1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]

Competitive Facility Location along a Highway*

Hee-Kap Ahn[1], Siu-Wing Cheng[2], Otfried Cheong[1], Mordecai Golin[2], and René van Oostrum[1]

[1] Department of Computer Science, Utrecht University, Netherlands, {heekap,otfried,rene}@cs.uu.nl
[2] Department of Computer Science, HKUST, Hong Kong, {scheng,golin}@cs.ust.hk

Abstract. We consider a competitive facility location problem with two players. Players alternate placing points, one at a time, into the playing arena, until each of them has placed n points. The arena is then subdivided according to the nearest-neighbor rule, and the player whose points control the larger area wins. We present a winning strategy for the second player, where the arena is a circle or a line segment.

1 Introduction

The classical facility location problem [5] asks for the optimum location of a new facility (police station, super market, transmitter, etc.) with respect to a given set of customers. Typically, the function to be optimized is the maximum distance from customers to the facility — this results in the minimum enclosing disk problem studied by Megiddo [8], Welzl [12] and Aronov et al. [2].

Competitive facility location deals with the placement of sites by competing market players. Geometric arguments are combined with arguments from *game theory* to see how the behavior of these decision makers affect each other. Competitive location models have been studied in many different fields, such as spatial economics and industrial organization [1,9], mathematics [6] and operations research [3,7,11]. Comprehensive overviews of competitive facility locations models are the surveys by Friesz et al. [11], Eiselt and Laporte [3] and Eiselt et al. [4].

We consider a model where the behavior of the customers is deterministic in the sense that a facility can determine the set of customers more attracted to it than to any other facility. This set is called the *market area* of the facility. The collection of market areas forms a tessellation of the underlying space. If customers choose the facility on the basis of distance in some metric, the tessellation is the Voronoi Diagram of the set of facilities [10].

We address a competitive facility location problem that we call the *Voronoi Game*. It is played by two players, Blue and Red, who place a specified number, n,

* Part of the work was done while the first, third, and fifth authors were at the Dept. of Computer Science, HKUST, Hong Kong. The work described in this paper has been supported by the Research Grants Council of Hong Kong, China (HKUST6074/97E, HKUST6088/99E, HKUST6094/99E, HKUST6162/00E, and HKUST6137/98E).

J. Wang (Ed.): COCOON 2001, LNCS 2108, pp. 237–246, 2001.

The most natural Voronoi Game is played in a two-dimensional arena U using the Euclidean metric. Unfortunately nobody knows how to win this game, even for very restricted regions U. In this note we present strategies for winning one-dimensional versions of the game, where the arena is a circle or a line segment, and variations. In other words, we consider competitive facility location along an Australian highway.

- **Blue and Red**
- $n > 1$
- **Circle or line segment**
- **„Keypoints“**

Technische Universität Braunschweig

Competitive Facility Location along a Highway*

Hee-Kap Ahn[1], Siu-Wing Cheng[2], Otfried Cheong[1], Mordecai Golin[2], and René van Oostrum[1]

[1] Department of Computer Science, Utrecht University, Netherlands, {heekap,otfried,rene}@cs.uu.nl
[2] Department of Computer Science, HKUST, Hong Kong, {scheng,golin}@cs.ust.hk

Abstract. We consider a competitive facility location problem with two players. Players alternate placing points, one at a time, into the playing arena, until each of them has placed n points. The arena is then subdivided according to the nearest-neighbor rule, and the player whose points control the larger area wins. We present a winning strategy for the second player, where the arena is a circle or a line segment.

1 Introduction

The classical facility location problem [5] asks for the optimum location of a new facility (police station, super market, transmitter, etc.) with respect to a given set of customers. Typically, the function to be optimized is the maximum distance from customers to the facility — this results in the minimum enclosing disk problem studied by Megiddo [8], Welzl [12] and Aronov et al. [2].

Competitive facility location deals with the placement of sites by competing market players. Geometric arguments are combined with arguments from *game theory* to see how the behavior of these decision makers affect each other. Competitive location models have been studied in many different fields, such as spatial economics and industrial organization [1,9], mathematics [6] and operations research [3,7,11]. Comprehensive overviews of competitive facility locations models are the surveys by Friesz et al. [11], Eiselt and Laporte [3] and Eiselt et al. [4].

We consider a model where the behavior of the customers is deterministic in the sense that a facility can determine the set of customers more attracted to it than to any other facility. This set is called the *market area* of the facility. The collection of market areas forms a tessellation of the underlying space. If customers choose the facility on the basis of distance in some metric, the tessellation is the Voronoi Diagram of the set of facilities [10].

We address a competitive facility location problem that we call the *Voronoi Game*. It is played by two players, Blue and Red, who place a specified number, n,

* Part of the work was done while the first, third, and fifth authors were at the Dept. of Computer Science, HKUST, Hong Kong. The work described in this paper has been supported by the Research Grants Council of Hong Kong, China (HKUST6074/97E, HKUST6088/99E, HKUST6094/99E, HKUST6162/00E, and HKUST6137/98E).

J. Wang (Ed.): COCOON 2001, LNCS 2108, pp. 237–246, 2001.

The most natural Voronoi Game is played in a two-dimensional arena U using the Euclidean metric. Unfortunately nobody knows how to win this game, even for very restricted regions U. In this note we present strategies for winning one-dimensional versions of the game, where the arena is a circle or a line segment, and variations. In other words, we consider competitive facility location along an Australian highway.

- **Blue and Red**
- $n > 1$
- **Circle or line segment**
- **„Keypoints“**

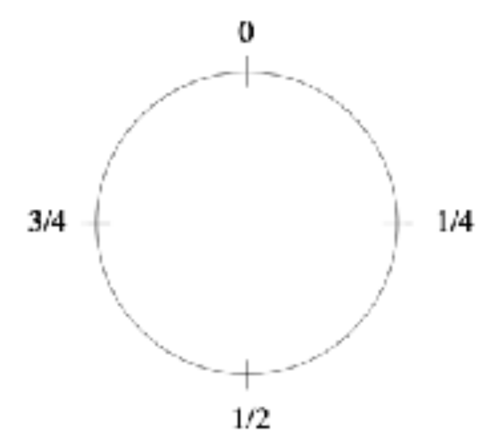
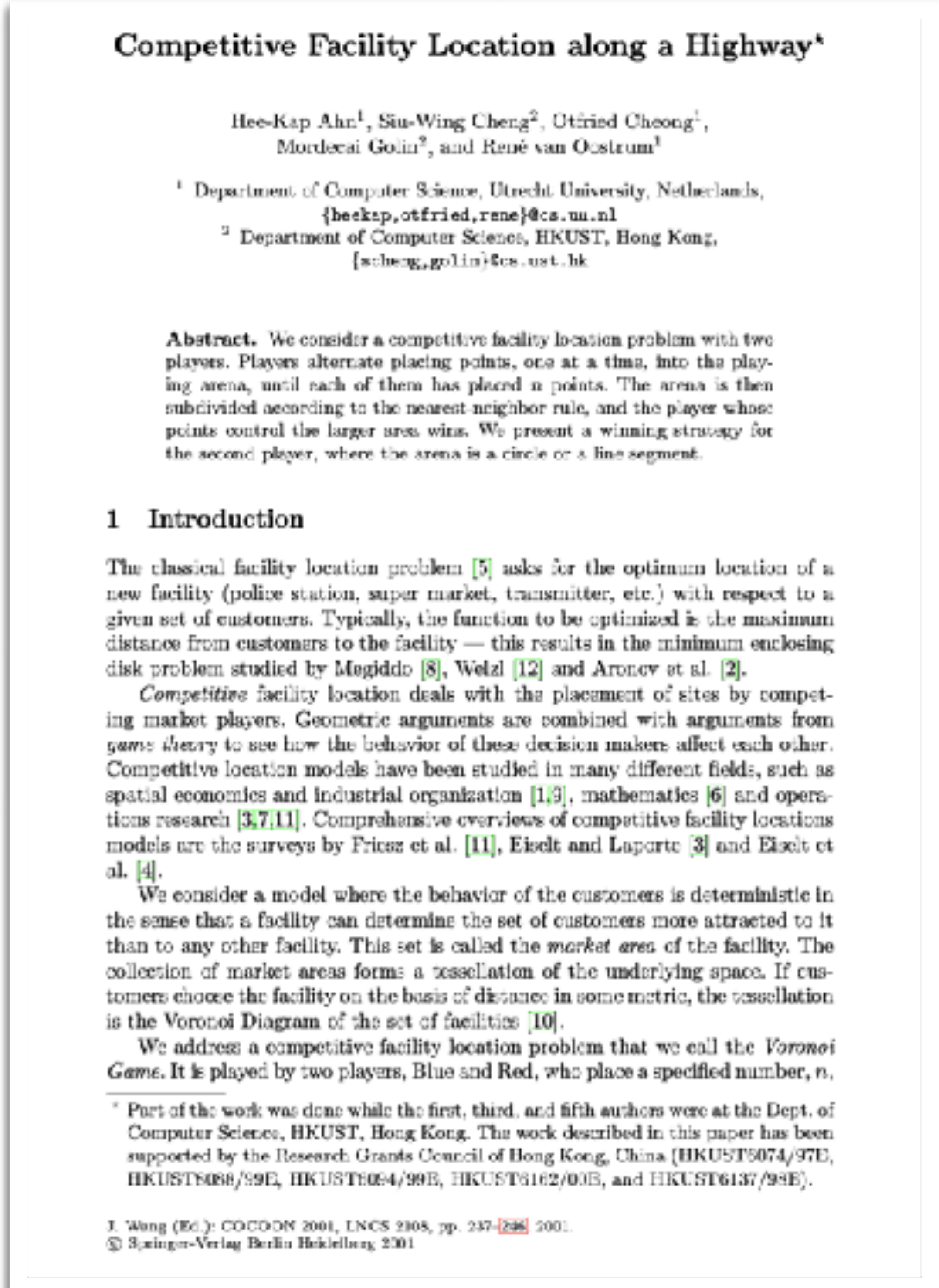


Figure 1: There are four keypoints when $n = 4$.



Competitive Facility Location along a Highway*

Hee-Kap Ahn[1], Siu-Wing Cheng[2], Otfried Cheong[1],
Mordecai Golin[2], and René van Oostrum[1]

[1] Department of Computer Science, Utrecht University, Netherlands,
{heekap,otfried,rene}@cs.uu.nl
[2] Department of Computer Science, HKUST, Hong Kong,
{scheng,golin}@cs.ust.hk

Abstract. We consider a competitive facility location problem with two players. Players alternate placing points, one at a time, into the playing arena, until each of them has placed n points. The arena is then subdivided according to the nearest-neighbor rule, and the player whose points control the larger area wins. We present a winning strategy for the second player, where the arena is a circle or a line segment.

1 Introduction

The classical facility location problem [5] asks for the optimum location of a new facility (police station, super market, transmitter, etc.) with respect to a given set of customers. Typically, the function to be optimized is the maximum distance from customers to the facility — this results in the minimum enclosing disk problem studied by Megiddo [8], Welzl [12] and Aronov et al. [2].

Competitive facility location deals with the placement of sites by competing market players. Geometric arguments are combined with arguments from *game theory* to see how the behavior of these decision makers affect each other. Competitive location models have been studied in many different fields, such as spatial economics and industrial organization [1,9], mathematics [6] and operations research [3,7,11]. Comprehensive overviews of competitive facility locations models are the surveys by Friesz et al. [11], Eiselt and Laporte [3] and Eiselt et al. [4].

We consider a model where the behavior of the customers is deterministic in the sense that a facility can determine the set of customers more attracted to it than to any other facility. This set is called the *market area* of the facility. The collection of market areas forms a tessellation of the underlying space. If customers choose the facility on the basis of distance in some metric, the tessellation is the Voronoi Diagram of the set of facilities [10].

We address a competitive facility location problem that we call the *Voronoi Game*. It is played by two players, Blue and Red, who place a specified number, *n*,

* Part of the work was done while the first, third, and fifth authors were at the Dept. of Computer Science, HKUST, Hong Kong. The work described in this paper has been supported by the Research Grants Council of Hong Kong, China (HKUST6074/97E, HKUST6088/99E, HKUST6094/99E, HKUST6162/00E, and HKUST6137/98E).

J. Wang (Ed.): COCOON 2001, LNCS 2108, pp. 237–246, 2001.

The most natural Voronoi Game is played in a two-dimensional arena U using the Euclidean metric. Unfortunately nobody knows how to win this game, even for very restricted regions U. In this note we present strategies for winning one-dimensional versions of the game, where the arena is a circle or a line segment, and variations. In other words, we consider competitive facility location along an Australian highway.

- **Blue and Red**
- $n > 1$
- **Circle or line segment**
- **„Keypoints“**

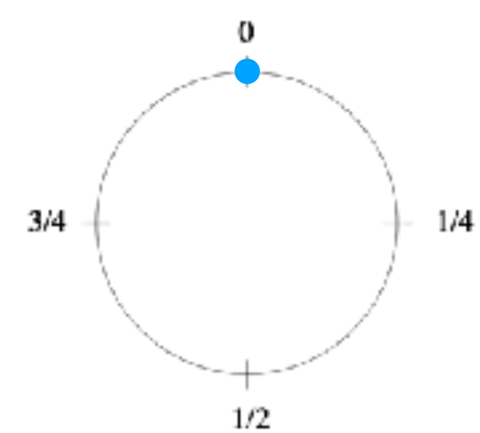


Figure 1: There are four keypoints when $n = 4$.

1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]

Competitive Facility Location along a Highway*

Hee-Kap Ahn[1], Siu-Wing Cheng[2], Otfried Cheong[1], Mordecai Golin[2], and René van Oostrum[1]

[1] Department of Computer Science, Utrecht University, Netherlands, {heekap,otfried,rene}@cs.uu.nl
[2] Department of Computer Science, HKUST, Hong Kong, {scheng,golin}@cs.ust.hk

Abstract. We consider a competitive facility location problem with two players. Players alternate placing points, one at a time, into the playing arena, until each of them has placed n points. The arena is then subdivided according to the nearest-neighbor rule, and the player whose points control the larger area wins. We present a winning strategy for the second player, where the arena is a circle or a line segment.

1 Introduction

The classical facility location problem [5] asks for the optimum location of a new facility (police station, super market, transmitter, etc.) with respect to a given set of customers. Typically, the function to be optimized is the maximum distance from customers to the facility — this results in the minimum enclosing disk problem studied by Megiddo [8], Welzl [12] and Aronov et al. [2].

Competitive facility location deals with the placement of sites by competing market players. Geometric arguments are combined with arguments from *game theory* to see how the behavior of these decision makers affect each other. Competitive location models have been studied in many different fields, such as spatial economics and industrial organization [1,9], mathematics [6] and operations research [3,7,11]. Comprehensive overviews of competitive facility locations models are the surveys by Friesz et al. [11], Eiselt and Laporte [3] and Eiselt et al. [4].

We consider a model where the behavior of the customers is deterministic in the sense that a facility can determine the set of customers more attracted to it than to any other facility. This set is called the *market area* of the facility. The collection of market areas forms a tessellation of the underlying space. If customers choose the facility on the basis of distance in some metric, the tessellation is the Voronoi Diagram of the set of facilities [10].

We address a competitive facility location problem that we call the *Voronoi Game*. It is played by two players, Blue and Red, who place a specified number, n,

* Part of the work was done while the first, third, and fifth authors were at the Dept. of Computer Science, HKUST, Hong Kong. The work described in this paper has been supported by the Research Grants Council of Hong Kong, China (HKUST6074/97E, HKUST6088/99E, HKUST6094/99E, HKUST6162/00E, and HKUST6137/98E).

J. Wang (Ed.): COCOON 2001, LNCS 2108, pp. 237–246, 2001.

The most natural Voronoi Game is played in a two-dimensional arena U using the Euclidean metric. Unfortunately nobody knows how to win this game, even for very restricted regions U. In this note we present strategies for winning one-dimensional versions of the game, where the arena is a circle or a line segment, and variations. In other words, we consider competitive facility location along an Australian highway.

- **Blue and Red**
- ***n > 1***
- **Circle or line segment**
- **„Keypoints“**

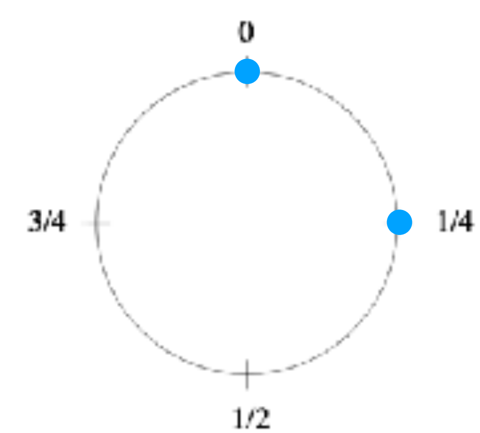


Figure 1: There are four keypoints when $n = 4$.

1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]

Competitive Facility Location along a Highway*

Hee-Kap Ahn[1], Siu-Wing Cheng[2], Otfried Cheong[1], Mordecai Golin[2], and René van Oostrum[1]

[1] Department of Computer Science, Utrecht University, Netherlands, {heekap,otfried,rene}@cs.uu.nl
[2] Department of Computer Science, HKUST, Hong Kong, {scheng,golin}@cs.ust.hk

Abstract. We consider a competitive facility location problem with two players. Players alternate placing points, one at a time, into the playing arena, until each of them has placed n points. The arena is then subdivided according to the nearest-neighbor rule, and the player whose points control the larger area wins. We present a winning strategy for the second player, where the arena is a circle or a line segment.

1 Introduction

The classical facility location problem [5] asks for the optimum location of a new facility (police station, super market, transmitter, etc.) with respect to a given set of customers. Typically, the function to be optimized is the maximum distance from customers to the facility — this results in the minimum enclosing disk problem studied by Megiddo [8], Welzl [12] and Aronov et al. [2].

Competitive facility location deals with the placement of sites by competing market players. Geometric arguments are combined with arguments from *game theory* to see how the behavior of these decision makers affect each other. Competitive location models have been studied in many different fields, such as spatial economics and industrial organization [1,3], mathematics [6] and operations research [3,7,11]. Comprehensive overviews of competitive facility locations models are the surveys by Friesz et al. [11], Eiselt and Laporte [3] and Eiselt et al. [4].

We consider a model where the behavior of the customers is deterministic in the sense that a facility can determine the set of customers more attracted to it than to any other facility. This set is called the *market area* of the facility. The collection of market areas forms a tessellation of the underlying space. If customers choose the facility on the basis of distance in some metric, the tessellation is the Voronoi Diagram of the set of facilities [10].

We address a competitive facility location problem that we call the *Voronoi Game*. It is played by two players, Blue and Red, who place a specified number, n,

* Part of the work was done while the first, third, and fifth authors were at the Dept. of Computer Science, HKUST, Hong Kong. The work described in this paper has been supported by the Research Grants Council of Hong Kong, China (HKUST6074/97E, HKUST6088/99E, HKUST6094/99E, HKUST6162/00E, and HKUST6137/98E).

J. Wang (Ed.): COCOON 2001, LNCS 2108, pp. 237–246, 2001.

The most natural Voronoi Game is played in a two-dimensional arena U using the Euclidean metric. Unfortunately nobody knows how to win this game, even for very restricted regions U. In this note we present strategies for winning one-dimensional versions of the game, where the arena is a circle or a line segment, and variations. In other words, we consider competitive facility location along an Australian highway.

- **Blue and Red**
- $n > 1$
- **Circle or line segment**
- **„Keypoints“**

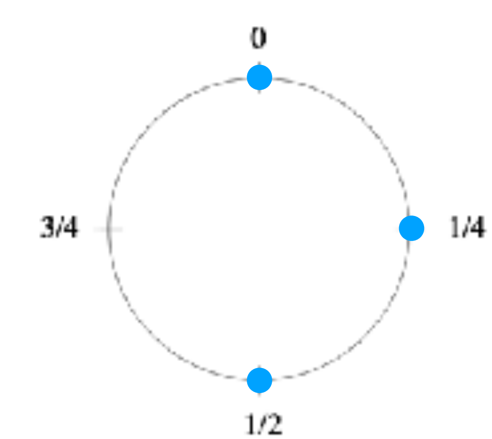


Figure 1: There are four keypoints when $n = 4$.

Technische Universität Braunschweig

1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]

Competitive Facility Location along a Highway*

Hee-Kap Ahn[1], Siu-Wing Cheng[2], Otfried Cheong[1], Mordecai Golin[2], and René van Oostrum[1]

[1] Department of Computer Science, Utrecht University, Netherlands, {heekap,otfried,rene}@cs.uu.nl
[2] Department of Computer Science, HKUST, Hong Kong, {scheng,golin}@cs.ust.hk

Abstract. We consider a competitive facility location problem with two players. Players alternate placing points, one at a time, into the playing arena, until each of them has placed n points. The arena is then subdivided according to the nearest-neighbor rule, and the player whose points control the larger area wins. We present a winning strategy for the second player, where the arena is a circle or a line segment.

1 Introduction

The classical facility location problem [5] asks for the optimum location of a new facility (police station, super market, transmitter, etc.) with respect to a given set of customers. Typically, the function to be optimized is the maximum distance from customers to the facility — this results in the minimum enclosing disk problem studied by Megiddo [8], Welzl [12] and Aronov et al. [2].

Competitive facility location deals with the placement of sites by competing market players. Geometric arguments are combined with arguments from *game theory* to see how the behavior of these decision makers affect each other. Competitive location models have been studied in many different fields, such as spatial economics and industrial organization [1,9], mathematics [6] and operations research [3,7,11]. Comprehensive overviews of competitive facility locations models are the surveys by Friesz et al. [11], Eiselt and Laporte [3] and Eiselt et al. [4].

We consider a model where the behavior of the customers is deterministic in the sense that a facility can determine the set of customers more attracted to it than to any other facility. This set is called the *market area* of the facility. The collection of market areas forms a tessellation of the underlying space. If customers choose the facility on the basis of distance in some metric, the tessellation is the Voronoi Diagram of the set of facilities [10].

We address a competitive facility location problem that we call the *Voronoi Game*. It is played by two players, Blue and Red, who place a specified number, n,

* Part of the work was done while the first, third, and fifth authors were at the Dept. of Computer Science, HKUST, Hong Kong. The work described in this paper has been supported by the Research Grants Council of Hong Kong, China (HKUST6074/97E, HKUST6088/99E, HKUST6094/99E, HKUST6162/00E, and HKUST6137/98E).

J. Wang (Ed.): COCOON 2001, LNCS 2108, pp. 237–246, 2001.

The most natural Voronoi Game is played in a two-dimensional arena U using the Euclidean metric. Unfortunately nobody knows how to win this game, even for very restricted regions U. In this note we present strategies for winning one-dimensional versions of the game, where the arena is a circle or a line segment, and variations. In other words, we consider competitive facility location along an Australian highway.

- Blue and Red
- $n > 1$
- Circle or line segment
- „Keypoints“

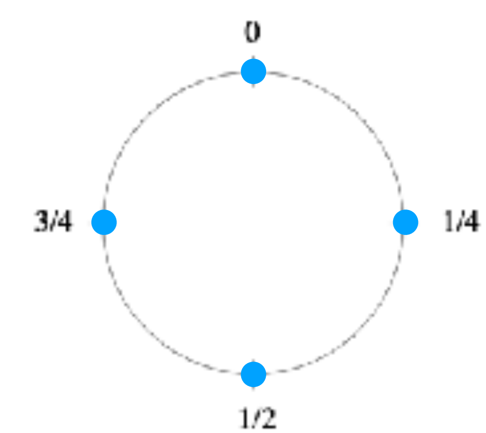


Figure 1: There are four keypoints when $n = 4$.

1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]

Competitive Facility Location along a Highway*

Hee-Kap Ahn[1], Siu-Wing Cheng[2], Otfried Cheong[1], Mordecai Golin[2], and René van Oostrum[1]

[1] Department of Computer Science, Utrecht University, Netherlands, {heekap,otfried,rene}@cs.uu.nl
[2] Department of Computer Science, HKUST, Hong Kong, {scheng,golin}@cs.ust.hk

Abstract. We consider a competitive facility location problem with two players. Players alternate placing points, one at a time, into the playing arena, until each of them has placed n points. The arena is then subdivided according to the nearest-neighbor rule, and the player whose points control the larger area wins. We present a winning strategy for the second player, where the arena is a circle or a line segment.

1 Introduction

The classical facility location problem [5] asks for the optimum location of a new facility (police station, super market, transmitter, etc.) with respect to a given set of customers. Typically, the function to be optimized is the maximum distance from customers to the facility — this results in the minimum enclosing disk problem studied by Megiddo [8], Welzl [12] and Aronov et al. [2].

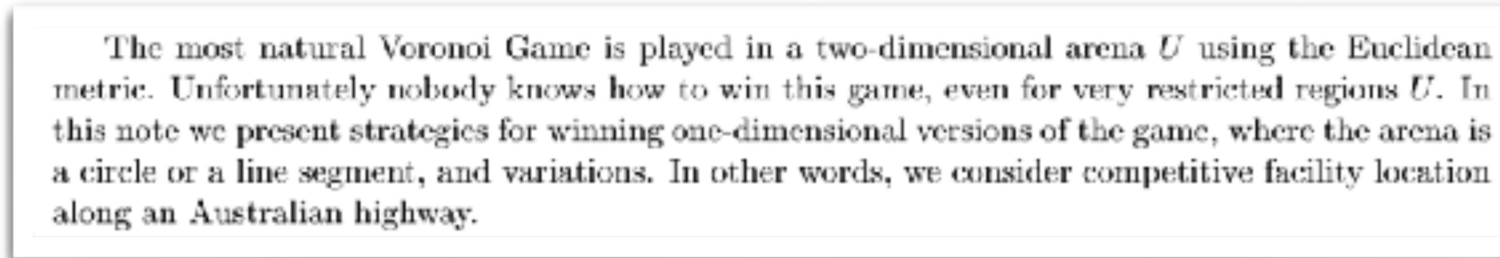
Competitive facility location deals with the placement of sites by competing market players. Geometric arguments are combined with arguments from *game theory* to see how the behavior of these decision makers affect each other. Competitive location models have been studied in many different fields, such as spatial economics and industrial organization [1,9], mathematics [6] and operations research [3,7,11]. Comprehensive overviews of competitive facility locations models are the surveys by Friesz et al. [11], Eiselt and Laporte [3] and Eiselt et al. [4].

We consider a model where the behavior of the customers is deterministic in the sense that a facility can determine the set of customers more attracted to it than to any other facility. This set is called the *market area* of the facility. The collection of market areas forms a tessellation of the underlying space. If customers choose the facility on the basis of distance in some metric, the tessellation is the Voronoi Diagram of the set of facilities [10].

We address a competitive facility location problem that we call the *Voronoi Game*. It is played by two players, Blue and Red, who place a specified number, n,

* Part of the work was done while the first, third, and fifth authors were at the Dept. of Computer Science, HKUST, Hong Kong. The work described in this paper has been supported by the Research Grants Council of Hong Kong, China (HKUST6074/97E, HKUST6088/99E, HKUST6094/99E, HKUST6162/00E, and HKUST6137/98E).

J. Wang (Ed.): COCOON 2001, LNCS 2108, pp. 237–246, 2001.



The most natural Voronoi Game is played in a two-dimensional arena U using the Euclidean metric. Unfortunately nobody knows how to win this game, even for very restricted regions U. In this note we present strategies for winning one-dimensional versions of the game, where the arena is a circle or a line segment, and variations. In other words, we consider competitive facility location along an Australian highway.

- **Blue and Red**
- $n > 1$
- **Circle or line segment**
- **„Keypoints“**

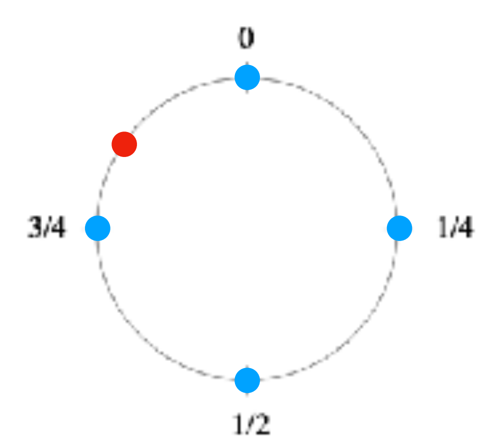


Figure 1: There are four keypoints when $n = 4$.

Technische Universität Braunschweig

Competitive Facility Location along a Highway*

Hee-Kap Ahn[1], Siu-Wing Cheng[2], Otfried Cheong[1], Mordecai Golin[2], and René van Oostrum[1]

[1] Department of Computer Science, Utrecht University, Netherlands, {heekap,otfried,rene}@cs.uu.nl
[2] Department of Computer Science, HKUST, Hong Kong, {scheng,golin}@cs.ust.hk

Abstract. We consider a competitive facility location problem with two players. Players alternate placing points, one at a time, into the playing arena, until each of them has placed n points. The arena is then subdivided according to the nearest-neighbor rule, and the player whose points control the larger area wins. We present a winning strategy for the second player, where the arena is a circle or a line segment.

1 Introduction

The classical facility location problem [5] asks for the optimum location of a new facility (police station, super market, transmitter, etc.) with respect to a given set of customers. Typically, the function to be optimized is the maximum distance from customers to the facility — this results in the minimum enclosing disk problem studied by Megiddo [8], Welzl [12] and Aronov et al. [2].

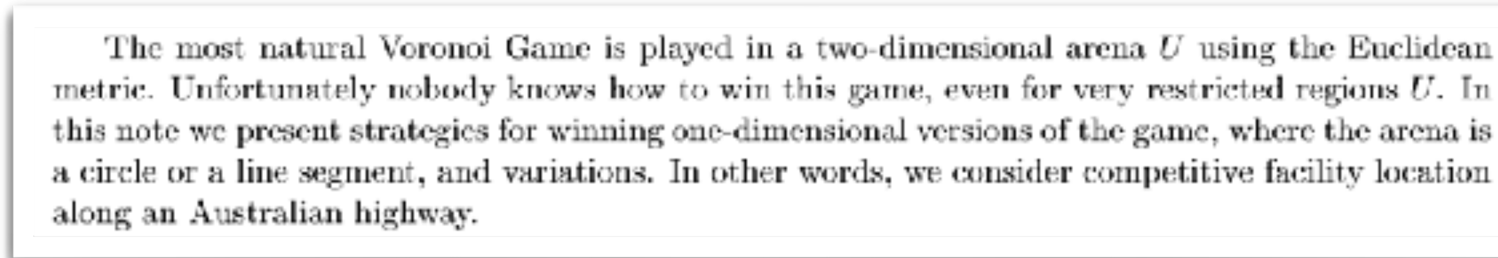
Competitive facility location deals with the placement of sites by competing market players. Geometric arguments are combined with arguments from *game theory* to see how the behavior of these decision makers affect each other. Competitive location models have been studied in many different fields, such as spatial economics and industrial organization [1,9], mathematics [6] and operations research [3,7,11]. Comprehensive overviews of competitive facility locations models are the surveys by Friesz et al. [11], Eiselt and Laporte [3] and Eiselt et al. [4].

We consider a model where the behavior of the customers is deterministic in the sense that a facility can determine the set of customers more attracted to it than to any other facility. This set is called the *market area* of the facility. The collection of market areas forms a tessellation of the underlying space. If customers choose the facility on the basis of distance in some metric, the tessellation is the Voronoi Diagram of the set of facilities [10].

We address a competitive facility location problem that we call the *Voronoi Game*. It is played by two players, Blue and Red, who place a specified number, n,

* Part of the work was done while the first, third, and fifth authors were at the Dept. of Computer Science, HKUST, Hong Kong. The work described in this paper has been supported by the Research Grants Council of Hong Kong, China (HKUST6074/97E, HKUST6088/99E, HKUST6094/99E, HKUST6162/00E, and HKUST6137/98E).

J. Wang (Ed.): COCOON 2001, LNCS 2108, pp. 237–246, 2001.



The most natural Voronoi Game is played in a two-dimensional arena U using the Euclidean metric. Unfortunately nobody knows how to win this game, even for very restricted regions U. In this note we present strategies for winning one-dimensional versions of the game, where the arena is a circle or a line segment, and variations. In other words, we consider competitive facility location along an Australian highway.

- **Blue and Red**
- $n > 1$
- **Circle or line segment**
- **„Keypoints“**

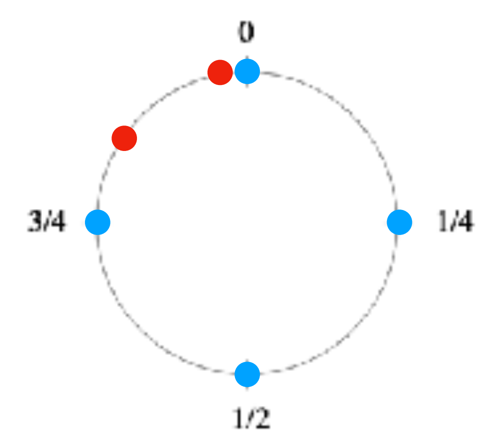
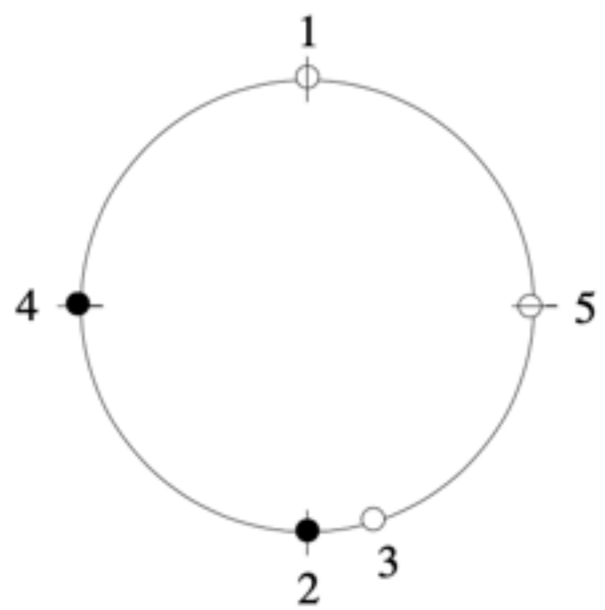


Figure 1: There are four keypoints when $n = 4$.

1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]

1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]

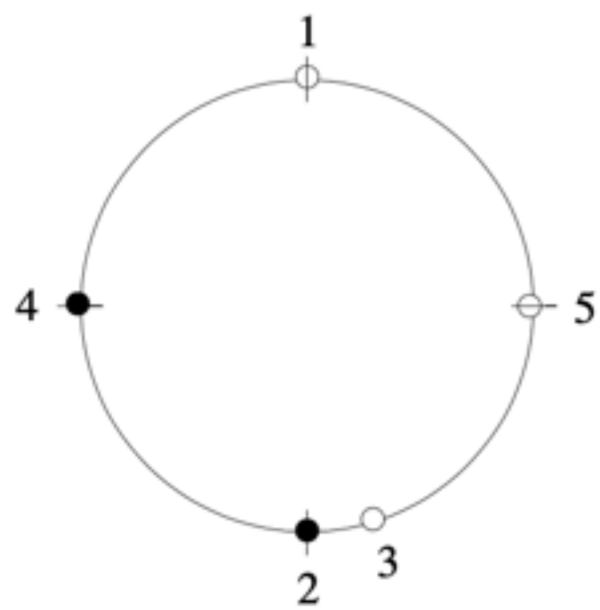
- **Stage I: Play keypoint while available**



Stage I

- **Stage I: Play keypoint while available**

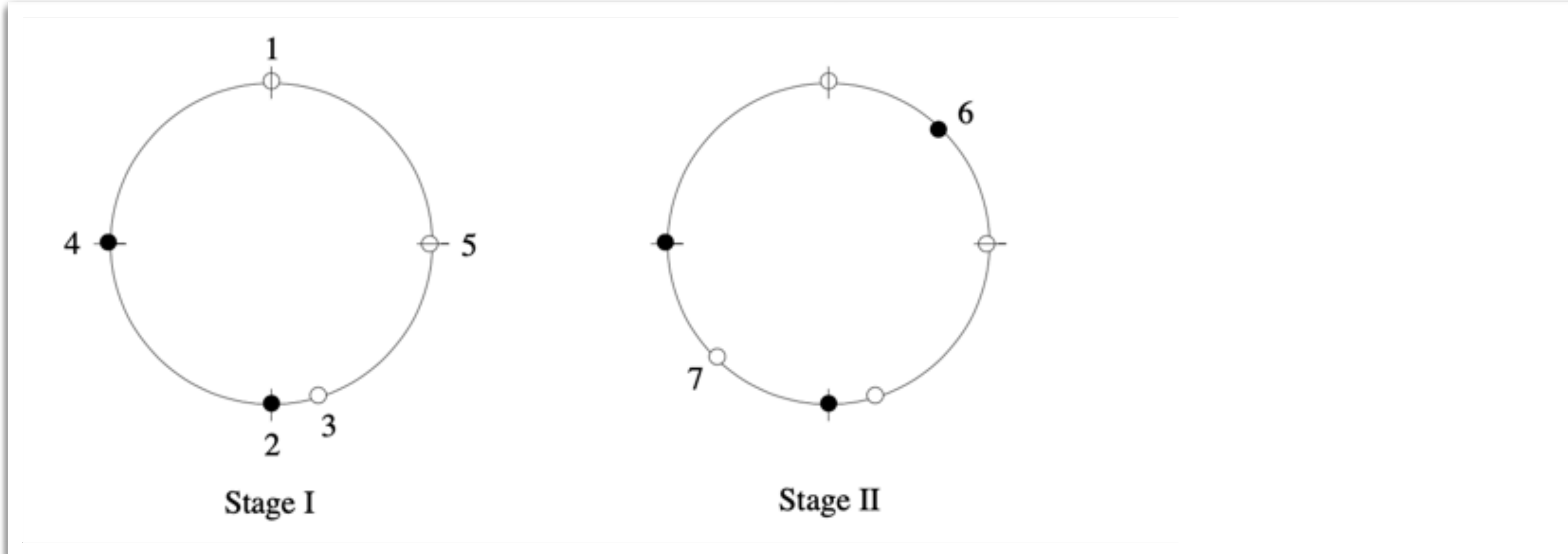
1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]



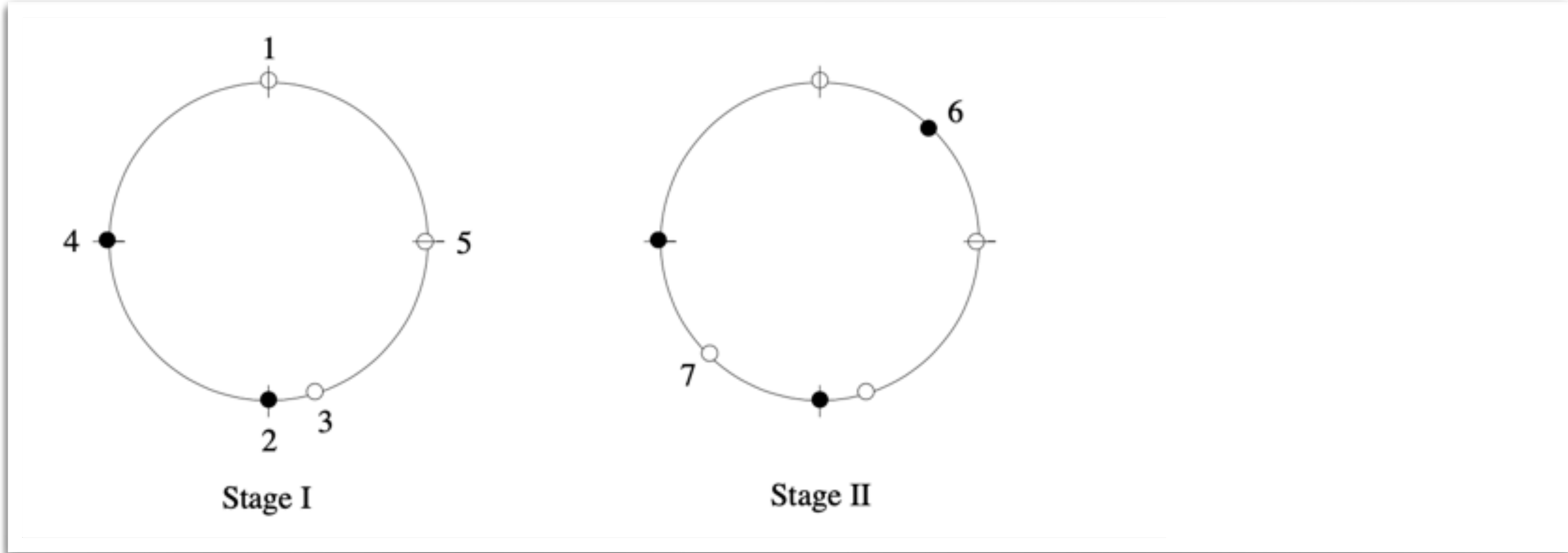
Stage I

- Stage I: Play keypoint while available
- **Stage II: Play in largest blue interval**

1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]

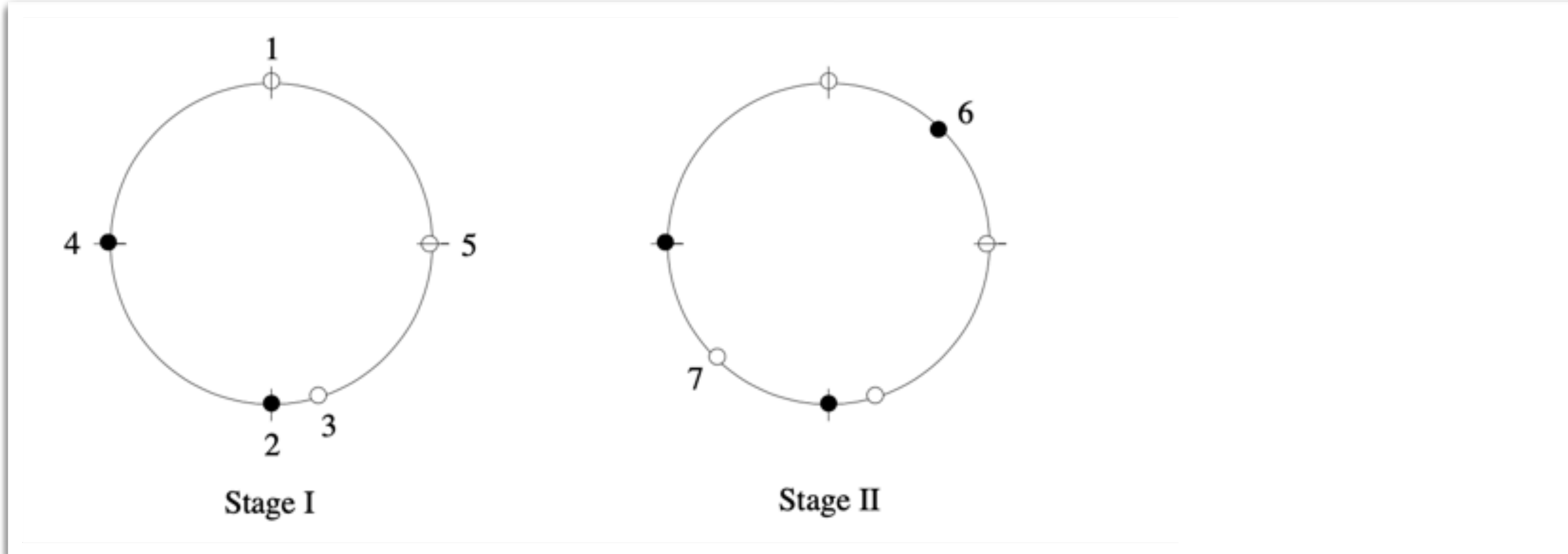


- Stage I: Play keypoint while available
- **Stage II: Play in largest blue interval**



- Stage I: Play keypoint while available
- Stage II: Play in largest blue interval
- **Stage III: Place last point to win:**

1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]

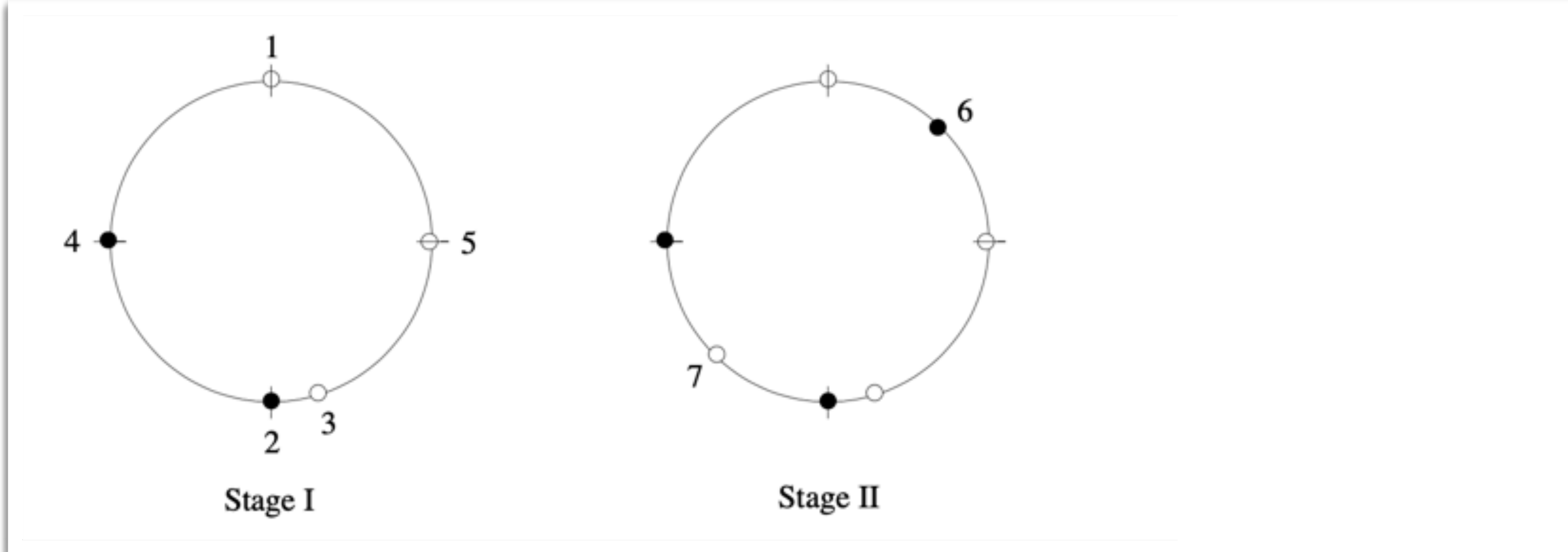


- Stage I: Play keypoint while available
- Stage II: Play in largest blue interval
- Stage III: Place last point to win:

(i) Play in largest blue interval

Technische Universität Braunschweig

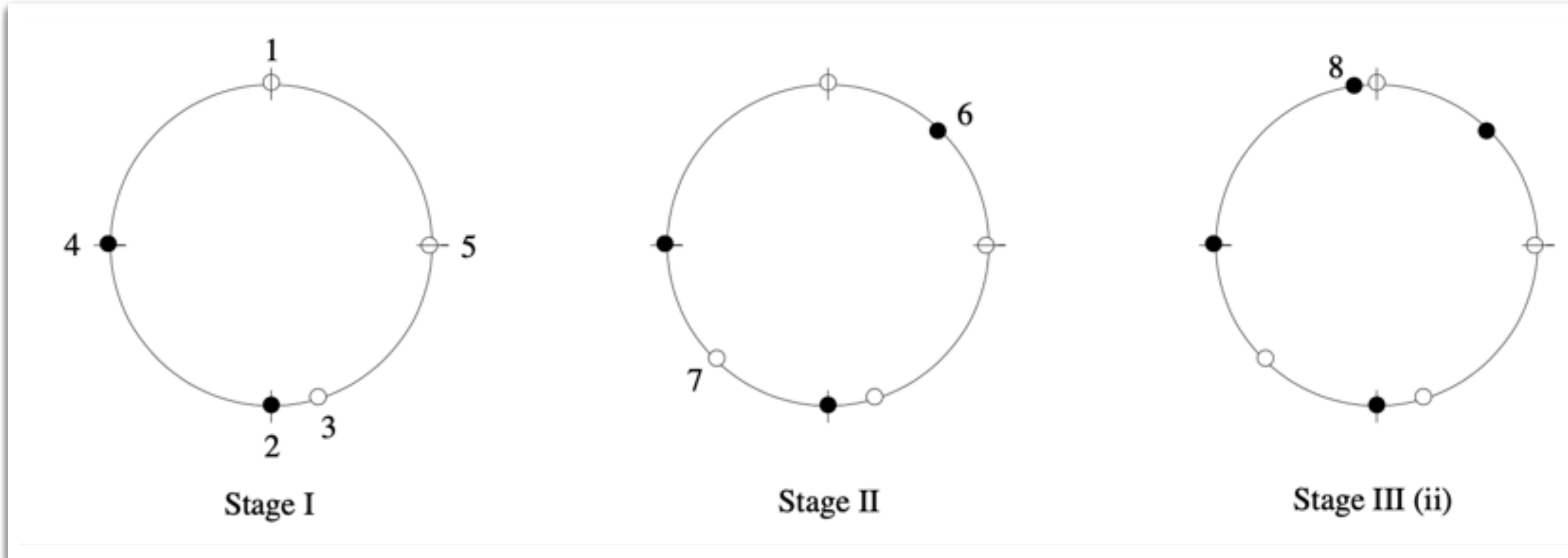
1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]



- Stage I: Play keypoint while available
- Stage II: Play in largest blue interval
- Stage III: Place last point to win:

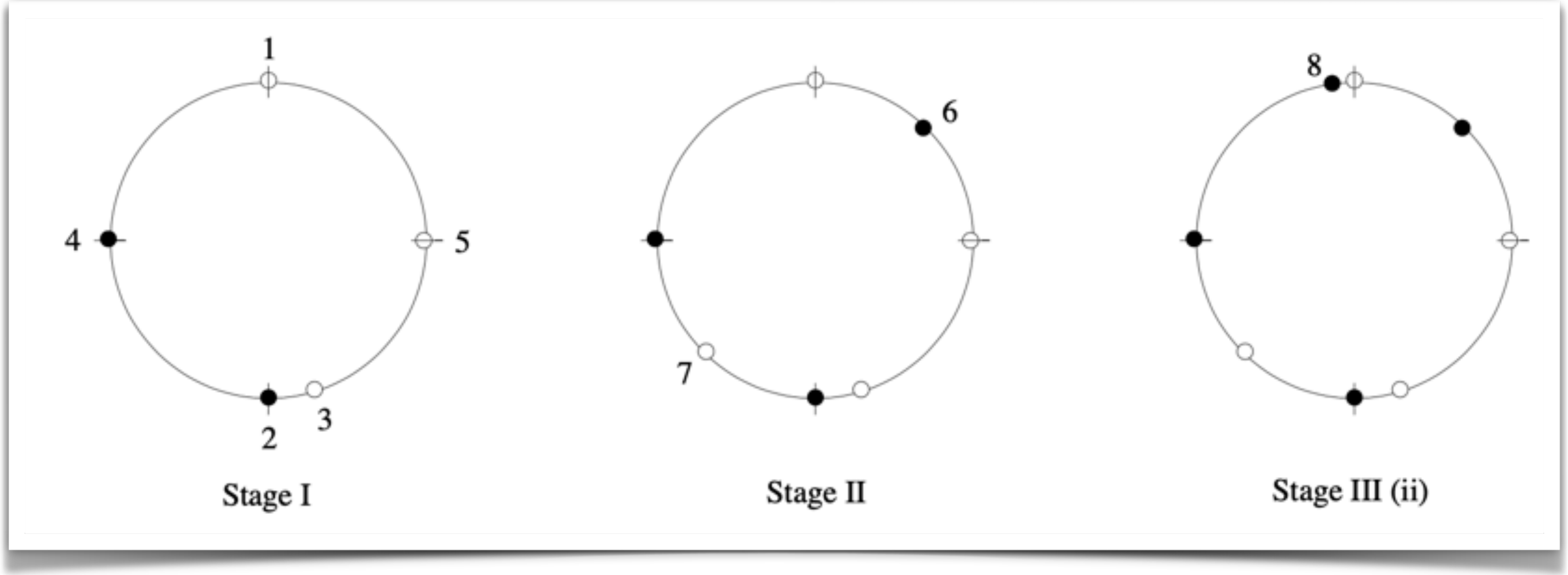
  (i) Play in largest blue interval

  (ii) Play in large mixed interval

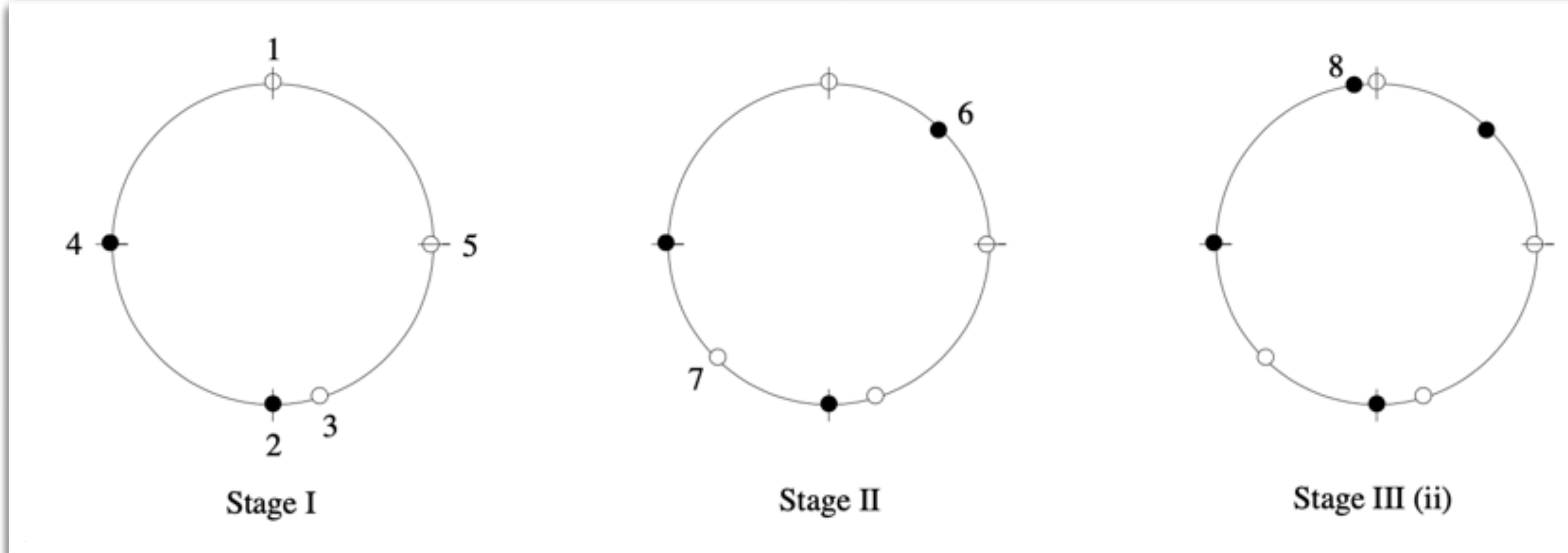


- Stage I: Play keypoint while available
- Stage II: Play in largest blue interval
- Stage III: Place last point to win:
  - (i) Play in largest blue interval
  - **(ii) Play in large mixed interval**

1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]

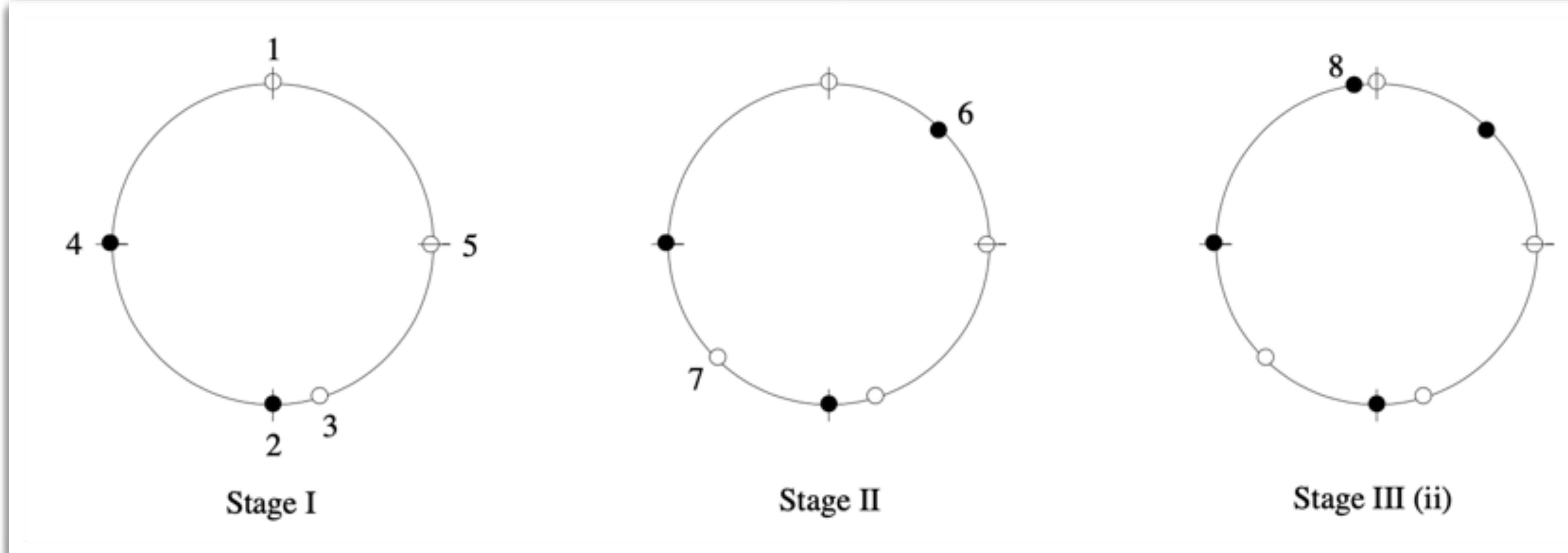


1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]



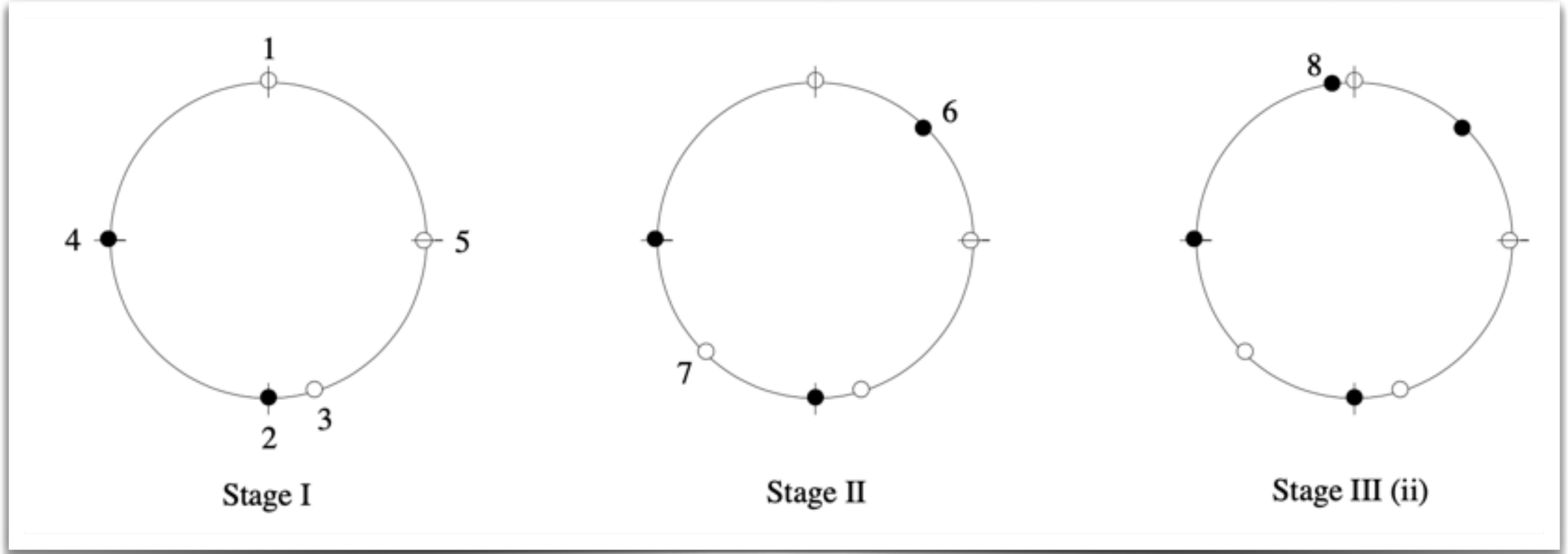
Observations:

Technische Universität Braunschweig



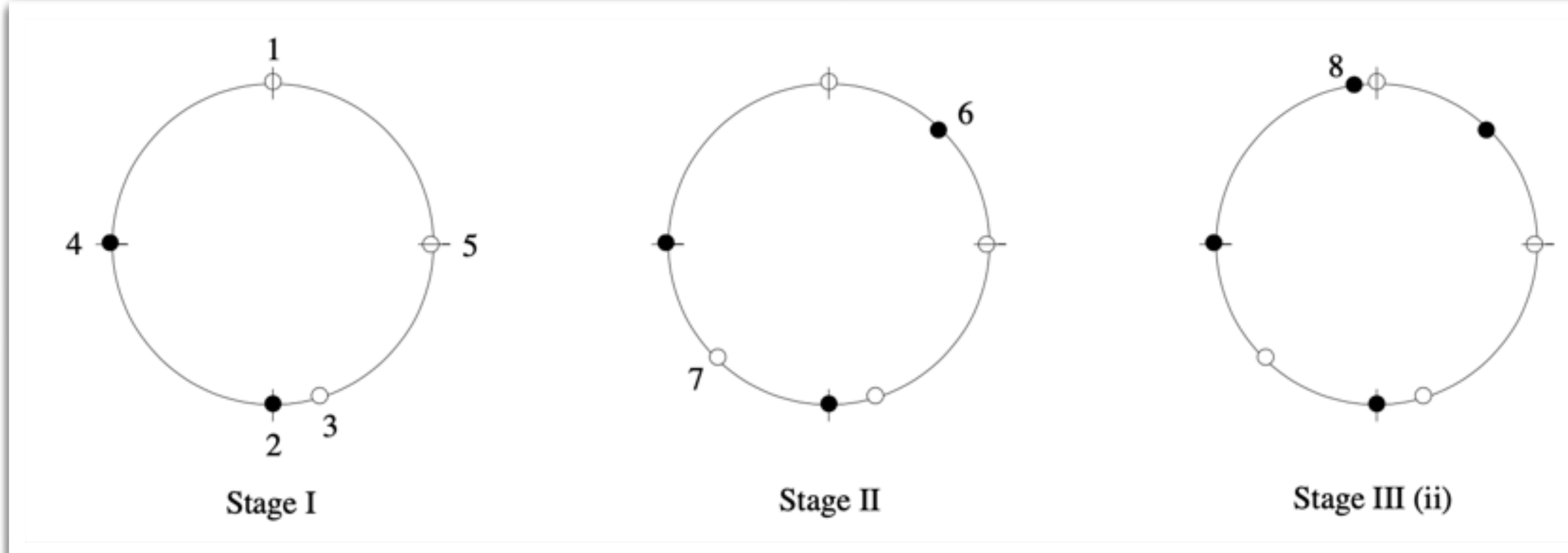
Observations:

- **Playing keypoints ensures that blue intervals are uneven.**



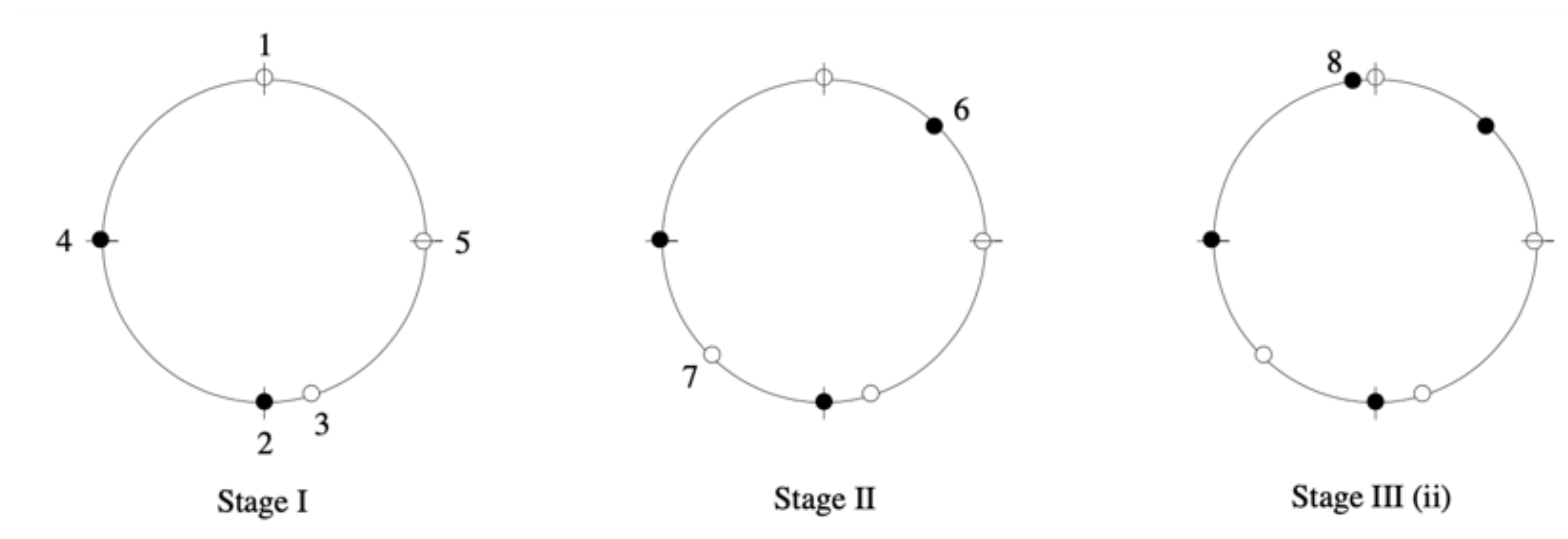
Observations:

- **Playing keypoints ensures that blue intervals are uneven.**
- **Playing into large blue intervals ensures never falling behind.**



Observations:

- **Playing keypoints ensures that blue intervals are uneven.**
- **Playing into large blue intervals ensures never falling behind.**
- **Placing the last point (exploiting uneven blue intervals) ensures win.**



Observations:

- **Playing keypoints ensures that blue intervals are uneven.**
- **Playing into large blue intervals ensures never falling behind.**
- **Placing the last point (exploiting uneven blue intervals) ensures win.**

Theorem 1 *The keypoint strategy is a well-defined winning strategy for Red.*

1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]

Technische Universität Braunschweig

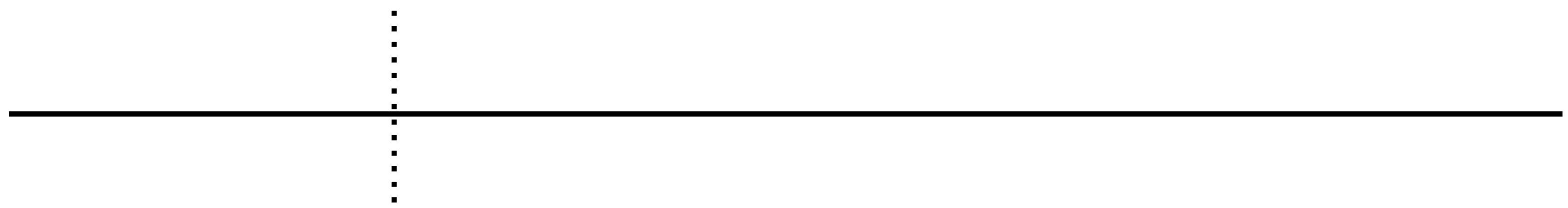
Modifications for game on a line segment:

Modifications for game on a line segment:

- **Keypoints are predefined by breaking line segments into equal pieces.**

Modifications for game on a line segment:

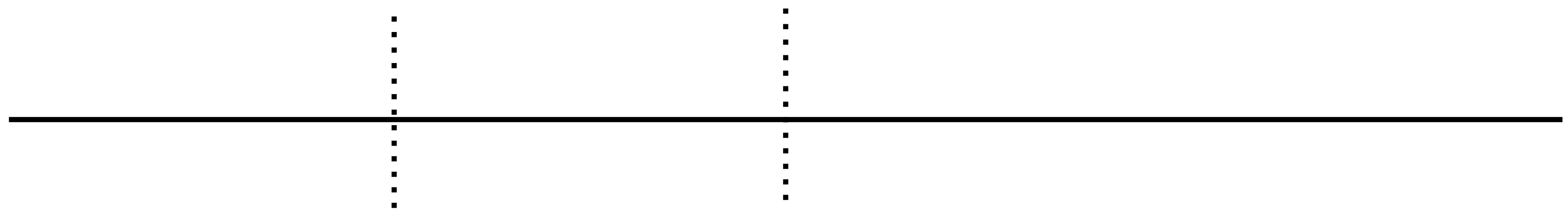
- **Keypoints are predefined by breaking line segments into equal pieces.**



Modifications for game on a line segment:

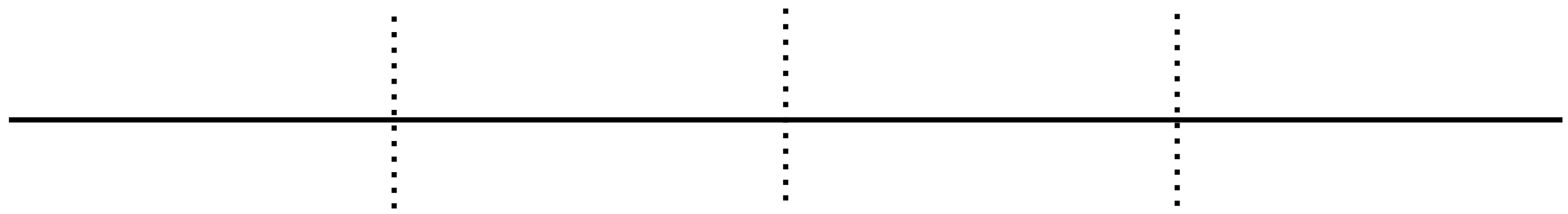
- **Keypoints are predefined by breaking line segments into equal pieces.**

1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]



Modifications for game on a line segment:

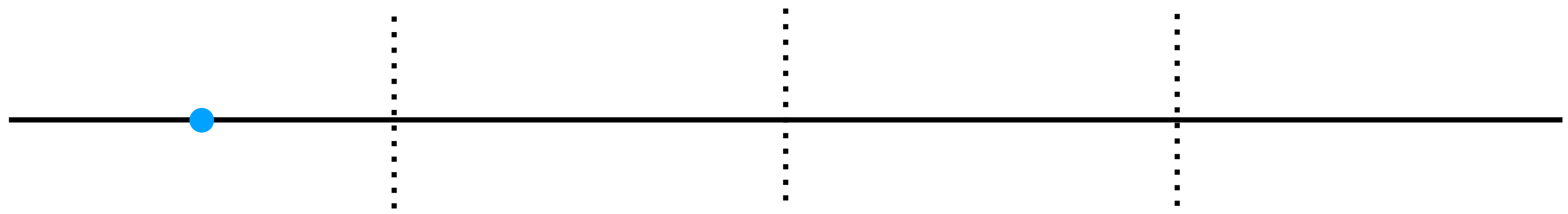
- **Keypoints are predefined by breaking line segments into equal pieces.**



Modifications for game on a line segment:

- **Keypoints are predefined by breaking line segments into equal pieces.**

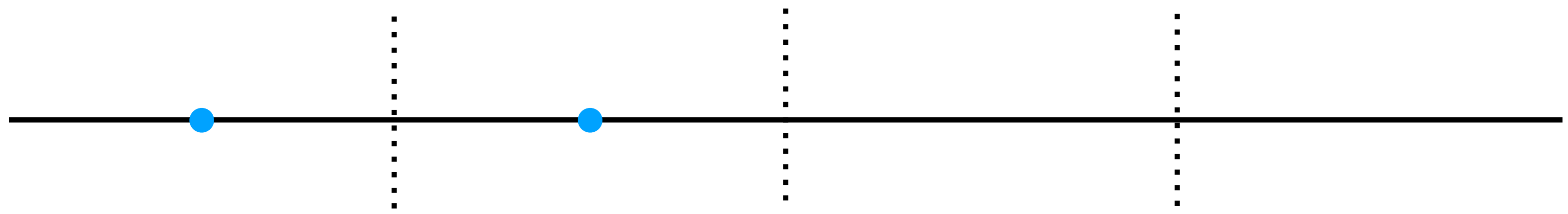
1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]



Modifications for game on a line segment:

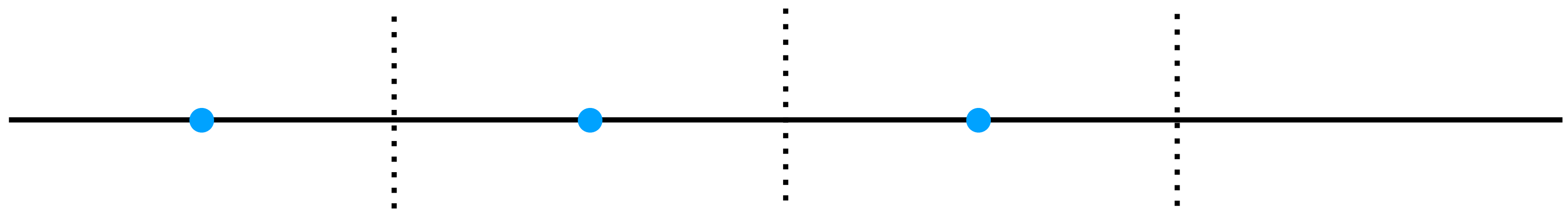
- **Keypoints are predefined by breaking line segments into equal pieces.**

Technische Universität Braunschweig



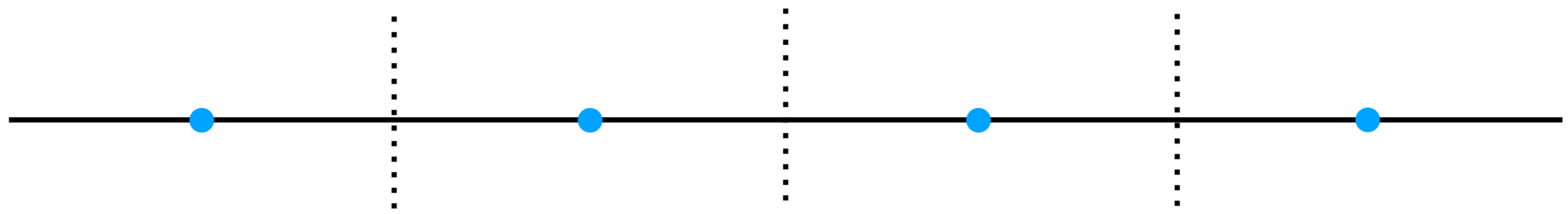
Modifications for game on a line segment:

- **Keypoints are predefined by breaking line segments into equal pieces.**



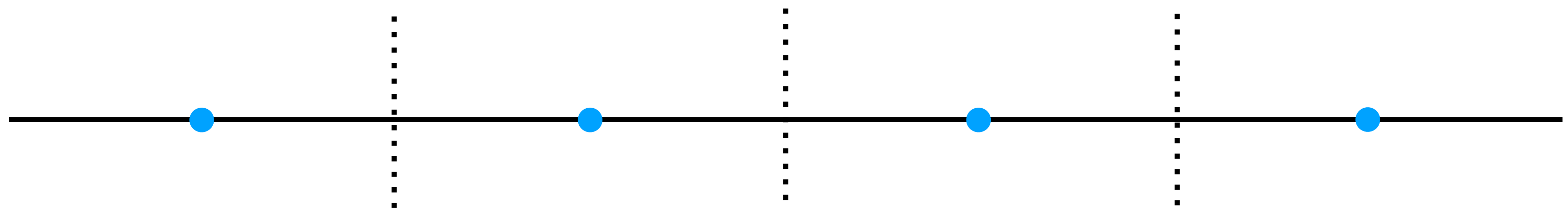
Modifications for game on a line segment:

- **Keypoints are predefined by breaking line segments into equal pieces.**



Modifications for game on a line segment:

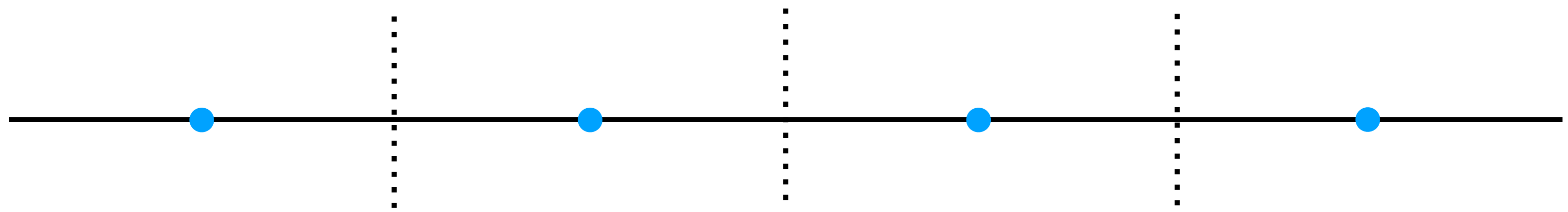
- **Keypoints are predefined by breaking line segments into equal pieces.**



Definition 2 *The n points $u_i = \frac{1}{2n} + \frac{i}{n}$, $i = 0, 1, \ldots, n-1$ are* **keypoints**.

Modifications for game on a line segment:

- **Keypoints are predefined by breaking line segments into equal pieces.**

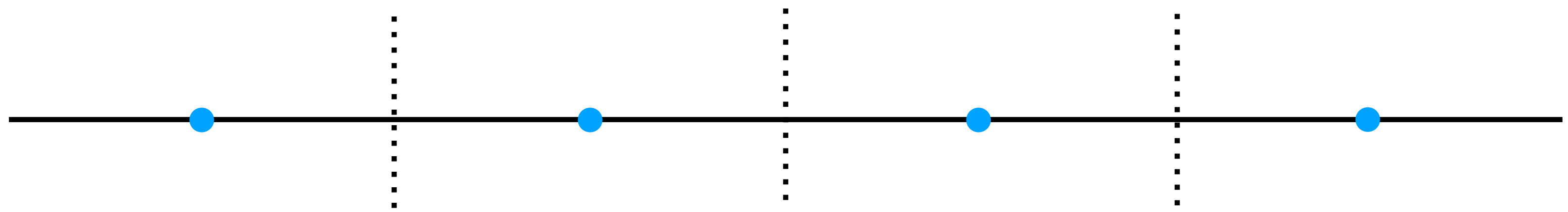


Definition 2 *The n points* $u_i = \frac{1}{2n} + \frac{i}{n}$, $i = 0, 1, \ldots, n-1$ *are* **keypoints**.

Modifications for game on a line segment:

- **Keypoints are predefined by breaking line segments into equal pieces.**
- **Distinguish interior and boundary intervals.**

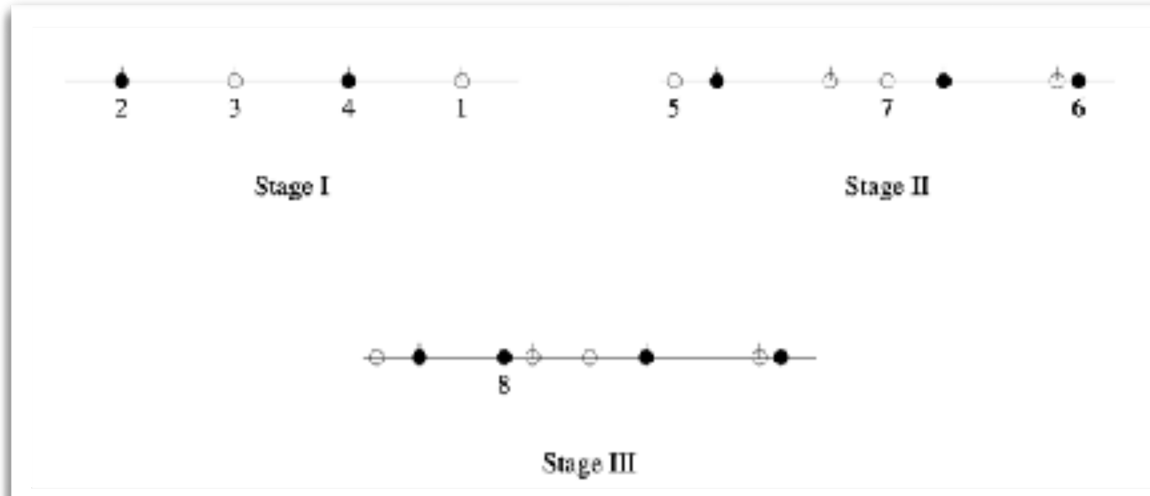
Technische Universität Braunschweig

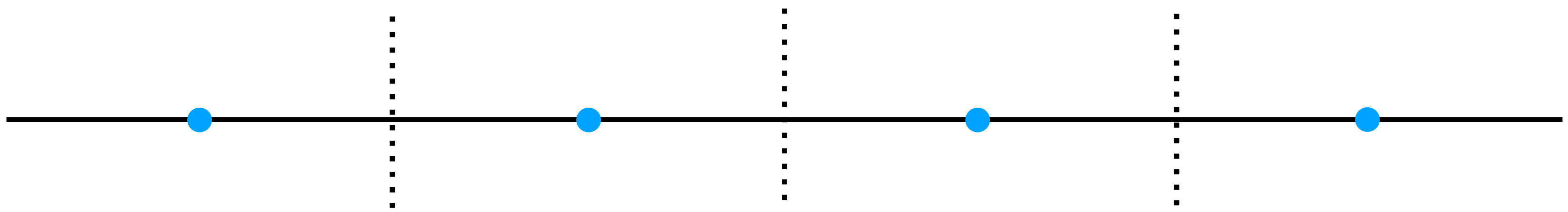


Definition 2 *The n points* $u_i = \frac{1}{2n} + \frac{i}{n}$, $i = 0, 1, \ldots, n-1$ *are* **keypoints**.

Modifications for game on a line segment:

- **Keypoints are predefined by breaking line segments into equal pieces.**
- **Distinguish interior and boundary intervals.**

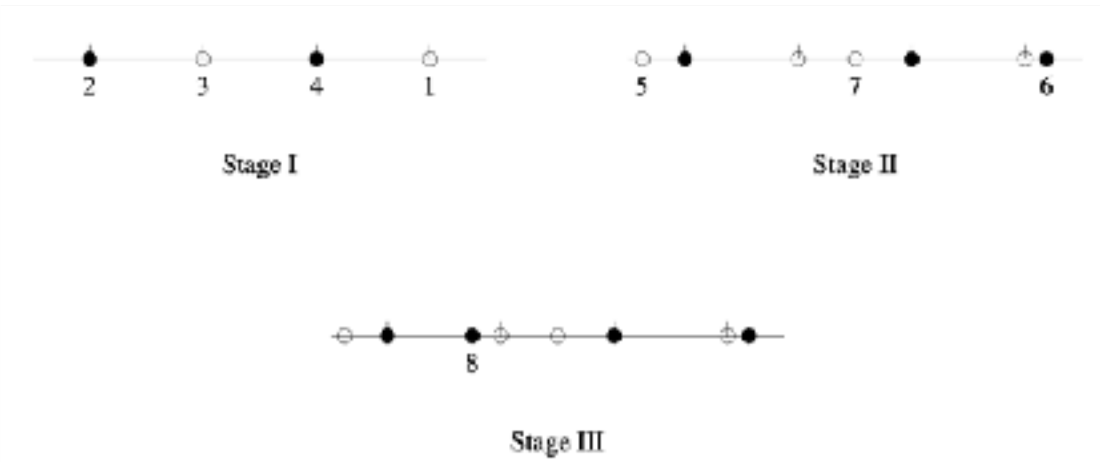




Definition 2 *The* n *points* $u_i = \frac{1}{2n} + \frac{i}{n}$, $i = 0, 1, \ldots, n-1$ *are* **keypoints**.

Modifications for game on a line segment:

- **Keypoints are predefined by breaking line segments into equal pieces.**
- **Distinguish interior and boundary intervals.**



Theorem 2 *The line strategy is a well-defined winning strategy for Red.*

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]

Technische Universität Braunschweig

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]

Discrete Comput Geom 31:125–138 (2004)
DOI: 10.1007/s00454-003-2951-4

Discrete & Computational
Geometry

The One-Round Voronoi Game*

Otfried Cheong,[1] Sariel Har-Peled,[2] Nathan Linial,[3] and Jiří Matoušek[4]

[1]Department of Mathematics and Computer Science, Eindhoven University of Technology,
P.O. Box 513, 5600 MB Eindhoven, The Netherlands
o.cheong@tue.nl

[2]Department of Computer Science, University of Illinois,
1304 W. Springfield Ave., Urbana, IL 61801, USA
sariel@cs.uiuc.edu

[3]Institute of Computer Science, Hebrew University,
Jerusalem 91904, Israel
nati@cs.huji.ac.il

[4]Department of Applied Mathematics and
Institute of Theoretical Computer Science (ITI), Charles University,
Malostranské nám. 25, 118 00 Praha 1, Czech Republic
matousek@kam.mff.cuni.cz

Abstract. In the one-round Voronoi game, the first player chooses an n-point set W in a square Q, and then the second player places another n-point set B into Q. The payoff for the second player is the fraction of the area of Q occupied by the regions of the points of B in the Voronoi diagram of $W \cup B$. We give a (randomized) strategy for the second player that always guarantees him a payoff of at least $\frac{1}{2} + \alpha$, for a constant $\alpha > 0$ and every large enough n. This contrasts with the one-dimensional situation, with $Q = [0, 1]$, where the first player can always win more than $\frac{1}{2}$.

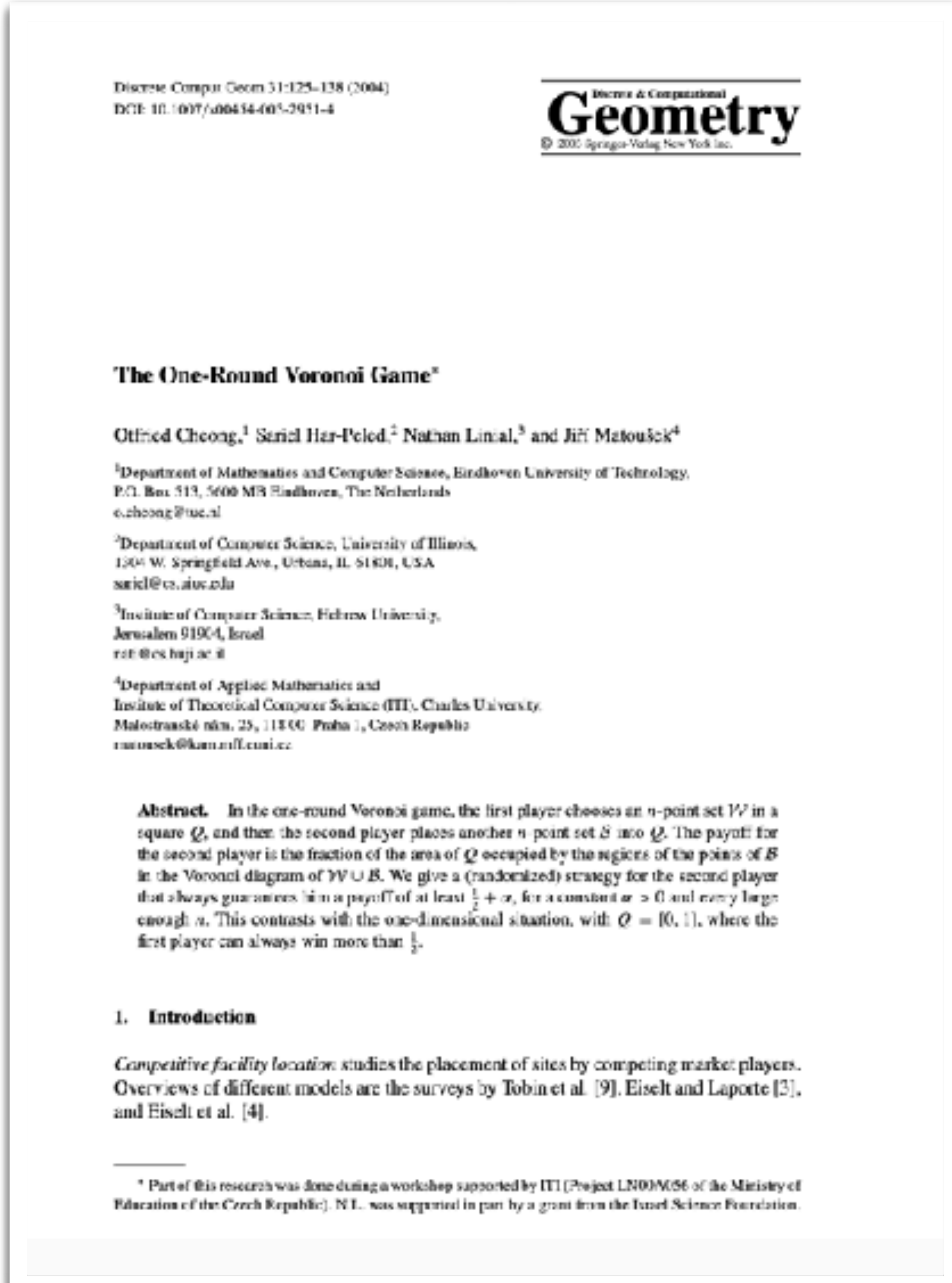
1. Introduction

Competitive facility location studies the placement of sites by competing market players. Overviews of different models are the surveys by Tobin et al. [9], Eiselt and Laporte [3], and Eiselt et al. [4].

* Part of this research was done during a workshop supported by ITI (Project LN00A056 of the Ministry of Education of the Czech Republic). N.L. was supported in part by a grant from the Israel Science Foundation.

Technische Universität Braunschweig

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]



Discrete Comput Geom 31:125–138 (2004)
DOI: 10.1007/s00454-003-2951-4

Discrete & Computational Geometry
© 2003 Springer-Verlag New York Inc.

The One-Round Voronoi Game*

Otfried Cheong,[1] Sariel Har-Peled,[2] Nathan Linial,[3] and Jiří Matoušek[4]

[1]Department of Mathematics and Computer Science, Eindhoven University of Technology,
P.O. Box 513, 5600 MB Eindhoven, The Netherlands
o.cheong@tue.nl

[2]Department of Computer Science, University of Illinois,
1304 W. Springfield Ave., Urbana, IL 61801, USA
sariel@cs.uiuc.edu

[3]Institute of Computer Science, Hebrew University,
Jerusalem 91904, Israel
nati@cs.huji.ac.il

[4]Department of Applied Mathematics and
Institute of Theoretical Computer Science (ITI), Charles University,
Malostranské nám. 25, 118 00 Praha 1, Czech Republic
matousek@kam.mff.cuni.cz

Abstract. In the one-round Voronoi game, the first player chooses an n-point set W in a square Q, and then the second player places another n-point set B into Q. The payoff for the second player is the fraction of the area of Q occupied by the regions of the points of B in the Voronoi diagram of $W \cup B$. We give a (randomized) strategy for the second player that always guarantees him a payoff of at least $\frac{1}{2} + \alpha$, for a constant $\alpha > 0$ and every large enough n. This contrasts with the one-dimensional situation, with $Q = [0, 1]$, where the first player can always win more than $\frac{1}{2}$.

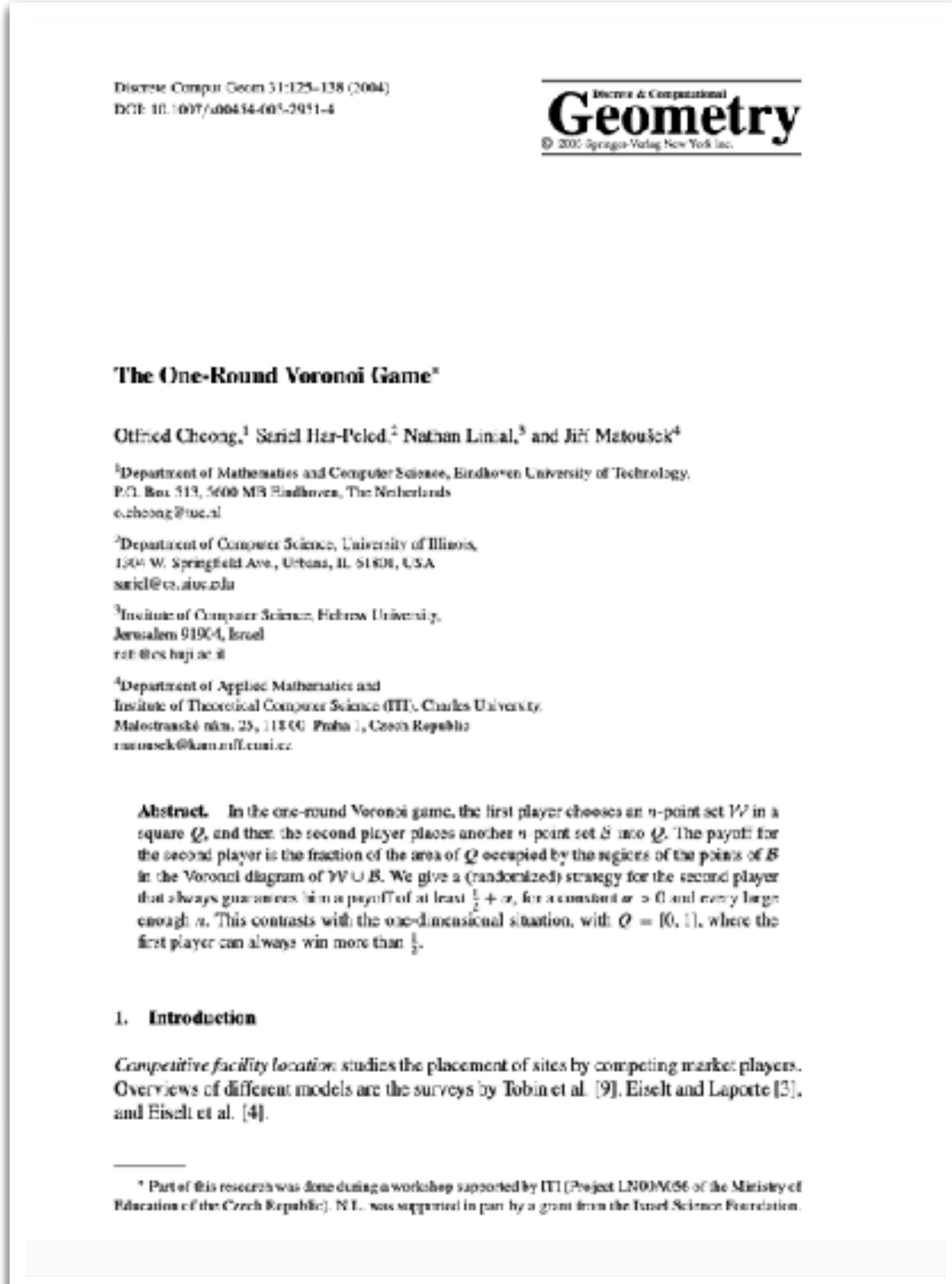
1. **Introduction**

Competitive facility location studies the placement of sites by competing market players. Overviews of different models are the surveys by Tobin et al. [9], Eiselt and Laporte [3], and Eiselt et al. [4].

* Part of this research was done during a workshop supported by ITI (Project LN00A056 of the Ministry of Education of the Czech Republic). N.L. was supported in part by a grant from the Israel Science Foundation.

- **A square**

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]



Discrete Comput Geom 31:125–138 (2004)
DOI: 10.1007/s00454-003-2951-4

Discrete & Computational Geometry

The One-Round Voronoi Game*

Otfried Cheong,[1] Sariel Har-Peled,[2] Nathan Linial,[3] and Jiří Matoušek[4]

[1]Department of Mathematics and Computer Science, Eindhoven University of Technology,
P.O. Box 513, 5600 MB Eindhoven, The Netherlands
o.cheong@tue.nl

[2]Department of Computer Science, University of Illinois,
1304 W. Springfield Ave., Urbana, IL 61801, USA
sariel@cs.uiuc.edu

[3]Institute of Computer Science, Hebrew University,
Jerusalem 91904, Israel
nati@cs.huji.ac.il

[4]Department of Applied Mathematics and
Institute of Theoretical Computer Science (ITI), Charles University,
Malostranské nám. 25, 118 00 Praha 1, Czech Republic
matousek@kam.mff.cuni.cz

Abstract. In the one-round Voronoi game, the first player chooses an n-point set W in a square Q, and then the second player places another n-point set B into Q. The payoff for the second player is the fraction of the area of Q occupied by the regions of the points of B in the Voronoi diagram of $W \cup B$. We give a (randomized) strategy for the second player that always guarantees him a payoff of at least $\frac{1}{2} + \alpha$, for a constant $\alpha > 0$ and every large enough n. This contrasts with the one-dimensional situation, with $Q = [0, 1]$, where the first player can always win more than $\frac{1}{2}$.

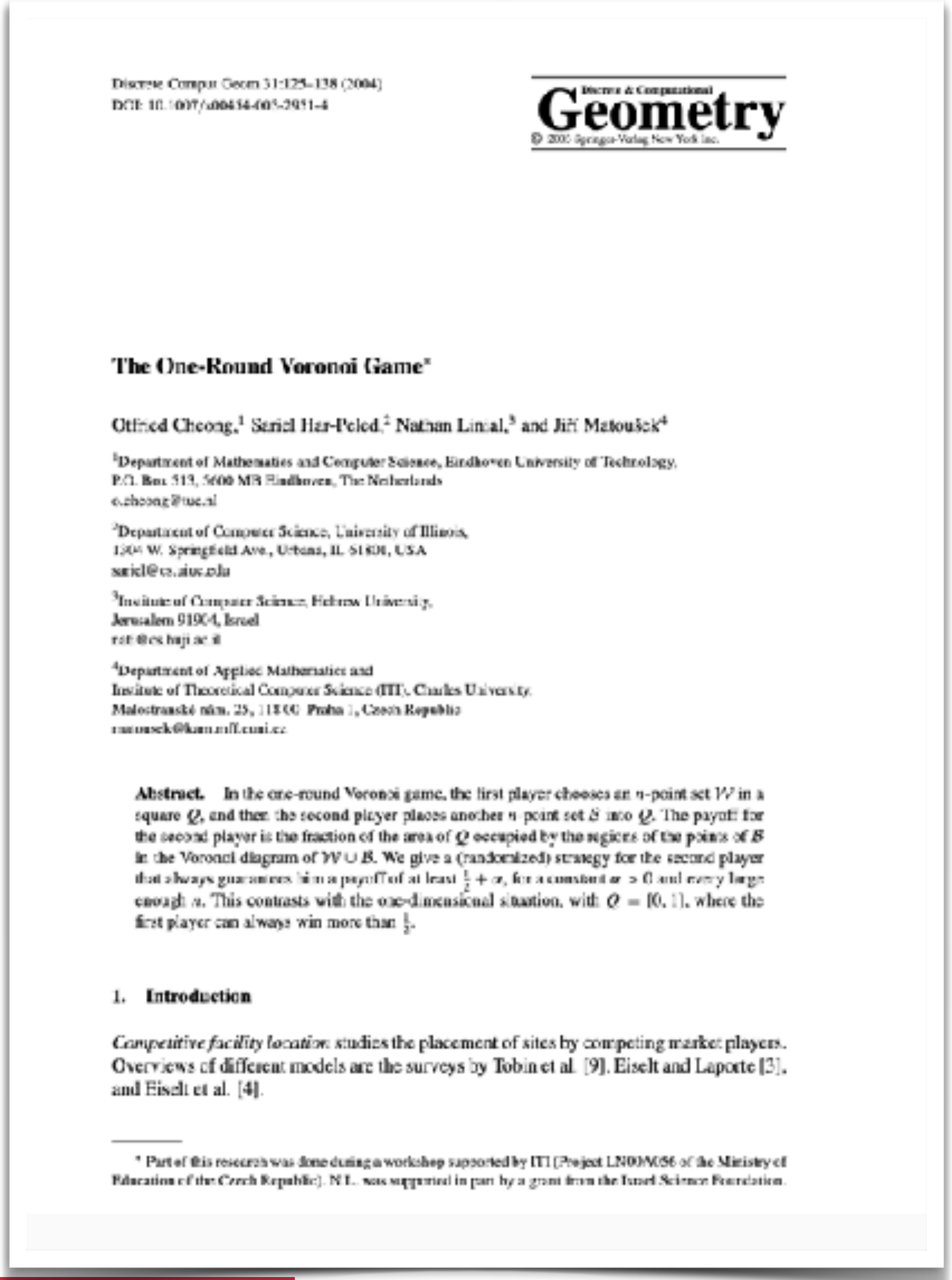
1. Introduction

Competitive facility location studies the placement of sites by competing market players. Overviews of different models are the surveys by Tobin et al. [9], Eiselt and Laporte [3], and Eiselt et al. [4].

* Part of this research was done during a workshop supported by ITI (Project LN00A056 of the Ministry of Education of the Czech Republic). N.L. was supported in part by a grant from the Israel Science Foundation.

- A square
- Two players, White and Black

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]



Discrete Comput Geom 31:125–138 (2004)
DOI: 10.1007/s00454-003-2951-4

Discrete & Computational Geometry

The One-Round Voronoi Game*

Otfried Cheong,[1] Sariel Har-Peled,[2] Nathan Linial,[3] and Jiří Matoušek[4]

[1] Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
o.cheong@tue.nl

[2] Department of Computer Science, University of Illinois, 1304 W. Springfield Ave., Urbana, IL 61801, USA
sariel@cs.uiuc.edu

[3] Institute of Computer Science, Hebrew University, Jerusalem 91904, Israel
nati@cs.huji.ac.il

[4] Department of Applied Mathematics and Institute of Theoretical Computer Science (ITI), Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic
matousek@kam.mff.cuni.cz

Abstract. In the one-round Voronoi game, the first player chooses an n-point set W in a square Q, and then the second player places another n-point set B into Q. The payoff for the second player is the fraction of the area of Q occupied by the regions of the points of B in the Voronoi diagram of $W \cup B$. We give a (randomized) strategy for the second player that always guarantees him a payoff of at least $\frac{1}{2} + \alpha$, for a constant $\alpha > 0$ and every large enough n. This contrasts with the one-dimensional situation, with $Q = [0, 1]$, where the first player can always win more than $\frac{1}{2}$.

1. Introduction

Competitive facility location studies the placement of sites by competing market players. Overviews of different models are the surveys by Tobin et al. [9], Eiselt and Laporte [3], and Eiselt et al. [4].

* Part of this research was done during a workshop supported by ITI (Project LN00A056 of the Ministry of Education of the Czech Republic). N.L. was supported in part by a grant from the Israel Science Foundation.

- A square
- Two players, White and Black
- Players place all points at once

Technische Universität Braunschweig

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]

Discrete Comput Geom 31:125–138 (2004)
DOI: 10.1007/s00454-003-2951-4

Discrete & Computational Geometry

The One-Round Voronoi Game*

Otfried Cheong,[1] Sariel Har-Peled,[2] Nathan Linial,[3] and Jiří Matoušek[4]

[1]Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
o.cheong@tue.nl

[2]Department of Computer Science, University of Illinois, 1304 W. Springfield Ave., Urbana, IL 61801, USA
sariel@cs.uiuc.edu

[3]Institute of Computer Science, Hebrew University, Jerusalem 91904, Israel
nati@cs.huji.ac.il

[4]Department of Applied Mathematics and Institute of Theoretical Computer Science (ITI), Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic
matousek@kam.mff.cuni.cz

Abstract. In the one-round Voronoi game, the first player chooses an n-point set W in a square Q, and then the second player places another n-point set B into Q. The payoff for the second player is the fraction of the area of Q occupied by the regions of the points of B in the Voronoi diagram of $W \cup B$. We give a (randomized) strategy for the second player that always guarantees him a payoff of at least $\frac{1}{2} + \alpha$, for a constant $\alpha > 0$ and every large enough n. This contrasts with the one-dimensional situation, with $Q = [0, 1]$, where the first player can always win more than $\frac{1}{2}$.

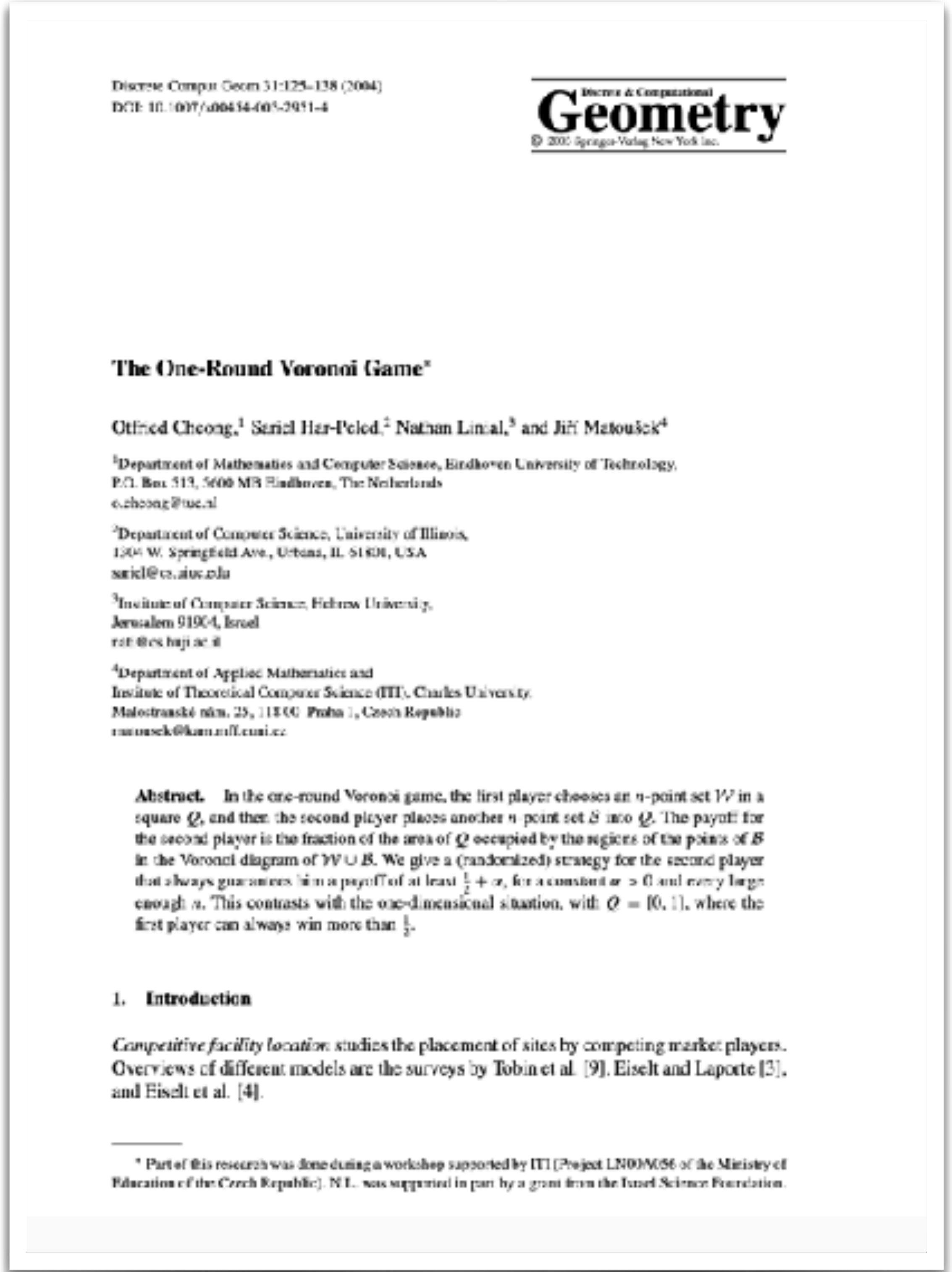
1. Introduction

Competitive facility location studies the placement of sites by competing market players. Overviews of different models are the surveys by Tobin et al. [9], Eiselt and Laporte [3], and Eiselt et al. [4].

* Part of this research was done during a workshop supported by ITI (Project LN00A056 of the Ministry of Education of the Czech Republic). N.L. was supported in part by a grant from the Israel Science Foundation.

- A square
- Two players, White and Black
- Players place all points at once
- Voronoi diagram is computed

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]



Discrete Comput Geom 31:125–138 (2004)
DOI: 10.1007/s00454-003-2951-4

Discrete & Computational
Geometry

The One-Round Voronoi Game*

Otfried Cheong,[1] Sariel Har-Peled,[2] Nathan Linial,[3] and Jiří Matoušek[4]

[1]Department of Mathematics and Computer Science, Eindhoven University of Technology,
P.O. Box 513, 5600 MB Eindhoven, The Netherlands
o.cheong@tue.nl

[2]Department of Computer Science, University of Illinois,
1304 W. Springfield Ave., Urbana, IL 61801, USA
sariel@cs.uiuc.edu

[3]Institute of Computer Science, Hebrew University,
Jerusalem 91904, Israel
nati@cs.huji.ac.il

[4]Department of Applied Mathematics and
Institute of Theoretical Computer Science (ITI), Charles University,
Malostranské nám. 25, 118 00 Praha 1, Czech Republic
matousek@kam.mff.cuni.cz

Abstract. In the one-round Voronoi game, the first player chooses an n-point set W in a square Q, and then the second player places another n-point set B into Q. The payoff for the second player is the fraction of the area of Q occupied by the regions of the points of B in the Voronoi diagram of $W \cup B$. We give a (randomized) strategy for the second player that always guarantees him a payoff of at least $\frac{1}{2} + \alpha$, for a constant $\alpha > 0$ and every large enough n. This contrasts with the one-dimensional situation, with $Q = [0, 1]$, where the first player can always win more than $\frac{1}{2}$.

1. Introduction

Competitive facility location studies the placement of sites by competing market players. Overviews of different models are the surveys by Tobin et al. [9], Eiselt and Laporte [3], and Eiselt et al. [4].

* Part of this research was done during a workshop supported by ITI (Project LN00A056 of the Ministry of Education of the Czech Republic). N.L. was supported in part by a grant from the Israel Science Foundation.

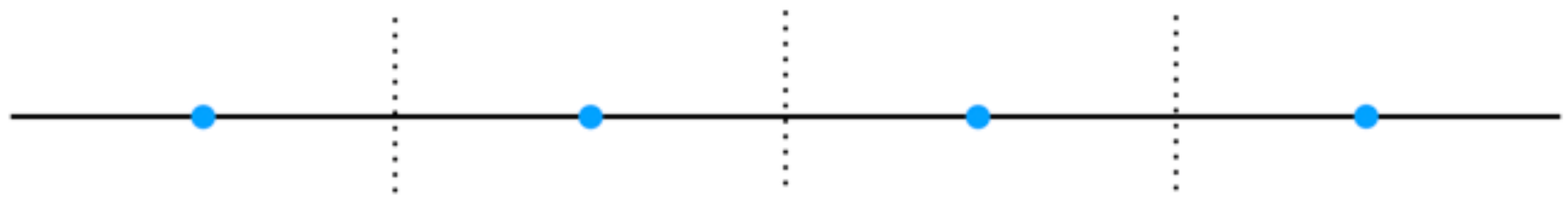
- A square
- Two players, White and Black
- Players place all points at once
- Voronoi diagram is computed
- Player with larger area wins

Technische Universität Braunschweig

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]

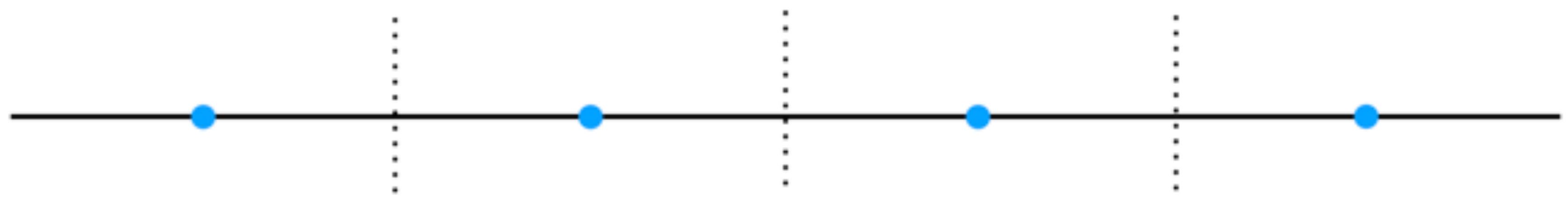
Technische Universität Braunschweig

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]



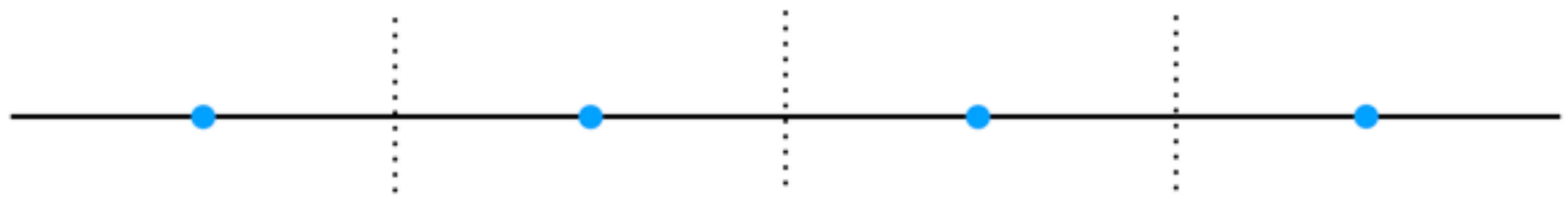
Technische Universität Braunschweig

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]



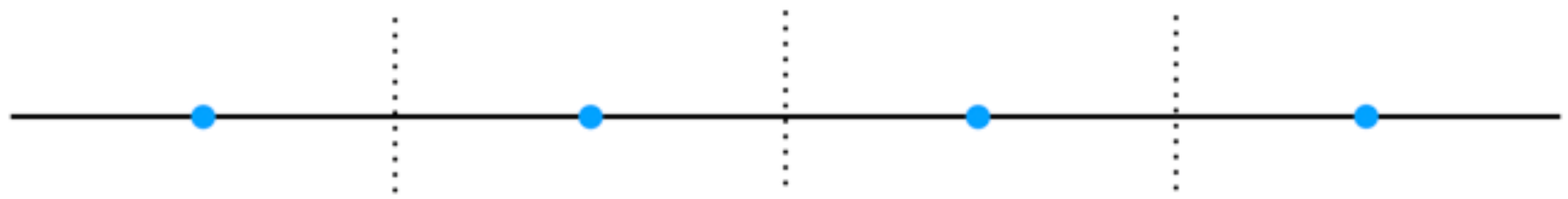
- **1D: First player wins**

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]



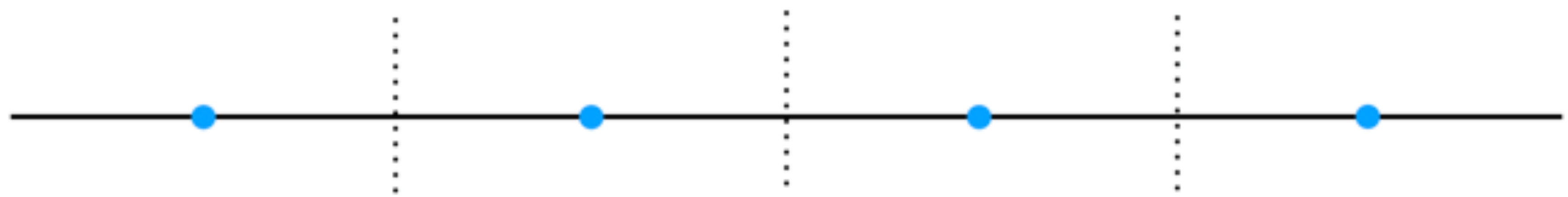
- 1D: First player wins
- 2D: Second player wins for large n

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]



- 1D: First player wins
- 2D: Second player wins for large n
- Complicated randomized arguments

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]



- **1D: First player wins**
- **2D: Second player wins for large n**
- **Complicated randomized arguments**

Lemma 4. *For every sufficiently large constant D, there exist constants $\beta_1 > 0, \delta > 0$, and n_0 such that for every n-point set $\mathcal{W} \subset Q$, $n \geq n_0$, if $\mathcal{B} \subset Q$ is obtained by δn independent random draws from the uniform distribution on Q, then*

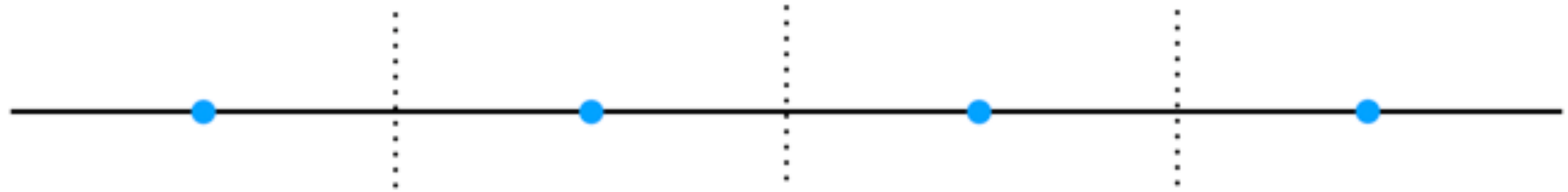
$$\mathrm{E}[\mathrm{vol}(R(\mathcal{B}, \mathcal{W}))] \geq (\tfrac{1}{2} + \beta_1)\delta n.$$

If the total area A_ℓ of the long regions (of diameter at least D) exceeds $n/2D$, then

$$\mathrm{E}[\mathrm{vol}(R(\mathcal{B}, \mathcal{W}))] \geq 2\delta n.$$

Technische Universität Braunschweig

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]



- 1D: First player wins
- 2D: Second player wins for large n
- Complicated randomized arguments

$$\mathbf{E}[\mathrm{vol}(R(x,\mathcal{W}))] = \frac{1}{\mathrm{vol}(Q)}\int_Q\int_Q I_{R(x,\mathcal{W})}(y)\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{n}\int_Q \mathrm{vol}(\{x \in Q : y \in R(x,\mathcal{W})\})\,\mathrm{d}y$$

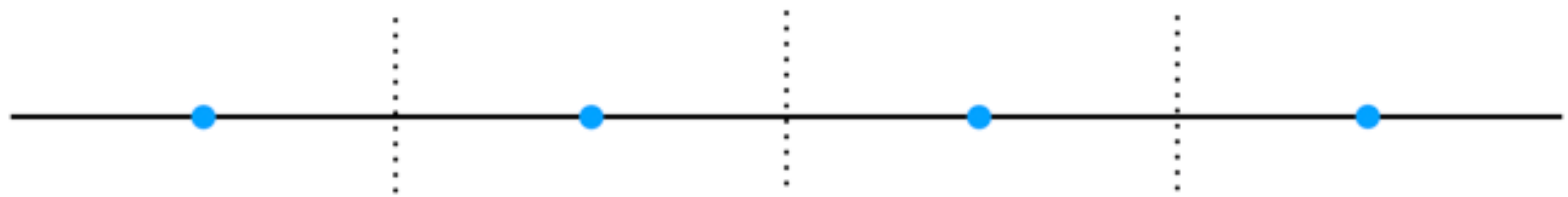
Lemma 4. *For every sufficiently large constant D, there exist constants $\beta_1 > 0, \delta > 0$, and n_0 such that for every n-point set $\mathcal{W} \subset Q$, $n \geq n_0$, if $\mathcal{B} \subset Q$ is obtained by δn independent random draws from the uniform distribution on Q, then*

$$\mathbf{E}[\mathrm{vol}(R(\mathcal{B},\mathcal{W}))] \geq (\tfrac{1}{2} + \beta_1)\delta n.$$

If the total area A_ℓ of the long regions (of diameter at least D) exceeds $n/2D$, then

$$\mathbf{E}[\mathrm{vol}(R(\mathcal{B},\mathcal{W}))] \geq 2\delta n.$$

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]



- 1D: First player wins
- 2D: Second player wins for large n
- Complicated randomized arguments

Lemma 4. *For every sufficiently large constant D, there exist constants $\beta_1 > 0, \delta > 0$, and n_0 such that for every n-point set $\mathcal{W} \subset Q$, $n \geq n_0$, if $\mathcal{B} \subset Q$ is obtained by δn independent random draws from the uniform distribution on Q, then*

$$\mathbf{E}[\mathrm{vol}(R(\mathcal{B}, \mathcal{W}))] \geq (\tfrac{1}{2} + \beta_1)\delta n.$$

If the total area A_ℓ of the long regions (of diameter at least D) exceeds $n/2D$, then

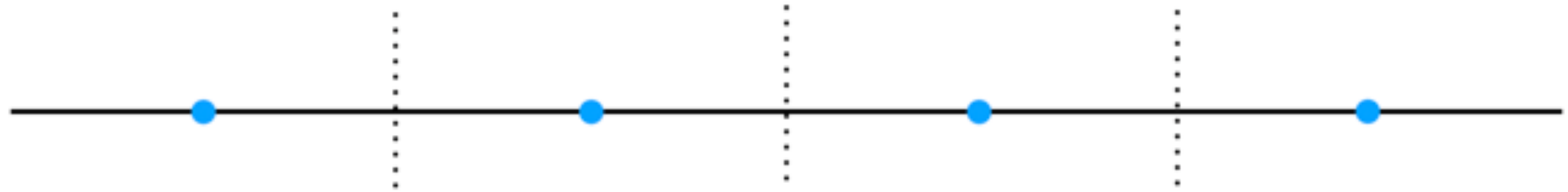
$$\mathbf{E}[\mathrm{vol}(R(\mathcal{B}, \mathcal{W}))] \geq 2\delta n.$$

$$\mathbf{E}[\mathrm{vol}(R(x, \mathcal{W}))] = \frac{1}{\mathrm{vol}(Q)} \int_Q \int_Q I_{R(x,\mathcal{W})}(y)\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{n} \int_Q \mathrm{vol}(\{x \in Q : y \in R(x, \mathcal{W})\})\,\mathrm{d}y$$

$$\int_{C_0} \mathrm{dist}(y, w)^2\,\mathrm{d}y = \int_0^{\sqrt{a/\pi}} r^2 \cdot 2\pi r\,\mathrm{d}r = \frac{a^2}{2\pi}.$$

Technische Universität Braunschweig

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]



- 1D: First player wins
- 2D: Second player wins for large n
- Complicated randomized arguments

Lemma 4. *For every sufficiently large constant D, there exist constants $\beta_1 > 0, \delta > 0$, and n_0 such that for every n-point set $\mathcal{W} \subset Q$, $n \geq n_0$, if $\mathcal{B} \subset Q$ is obtained by δn independent random draws from the uniform distribution on Q, then*

$$\mathbf{E}[\mathrm{vol}(R(\mathcal{B}, \mathcal{W}))] \geq (\tfrac{1}{2} + \beta_1)\delta n.$$

If the total area A_ℓ of the long regions (of diameter at least D) exceeds $n/2D$, then

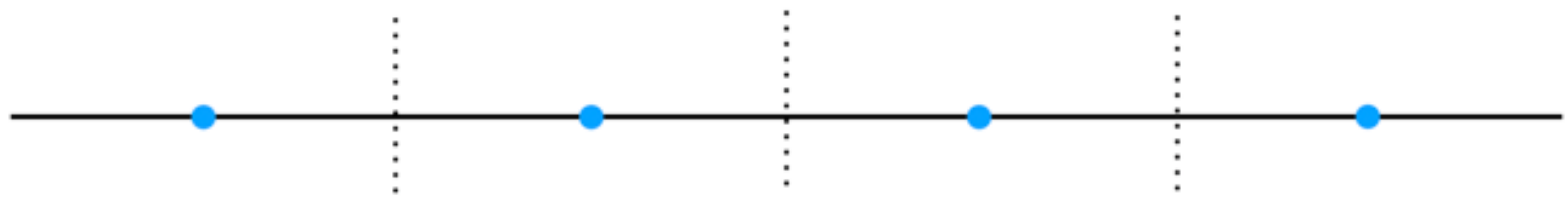
$$\mathbf{E}[\mathrm{vol}(R(\mathcal{B}, \mathcal{W}))] \geq 2\delta n.$$

$$\mathbf{E}[\mathrm{vol}(R(x, \mathcal{W}))] = \frac{1}{\mathrm{vol}(Q)} \int_Q \int_Q I_{R(x,\mathcal{W})}(y)\, \mathrm{d}y\, \mathrm{d}x$$
$$= \frac{1}{n} \int_Q \mathrm{vol}(\{x \in Q : y \in R(x, \mathcal{W})\})\, \mathrm{d}y$$

$$\int_{C_0} \mathrm{dist}(y, w)^2\, \mathrm{d}y = \int_0^{\sqrt{a/\pi}} r^2 \cdot 2\pi r\, \mathrm{d}r = \frac{a^2}{2\pi}.$$

$$F_0(\mathcal{W}) = \frac{\pi}{n} \sum_{w \in \mathcal{W}} \int_{\mathrm{cell}_{\mathcal{W}}(w)} \mathrm{dist}(y, w)^2\, \mathrm{d}y$$
$$\geq \frac{1}{2n} \sum_{w \in \mathcal{W}} a_w^2 \geq \frac{1}{2n} \frac{\left(\sum_{w \in \mathcal{W}} a_w\right)^2}{n} \geq \frac{1}{2}$$

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]



- 1D: First player wins
- 2D: Second player wins for large n
- Complicated randomized arguments

Lemma 4. *For every sufficiently large constant D, there exist constants $\beta_1 > 0, \delta > 0$, and n_0 such that for every n-point set $\mathcal{W} \subset Q$, $n \geq n_0$, if $\mathcal{B} \subset Q$ is obtained by δn independent random draws from the uniform distribution on Q, then*

$$\mathbf{E}[\mathrm{vol}(R(\mathcal{B}, \mathcal{W}))] \geq (\tfrac{1}{2} + \beta_1)\delta n.$$

If the total area A_ℓ of the long regions (of diameter at least D) exceeds $n/2D$, then

$$\mathbf{E}[\mathrm{vol}(R(\mathcal{B}, \mathcal{W}))] \geq 2\delta n.$$

$$\begin{aligned}\mathbf{E}[\mathrm{vol}(R(x, \mathcal{W}))] &= \frac{1}{\mathrm{vol}(Q)} \int_Q \int_Q I_{R(x,\mathcal{W})}(y)\, \mathrm{d}y\, \mathrm{d}x \\ &= \frac{1}{n} \int_Q \mathrm{vol}(\{x \in Q : y \in R(x, \mathcal{W})\})\, \mathrm{d}y\end{aligned}$$

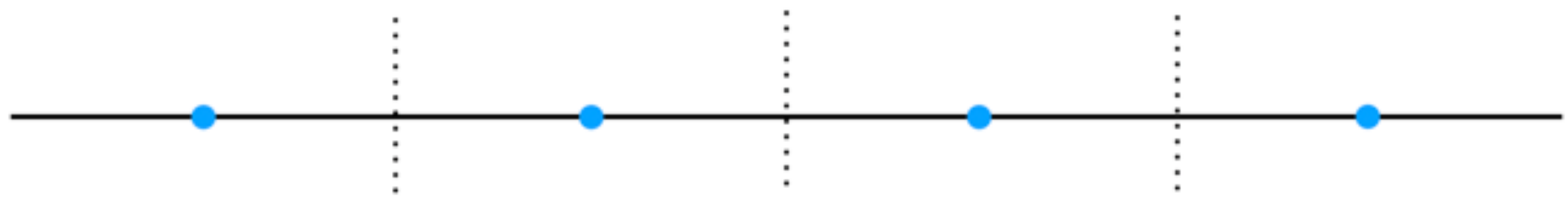
$$\int_{C_0} \mathrm{dist}(y, w)^2\, \mathrm{d}y = \int_0^{\sqrt{a/\pi}} r^2 \cdot 2\pi r\, \mathrm{d}r = \frac{a^2}{2\pi}.$$

$$\begin{aligned}F_0(\mathcal{W}) &= \frac{\pi}{n} \sum_{w \in \mathcal{W}} \int_{\mathrm{cell}_{\mathcal{W}}(w)} \mathrm{dist}(y, w)^2\, \mathrm{d}y \\ &\geq \frac{1}{2n} \sum_{w \in \mathcal{W}} a_w^2 \geq \frac{1}{2n} \frac{\left(\sum_{w \in \mathcal{W}} a_w\right)^2}{n} \geq \frac{1}{2}\end{aligned}$$

$$\begin{aligned}P(y) &= \mathrm{Prob}[\mathcal{B} \cap B(y, \mathrm{dist}(y, \mathcal{W})) \neq \emptyset] \\ &= 1 - (\mathrm{Prob}[x \notin B(y, \mathrm{dist}(y, \mathcal{W}))])^{\delta n} \\ &= 1 - \left(1 - \frac{1}{n} \cdot \mathrm{vol}(B(y, \mathrm{dist}(y, \mathcal{W})) \cap Q)\right)^{\delta n}.\end{aligned}$$

Technische Universität Braunschweig

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]



- **1D: First player wins**
- **2D: Second player wins for large n**
- **Complicated randomized arguments**

Lemma 4. *For every sufficiently large constant D, there exist constants $\beta_1 > 0, \delta > 0$, and n_0 such that for every n-point set $\mathcal{W} \subset Q$, $n \geq n_0$, if $\mathcal{B} \subset Q$ is obtained by δn independent random draws from the uniform distribution on Q, then*

$$\mathbf{E}[\mathrm{vol}(R(\mathcal{B}, \mathcal{W}))] \geq (\tfrac{1}{2} + \beta_1)\delta n.$$

If the total area A_ℓ of the long regions (of diameter at least D) exceeds $n/2D$, then

$$\mathbf{E}[\mathrm{vol}(R(\mathcal{B}, \mathcal{W}))] \geq 2\delta n.$$

Theorem 5. *There exist constants $\alpha > 0$ and n_0 such that for every $n \geq n_0$, Black can always win at least $\frac{1}{2} + \alpha$ in the Voronoi game. That is, for every n-point set $\mathcal{W} \subset Q$ there exists an n-point set $\mathcal{B} \subset Q \setminus \mathcal{W}$ with $\mathrm{vol}(R(\mathcal{B}, \mathcal{W})) \geq (\frac{1}{2} + \alpha)\,\mathrm{vol}(Q)$.*

$$\begin{aligned}\mathbf{E}[\mathrm{vol}(R(x, \mathcal{W}))] &= \frac{1}{\mathrm{vol}(Q)} \int_Q \int_Q I_{R(x,\mathcal{W})}(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{1}{n} \int_Q \mathrm{vol}(\{x \in Q : y \in R(x, \mathcal{W})\})\,\mathrm{d}y\end{aligned}$$

$$\int_{C_0} \mathrm{dist}(y, w)^2\,\mathrm{d}y = \int_0^{\sqrt{a/\pi}} r^2 \cdot 2\pi r\,\mathrm{d}r = \frac{a^2}{2\pi}.$$

$$\begin{aligned}F_0(\mathcal{W}) &= \frac{\pi}{n} \sum_{w \in \mathcal{W}} \int_{\mathrm{cell}_{\mathcal{W}}(w)} \mathrm{dist}(y, w)^2\,\mathrm{d}y \\ &\geq \frac{1}{2n} \sum_{w \in \mathcal{W}} a_w^2 \geq \frac{1}{2n} \frac{\left(\sum_{w \in \mathcal{W}} a_w\right)^2}{n} \geq \frac{1}{2}\end{aligned}$$

$$\begin{aligned}P(y) &= \mathrm{Prob}[\mathcal{B} \cap B(y, \mathrm{dist}(y, \mathcal{W})) \neq \emptyset] \\ &= 1 - (\mathrm{Prob}[x \notin B(y, \mathrm{dist}(y, \mathcal{W}))])^{\delta n} \\ &= 1 - \left(1 - \frac{1}{n} \cdot \mathrm{vol}(B(y, \mathrm{dist}(y, \mathcal{W})) \cap Q)\right)^{\delta n}.\end{aligned}$$

Technische Universität Braunschweig

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]

Discrete Comput Geom 31:125–138 (2004)
DOI: 10.1007/s00454-003-2951-4

Discrete & Computational Geometry

The One-Round Voronoi Game*

Otfried Cheong,[1] Sariel Har-Peled,[2] Nathan Linial,[3] and Jiří Matoušek[4]

[1]Department of Mathematics and Computer Science, Eindhoven University of Technology,
P.O. Box 513, 5600 MB Eindhoven, The Netherlands
o.cheong@tue.nl

[2]Department of Computer Science, University of Illinois,
1304 W. Springfield Ave., Urbana, IL 61801, USA
sariel@cs.uiuc.edu

[3]Institute of Computer Science, Hebrew University,
Jerusalem 91904, Israel
nati@cs.huji.ac.il

[4]Department of Applied Mathematics and
Institute of Theoretical Computer Science (ITI), Charles University,
Malostranské nám. 25, 118 00 Praha 1, Czech Republic
matousek@kam.mff.cuni.cz

Abstract. In the one-round Voronoi game, the first player chooses an n-point set W in a square Q, and then the second player places another n-point set B into Q. The payoff for the second player is the fraction of the area of Q occupied by the regions of the points of B in the Voronoi diagram of $W \cup B$. We give a (randomized) strategy for the second player that always guarantees him a payoff of at least $\frac{1}{2} + \alpha$, for a constant $\alpha > 0$ and every large enough n. This contrasts with the one-dimensional situation, with $Q = [0, 1]$, where the first player can always win more than $\frac{1}{2}$.

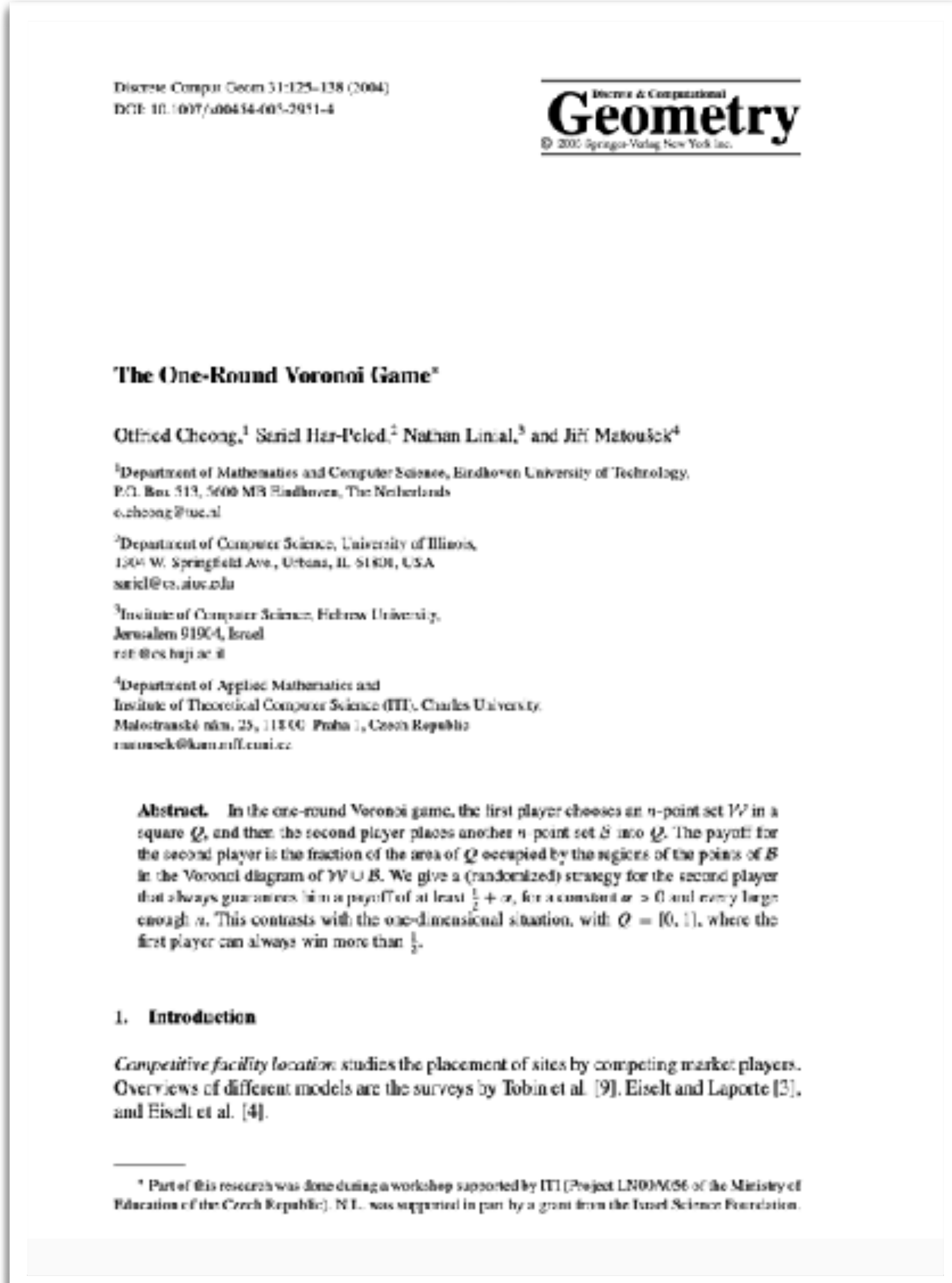
1. Introduction

Competitive facility location studies the placement of sites by competing market players. Overviews of different models are the surveys by Tobin et al. [9], Eiselt and Laporte [3], and Eiselt et al. [4].

* Part of this research was done during a workshop supported by ITI (Project LN00A056 of the Ministry of Education of the Czech Republic). N.L. was supported in part by a grant from the Israel Science Foundation.

Technische Universität Braunschweig

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]


Discrete Comput Geom 31:125–138 (2004)
DOI: 10.1007/s00454-003-2951-4

Discrete & Computational Geometry

The One-Round Voronoi Game*

Otfried Cheong,[1] Sariel Har-Peled,[2] Nathan Linial,[3] and Jiří Matoušek[4]

[1]Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
o.cheong@tue.nl

[2]Department of Computer Science, University of Illinois, 1304 W. Springfield Ave., Urbana, IL 61801, USA
sariel@cs.uiuc.edu

[3]Institute of Computer Science, Hebrew University, Jerusalem 91904, Israel
nati@cs.huji.ac.il

[4]Department of Applied Mathematics and Institute of Theoretical Computer Science (ITI), Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic
matousek@kam.mff.cuni.cz

Abstract. In the one-round Voronoi game, the first player chooses an n-point set W in a square Q, and then the second player places another n-point set B into Q. The payoff for the second player is the fraction of the area of Q occupied by the regions of the points of B in the Voronoi diagram of $W \cup B$. We give a (randomized) strategy for the second player that always guarantees him a payoff of at least $\frac{1}{2} + \alpha$, for a constant $\alpha > 0$ and every large enough n. This contrasts with the one-dimensional situation, with $Q = [0, 1]$, where the first player can always win more than $\frac{1}{2}$.

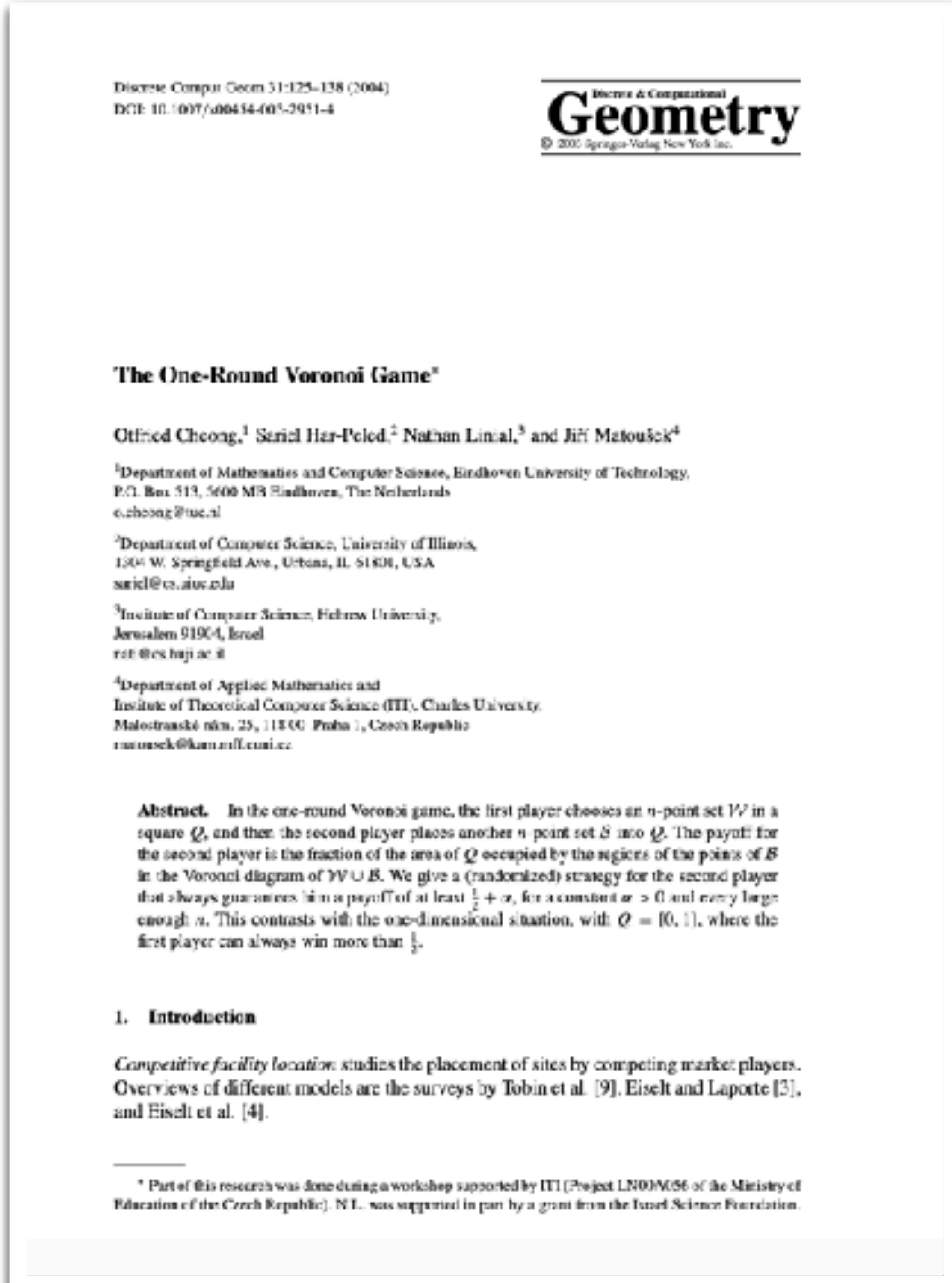
1. Introduction

Competitive facility location studies the placement of sites by competing market players. Overviews of different models are the surveys by Tobin et al. [9], Eiselt and Laporte [3], and Eiselt et al. [4].

* Part of this research was done during a workshop supported by ITI (Project LN00A056 of the Ministry of Education of the Czech Republic). N.L. was supported in part by a grant from the Israel Science Foundation.

Open:

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]


Discrete Comput Geom 31:125–138 (2004)
DOI: 10.1007/s00454-003-2951-4

Discrete & Computational Geometry
© 2003 Springer-Verlag New York Inc.

The One-Round Voronoi Game*

Otfried Cheong,[1] Sariel Har-Peled,[2] Nathan Linial,[3] and Jiří Matoušek[4]

[1]Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
o.cheong@tue.nl

[2]Department of Computer Science, University of Illinois, 1304 W. Springfield Ave., Urbana, IL 61801, USA
sariel@cs.uiuc.edu

[3]Institute of Computer Science, Hebrew University, Jerusalem 91904, Israel
nati@cs.huji.ac.il

[4]Department of Applied Mathematics and Institute of Theoretical Computer Science (ITI), Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic
matousek@kam.mff.cuni.cz

Abstract. In the one-round Voronoi game, the first player chooses an n-point set W in a square Q, and then the second player places another n-point set B into Q. The payoff for the second player is the fraction of the area of Q occupied by the regions of the points of B in the Voronoi diagram of $W \cup B$. We give a (randomized) strategy for the second player that always guarantees him a payoff of at least $\frac{1}{2} + \alpha$, for a constant $\alpha > 0$ and every large enough n. This contrasts with the one-dimensional situation, with $Q = [0, 1]$, where the first player can always win more than $\frac{1}{2}$.

1. **Introduction**

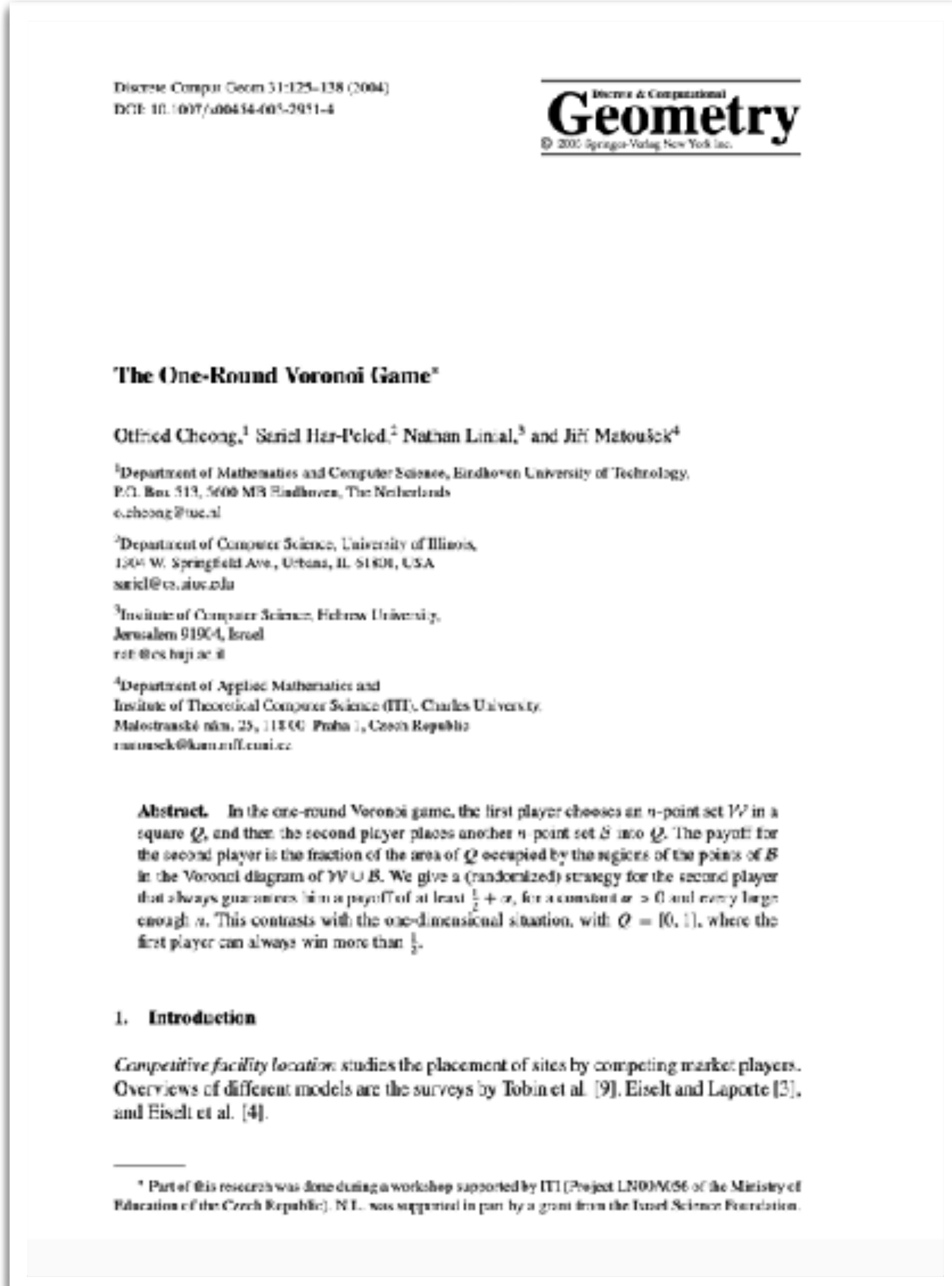
Competitive facility location studies the placement of sites by competing market players. Overviews of different models are the surveys by Tobin et al. [9], Eiselt and Laporte [3], and Eiselt et al. [4].

* Part of this research was done during a workshop supported by ITI (Project LN00A056 of the Ministry of Education of the Czech Republic). N.L. was supported in part by a grant from the Israel Science Foundation.

Open:

- **White wins for $n=1$, Black for large n**

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]



Discrete Comput Geom 31:125–138 (2004)
DOI: 10.1007/s00454-003-2951-4

Discrete & Computational Geometry

The One-Round Voronoi Game*

Otfried Cheong,[1] Sariel Har-Peled,[2] Nathan Linial,[3] and Jiří Matoušek[4]

[1]Department of Mathematics and Computer Science, Eindhoven University of Technology,
P.O. Box 513, 5600 MB Eindhoven, The Netherlands
o.cheong@tue.nl

[2]Department of Computer Science, University of Illinois,
1304 W. Springfield Ave., Urbana, IL 61801, USA
sariel@cs.uiuc.edu

[3]Institute of Computer Science, Hebrew University,
Jerusalem 91904, Israel
nati@cs.huji.ac.il

[4]Department of Applied Mathematics and
Institute of Theoretical Computer Science (ITI), Charles University,
Malostranské nám. 25, 118 00 Praha 1, Czech Republic
matousek@kam.mff.cuni.cz

Abstract. In the one-round Voronoi game, the first player chooses an n-point set W in a square Q, and then the second player places another n-point set B into Q. The payoff for the second player is the fraction of the area of Q occupied by the regions of the points of B in the Voronoi diagram of $W \cup B$. We give a (randomized) strategy for the second player that always guarantees him a payoff of at least $\frac{1}{2} + \alpha$, for a constant $\alpha > 0$ and every large enough n. This contrasts with the one-dimensional situation, with $Q = [0, 1]$, where the first player can always win more than $\frac{1}{2}$.

1. Introduction

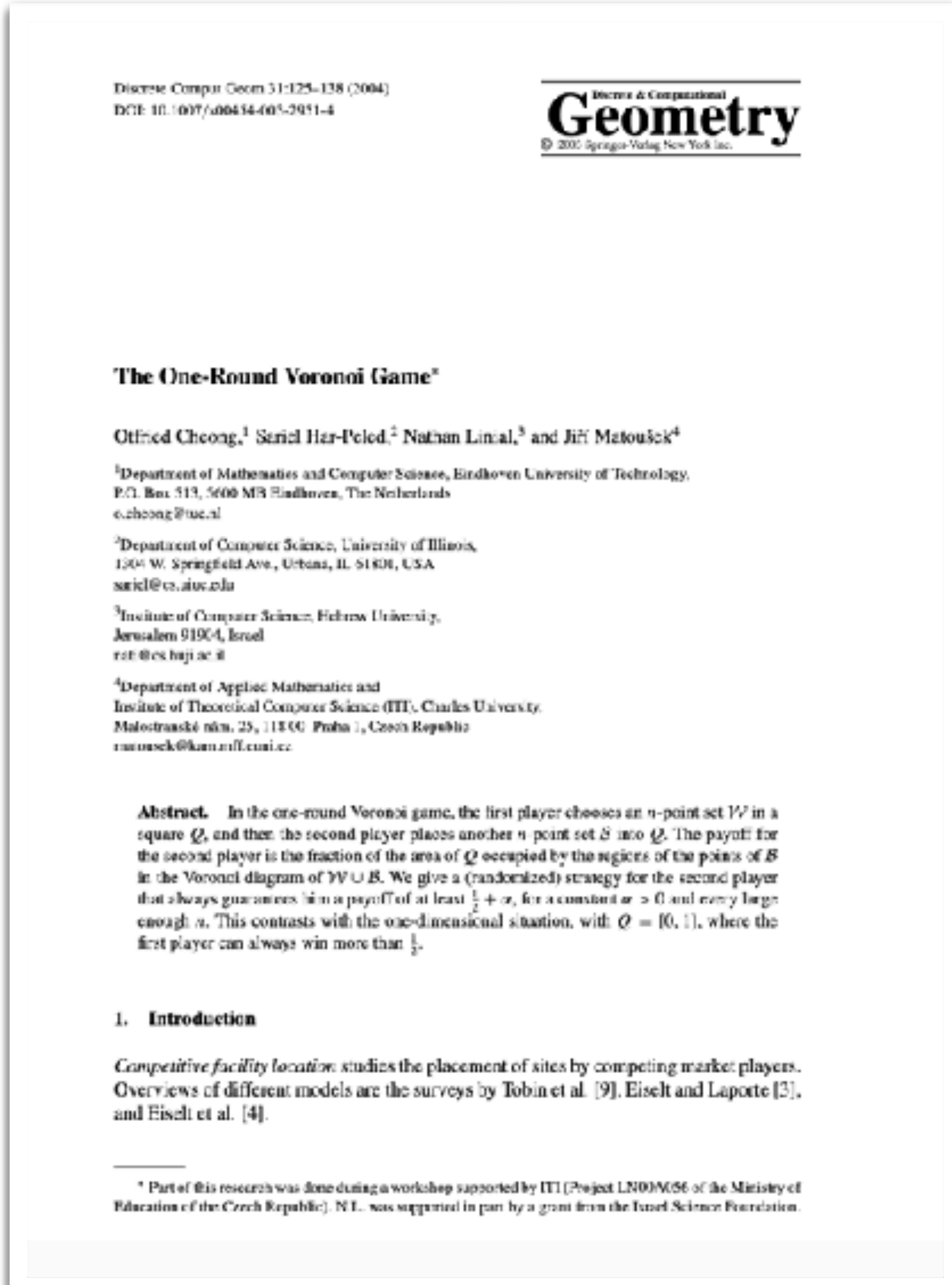
Competitive facility location studies the placement of sites by competing market players. Overviews of different models are the surveys by Tobin et al. [9], Eiselt and Laporte [3], and Eiselt et al. [4].

* Part of this research was done during a workshop supported by ITI (Project LN00A056 of the Ministry of Education of the Czech Republic). N.L. was supported in part by a grant from the Israel Science Foundation.

Open:

- **White wins for *n=1*, Black for large *n***
- **White wins for 1D, Black for 2D**

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]


Discrete Comput Geom 31:125–138 (2004)
DOI: 10.1007/s00454-003-2951-4

Discrete & Computational Geometry

The One-Round Voronoi Game*

Otfried Cheong,[1] Sariel Har-Peled,[2] Nathan Linial,[3] and Jiří Matoušek[4]

[1]Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
o.cheong@tue.nl

[2]Department of Computer Science, University of Illinois, 1304 W. Springfield Ave., Urbana, IL 61801, USA
sariel@cs.uiuc.edu

[3]Institute of Computer Science, Hebrew University, Jerusalem 91904, Israel
nati@cs.huji.ac.il

[4]Department of Applied Mathematics and Institute of Theoretical Computer Science (ITI), Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic
matousek@kam.mff.cuni.cz

Abstract. In the one-round Voronoi game, the first player chooses an n-point set W in a square Q, and then the second player places another n-point set B into Q. The payoff for the second player is the fraction of the area of Q occupied by the regions of the points of B in the Voronoi diagram of $W \cup B$. We give a (randomized) strategy for the second player that always guarantees him a payoff of at least $\frac{1}{2} + \alpha$, for a constant $\alpha > 0$ and every large enough n. This contrasts with the one-dimensional situation, with $Q = [0, 1]$, where the first player can always win more than $\frac{1}{2}$.

1. Introduction

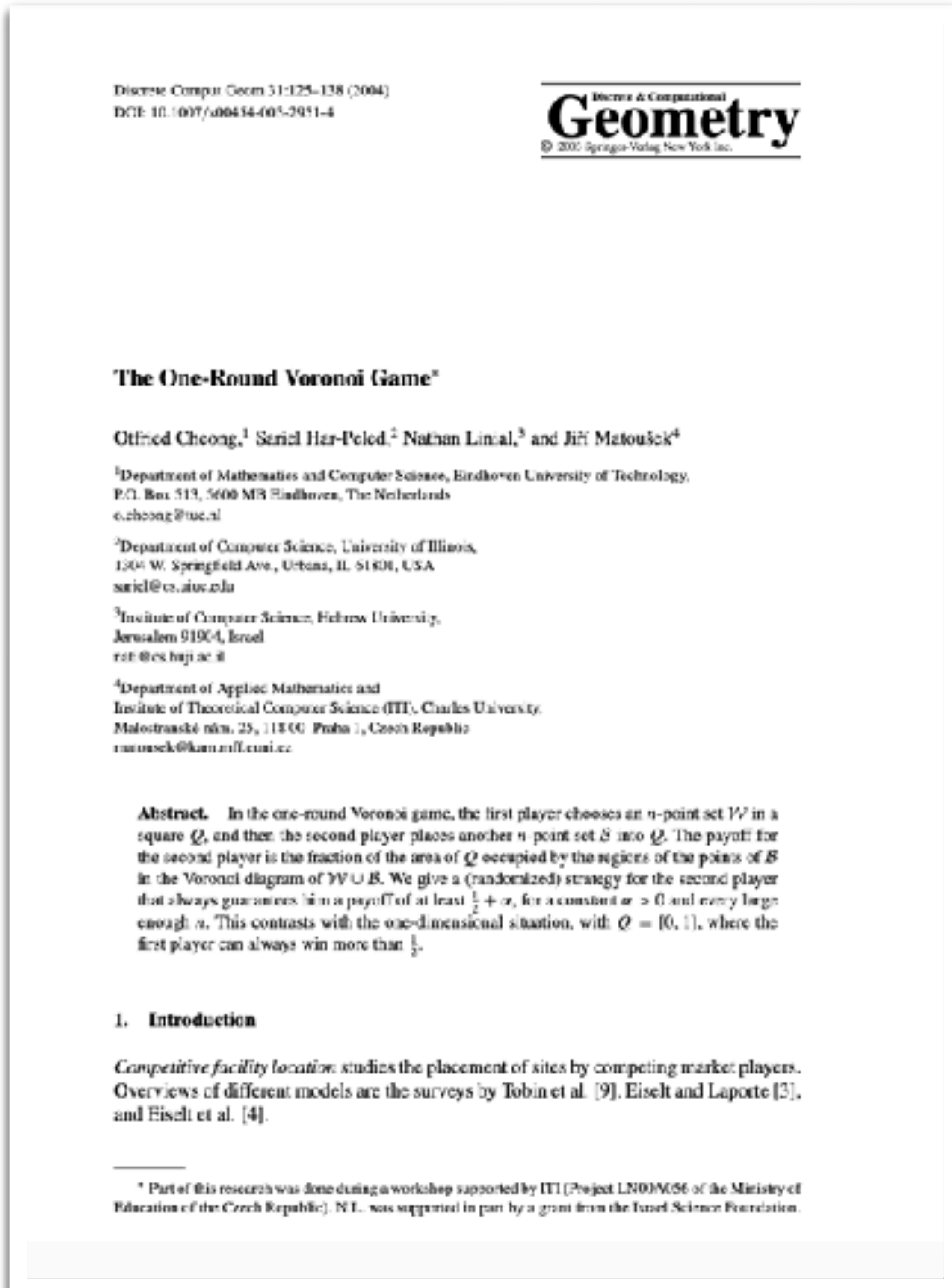
Competitive facility location studies the placement of sites by competing market players. Overviews of different models are the surveys by Tobin et al. [9], Eiselt and Laporte [3], and Eiselt et al. [4].

* Part of this research was done during a workshop supported by ITI (Project LN00A056 of the Ministry of Education of the Czech Republic). N.L. was supported in part by a grant from the Israel Science Foundation.

Open:

- White wins for $n=1$, Black for large n
- White wins for 1D, Black for 2D
- Strategy uses randomization

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]


Discrete Comput Geom 31:125–138 (2004)
DOI: 10.1007/s00454-003-2951-4

Discrete & Computational Geometry

The One-Round Voronoi Game*

Otfried Cheong,[1] Sariel Har-Peled,[2] Nathan Linial,[3] and Jiří Matoušek[4]

[1]Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
o.cheong@tue.nl

[2]Department of Computer Science, University of Illinois, 1304 W. Springfield Ave., Urbana, IL 61801, USA
sariel@cs.uiuc.edu

[3]Institute of Computer Science, Hebrew University, Jerusalem 91904, Israel
nati@cs.huji.ac.il

[4]Department of Applied Mathematics and Institute of Theoretical Computer Science (ITI), Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic
matousek@kam.mff.cuni.cz

Abstract. In the one-round Voronoi game, the first player chooses an n-point set W in a square Q, and then the second player places another n-point set B into Q. The payoff for the second player is the fraction of the area of Q occupied by the regions of the points of B in the Voronoi diagram of $W \cup B$. We give a (randomized) strategy for the second player that always guarantees him a payoff of at least $\frac{1}{2} + \alpha$, for a constant $\alpha > 0$ and every large enough n. This contrasts with the one-dimensional situation, with $Q = [0, 1]$, where the first player can always win more than $\frac{1}{2}$.

1. Introduction

Competitive facility location studies the placement of sites by competing market players. Overviews of different models are the surveys by Tobin et al. [9], Eiselt and Laporte [3], and Eiselt et al. [4].

* Part of this research was done during a workshop supported by ITI (Project LN00A056 of the Ministry of Education of the Czech Republic). N.L. was supported in part by a grant from the Israel Science Foundation.

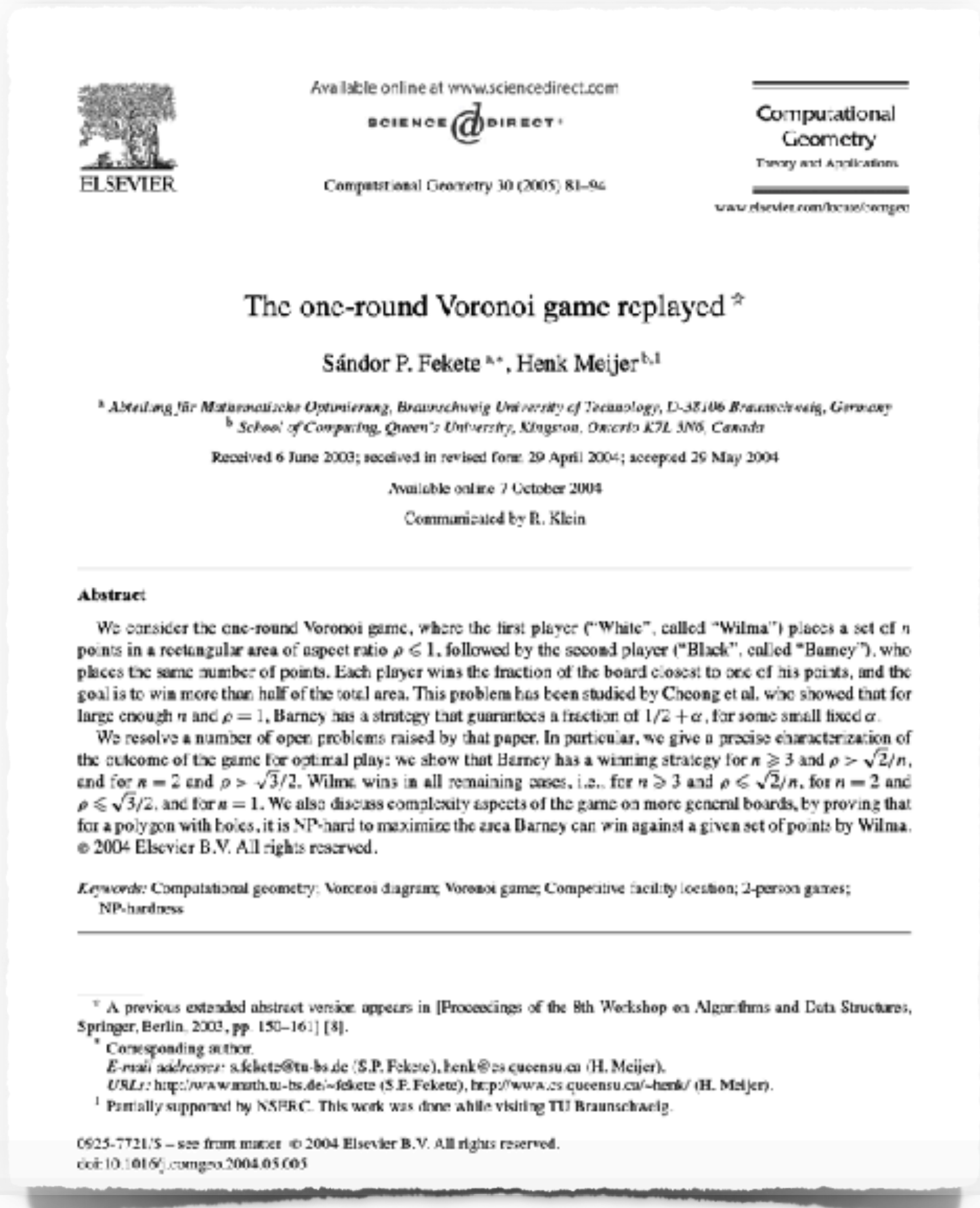
Open:

- **White wins for $n=1$, Black for large n**
- **White wins for 1D, Black for 2D**
- **Strategy uses randomization**
- **Explain and find simpler strategy!**

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Technische Universität Braunschweig

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]



ELSEVIER

Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

Computational Geometry 30 (2005) 81–96

Computational Geometry Theory and Applications

www.elsevier.com/locate/comgeo

The one-round Voronoi game replayed ☆

Sándor P. Fekete [a,*], Henk Meijer [b,1]

[a] Abteilung für Mathematische Optimierung, Braunschweig University of Technology, D-38106 Braunschweig, Germany
[b] School of Computing, Queen's University, Kingston, Ontario K7L 3N6, Canada

Received 6 June 2003; received in revised form 29 April 2004; accepted 29 May 2004

Available online 7 October 2004

Communicated by R. Klein

Abstract

We consider the one-round Voronoi game, where the first player ("White", called "Wilma") places a set of n points in a rectangular area of aspect ratio $\rho \leqslant 1$, followed by the second player ("Black", called "Barney"), who places the same number of points. Each player wins the fraction of the board closest to one of his points, and the goal is to win more than half of the total area. This problem has been studied by Cheong et al. who showed that for large enough n and $\rho = 1$, Barney has a strategy that guarantees a fraction of $1/2 + \alpha$, for some small fixed α.

We resolve a number of open problems raised by that paper. In particular, we give a precise characterization of the outcome of the game for optimal play: we show that Barney has a winning strategy for $n \geqslant 3$ and $\rho > \sqrt{2}/n$, and for $n = 2$ and $\rho > \sqrt{3}/2$. Wilma wins in all remaining cases, i.e., for $n \geqslant 3$ and $\rho \leqslant \sqrt{2}/n$, for $n = 2$ and $\rho \leqslant \sqrt{3}/2$, and for $n = 1$. We also discuss complexity aspects of the game on more general boards, by proving that for a polygon with holes, it is NP-hard to maximize the area Barney can win against a given set of points by Wilma.

Keywords: Computational geometry; Voronoi diagram; Voronoi game; Competitive facility location; 2-person games; NP-hardness

☆ A previous extended abstract version appears in [Proceedings of the 8th Workshop on Algorithms and Data Structures, Springer, Berlin, 2003, pp. 150–161] [8].

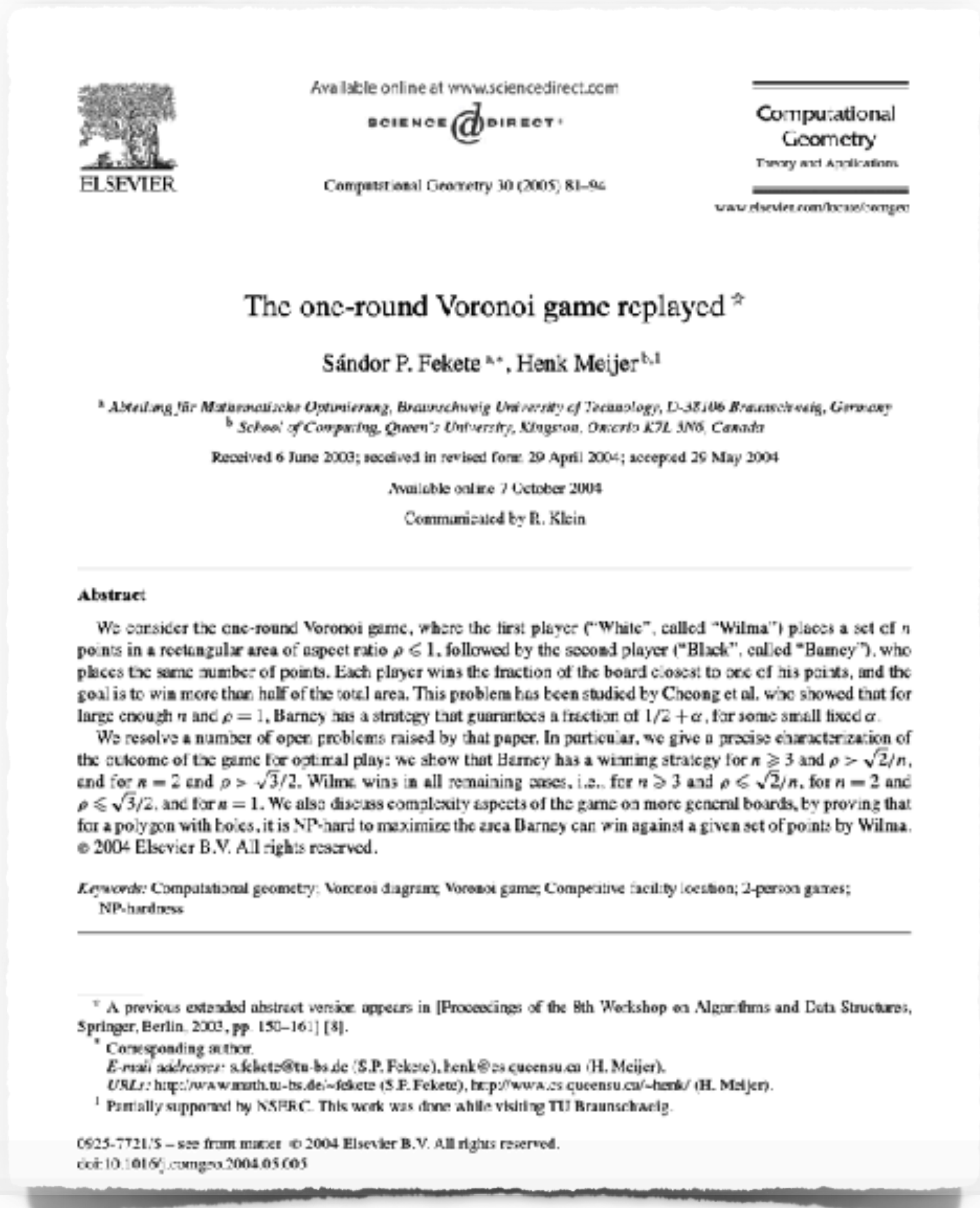
* Corresponding author.
E-mail addresses: s.fekete@tu-bs.de (S.P. Fekete), henk@cs.queensu.ca (H. Meijer).
URLs: http://www.math.tu-bs.de/~fekete (S.P. Fekete), http://www.cs.queensu.ca/~henk/ (H. Meijer).

1 Partially supported by NSERC. This work was done while visiting TU Braunschweig.

0925-7721/$ – see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.comgeo.2004.05.005

Technische Universität Braunschweig

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]



Available online at www.sciencedirect.com

SCIENCE @ DIRECT

ELSEVIER

Computational Geometry 30 (2005) 81–94

Computational Geometry
Theory and Applications

www.elsevier.com/locate/comgeo

The one-round Voronoi game replayed ☆

Sándor P. Fekete [a,*], Henk Meijer [b,1]

[a] Abteilung für Mathematische Optimierung, Braunschweig University of Technology, D-38106 Braunschweig, Germany
[b] School of Computing, Queen's University, Kingston, Ontario K7L 3N6, Canada

Received 6 June 2003; received in revised form 29 April 2004; accepted 29 May 2004

Available online 7 October 2004

Communicated by R. Klein

Abstract

We consider the one-round Voronoi game, where the first player ("White", called "Wilma") places a set of n points in a rectangular area of aspect ratio $\rho \leqslant 1$, followed by the second player ("Black", called "Barney"), who places the same number of points. Each player wins the fraction of the board closest to one of his points, and the goal is to win more than half of the total area. This problem has been studied by Cheong et al. who showed that for large enough n and $\rho = 1$, Barney has a strategy that guarantees a fraction of $1/2 + \alpha$, for some small fixed α.

We resolve a number of open problems raised by that paper. In particular, we give a precise characterization of the outcome of the game for optimal play: we show that Barney has a winning strategy for $n \geqslant 3$ and $\rho > \sqrt{2}/n$, and for $n = 2$ and $\rho > \sqrt{3}/2$. Wilma wins in all remaining cases, i.e., for $n \geqslant 3$ and $\rho \leqslant \sqrt{2}/n$, for $n = 2$ and $\rho \leqslant \sqrt{3}/2$, and for $n = 1$. We also discuss complexity aspects of the game on more general boards, by proving that for a polygon with holes, it is NP-hard to maximize the area Barney can win against a given set of points by Wilma.

Keywords: Computational geometry; Voronoi diagram; Voronoi game; Competitive facility location; 2-person games; NP-hardness

☆ A previous extended abstract version appears in [Proceedings of the 8th Workshop on Algorithms and Data Structures, Springer, Berlin, 2003, pp. 150–161] [8].

* Corresponding author.
E-mail addresses: s.fekete@tu-bs.de (S.P. Fekete), henk@cs.queensu.ca (H. Meijer).
URLs: http://www.math.tu-bs.de/~fekete (S.P. Fekete), http://www.cs.queensu.ca/~henk/ (H. Meijer).

[1] Partially supported by NSERC. This work was done while visiting TU Braunschweig.

0925-7721/$ – see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.comgeo.2004.05.005

Insights:

Technische Universität Braunschweig

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT
Computational Geometry 30 (2005) 81–94
Computational Geometry
Theory and Applications
www.elsevier.com/locate/comgeo

The one-round Voronoi game replayed ☆

Sándor P. Fekete [a,*], Henk Meijer [b,1]

[a] Abteilung für Mathematische Optimierung, Braunschweig University of Technology, D-38106 Braunschweig, Germany
[b] School of Computing, Queen's University, Kingston, Ontario K7L 3N6, Canada

Received 6 June 2003; received in revised form 29 April 2004; accepted 29 May 2004

Available online 7 October 2004

Communicated by R. Klein

Abstract

We consider the one-round Voronoi game, where the first player ("White", called "Wilma") places a set of n points in a rectangular area of aspect ratio $\rho \leqslant 1$, followed by the second player ("Black", called "Barney"), who places the same number of points. Each player wins the fraction of the board closest to one of his points, and the goal is to win more than half of the total area. This problem has been studied by Cheong et al. who showed that for large enough n and $\rho = 1$, Barney has a strategy that guarantees a fraction of $1/2 + \alpha$, for some small fixed α.

We resolve a number of open problems raised by that paper. In particular, we give a precise characterization of the outcome of the game for optimal play: we show that Barney has a winning strategy for $n \geqslant 3$ and $\rho > \sqrt{2}/n$, and for $n = 2$ and $\rho > \sqrt{3}/2$. Wilma wins in all remaining cases, i.e., for $n \geqslant 3$ and $\rho \leqslant \sqrt{2}/n$, for $n = 2$ and $\rho \leqslant \sqrt{3}/2$, and for $n = 1$. We also discuss complexity aspects of the game on more general boards, by proving that for a polygon with holes, it is NP-hard to maximize the area Barney can win against a given set of points by Wilma.

Keywords: Computational geometry; Voronoi diagram; Voronoi game; Competitive facility location; 2-person games; NP-hardness

☆ A previous extended abstract version appears in [Proceedings of the 8th Workshop on Algorithms and Data Structures, Springer, Berlin, 2003, pp. 150–161] [8].

* Corresponding author.
E-mail addresses: s.fekete@tu-bs.de (S.P. Fekete), henk@cs.queensu.ca (H. Meijer).
URLs: http://www.math.tu-bs.de/~fekete (S.P. Fekete), http://www.cs.queensu.ca/~henk/ (H. Meijer).
[1] Partially supported by NSERC. This work was done while visiting TU Braunschweig.

0925-7721/$ – see front matter
doi:10.1016/j.comgeo.2004.05.005

Insights:

- Consider rectangle

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Available online at www.sciencedirect.com

SCIENCE @ DIRECT

ELSEVIER

Computational Geometry 30 (2005) 81–96

Computational Geometry
Theory and Applications

www.elsevier.com/locate/comgeo

The one-round Voronoi game replayed [☆]

Sándor P. Fekete [a,*], Henk Meijer [b,1]

[a] Abteilung für Mathematische Optimierung, Braunschweig University of Technology, D-38106 Braunschweig, Germany
[b] School of Computing, Queen's University, Kingston, Ontario K7L 3N6, Canada

Received 6 June 2003; received in revised form 29 April 2004; accepted 29 May 2004

Available online 7 October 2004

Communicated by R. Klein

Abstract

We consider the one-round Voronoi game, where the first player ("White", called "Wilma") places a set of n points in a rectangular area of aspect ratio $\rho \leqslant 1$, followed by the second player ("Black", called "Barney"), who places the same number of points. Each player wins the fraction of the board closest to one of his points, and the goal is to win more than half of the total area. This problem has been studied by Cheong et al. who showed that for large enough n and $\rho = 1$, Barney has a strategy that guarantees a fraction of $1/2 + \alpha$, for some small fixed α.

We resolve a number of open problems raised by that paper. In particular, we give a precise characterization of the outcome of the game for optimal play: we show that Barney has a winning strategy for $n \geqslant 3$ and $\rho > \sqrt{2}/n$, and for $n = 2$ and $\rho > \sqrt{3}/2$. Wilma wins in all remaining cases, i.e., for $n \geqslant 3$ and $\rho \leqslant \sqrt{2}/n$, for $n = 2$ and $\rho \leqslant \sqrt{3}/2$, and for $n = 1$. We also discuss complexity aspects of the game on more general boards, by proving that for a polygon with holes, it is NP-hard to maximize the area Barney can win against a given set of points by Wilma.
© 2004 Elsevier B.V. All rights reserved.

Keywords: Computational geometry; Voronoi diagram; Voronoi game; Competitive facility location; 2-person games; NP-hardness

[☆] A previous extended abstract version appears in [Proceedings of the 8th Workshop on Algorithms and Data Structures, Springer, Berlin, 2003, pp. 150–161] [8].

[*] Corresponding author.
E-mail addresses: s.fekete@tu-bs.de (S.P. Fekete), henk@cs.queensu.ca (H. Meijer).
URLs: http://www.math.tu-bs.de/~fekete (S.P. Fekete), http://www.cs.queensu.ca/~henk/ (H. Meijer).

[1] Partially supported by NSERC. This work was done while visiting TU Braunschweig.

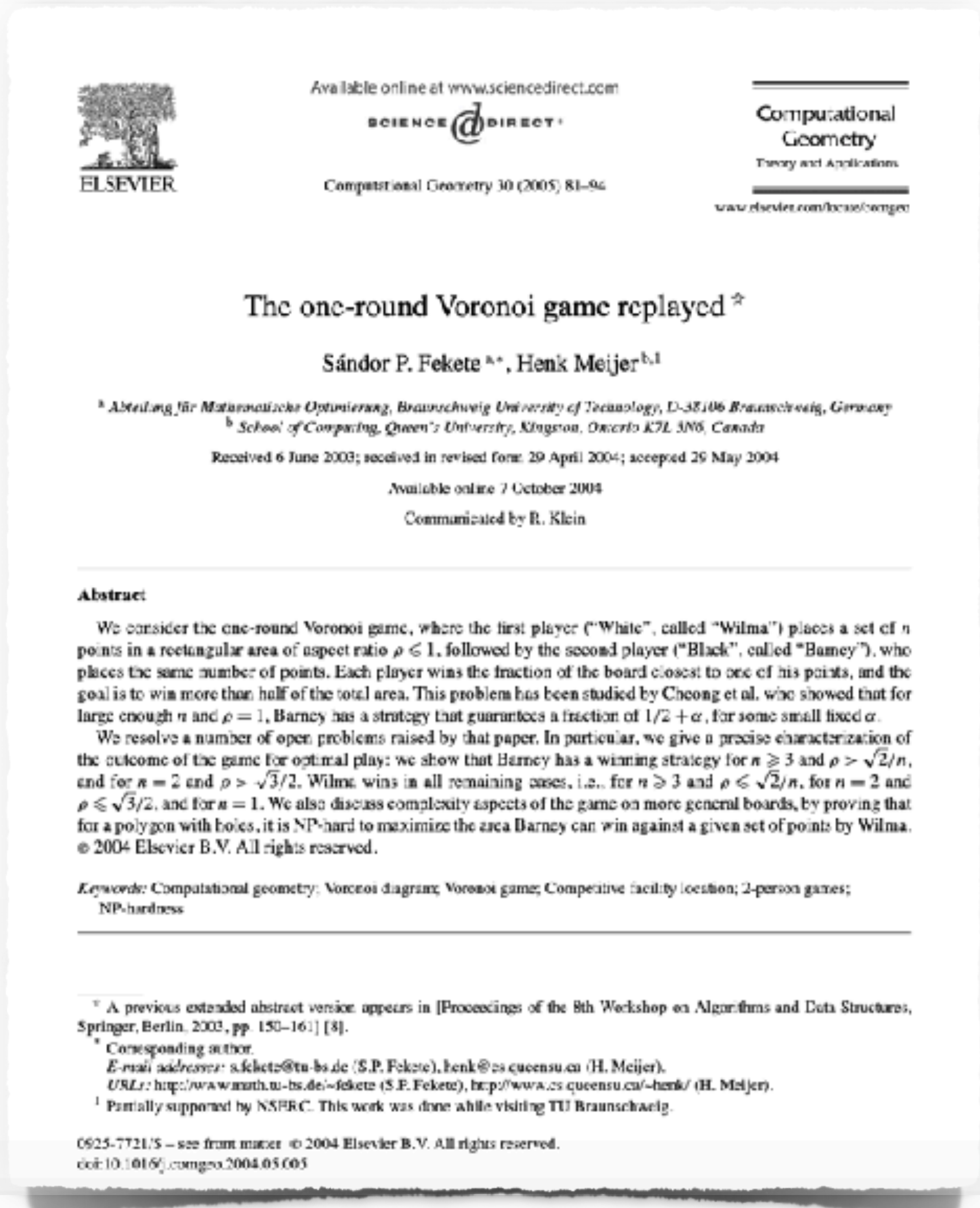
0925-7721/$ – see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.comgeo.2004.05.005

Insights:

- Consider rectangle
- Outcome depends on aspect ratio

Technische Universität Braunschweig

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]


Available online at www.sciencedirect.com
SCIENCE @ DIRECT
ELSEVIER
Computational Geometry 30 (2005) 81–94
Computational Geometry
Theory and Applications
www.elsevier.com/locate/comgeo

The one-round Voronoi game replayed ☆

Sándor P. Fekete [a,*], Henk Meijer [b,1]

[a] Abteilung für Mathematische Optimierung, Braunschweig University of Technology, D-38106 Braunschweig, Germany
[b] School of Computing, Queen's University, Kingston, Ontario K7L 3N6, Canada

Received 6 June 2003; received in revised form 29 April 2004; accepted 29 May 2004

Available online 7 October 2004

Communicated by R. Klein

Abstract

We consider the one-round Voronoi game, where the first player ("White", called "Wilma") places a set of n points in a rectangular area of aspect ratio $\rho \leqslant 1$, followed by the second player ("Black", called "Barney"), who places the same number of points. Each player wins the fraction of the board closest to one of his points, and the goal is to win more than half of the total area. This problem has been studied by Cheong et al. who showed that for large enough n and $\rho = 1$, Barney has a strategy that guarantees a fraction of $1/2 + \alpha$, for some small fixed α.

We resolve a number of open problems raised by that paper. In particular, we give a precise characterization of the outcome of the game for optimal play: we show that Barney has a winning strategy for $n \geqslant 3$ and $\rho > \sqrt{2}/n$, and for $n = 2$ and $\rho > \sqrt{3}/2$. Wilma wins in all remaining cases, i.e., for $n \geqslant 3$ and $\rho \leqslant \sqrt{2}/n$, for $n = 2$ and $\rho \leqslant \sqrt{3}/2$, and for $n = 1$. We also discuss complexity aspects of the game on more general boards, by proving that for a polygon with holes, it is NP-hard to maximize the area Barney can win against a given set of points by Wilma.
© 2004 Elsevier B.V. All rights reserved.

Keywords: Computational geometry; Voronoi diagram; Voronoi game; Competitive facility location; 2-person games; NP-hardness

☆ A previous extended abstract version appears in [Proceedings of the 8th Workshop on Algorithms and Data Structures, Springer, Berlin, 2003, pp. 150–161] [8].

* Corresponding author.
E-mail addresses: s.fekete@tu-bs.de (S.P. Fekete), henk@cs.queensu.ca (H. Meijer).
URLs: http://www.math.tu-bs.de/~fekete (S.P. Fekete), http://www.cs.queensu.ca/~henk/ (H. Meijer).

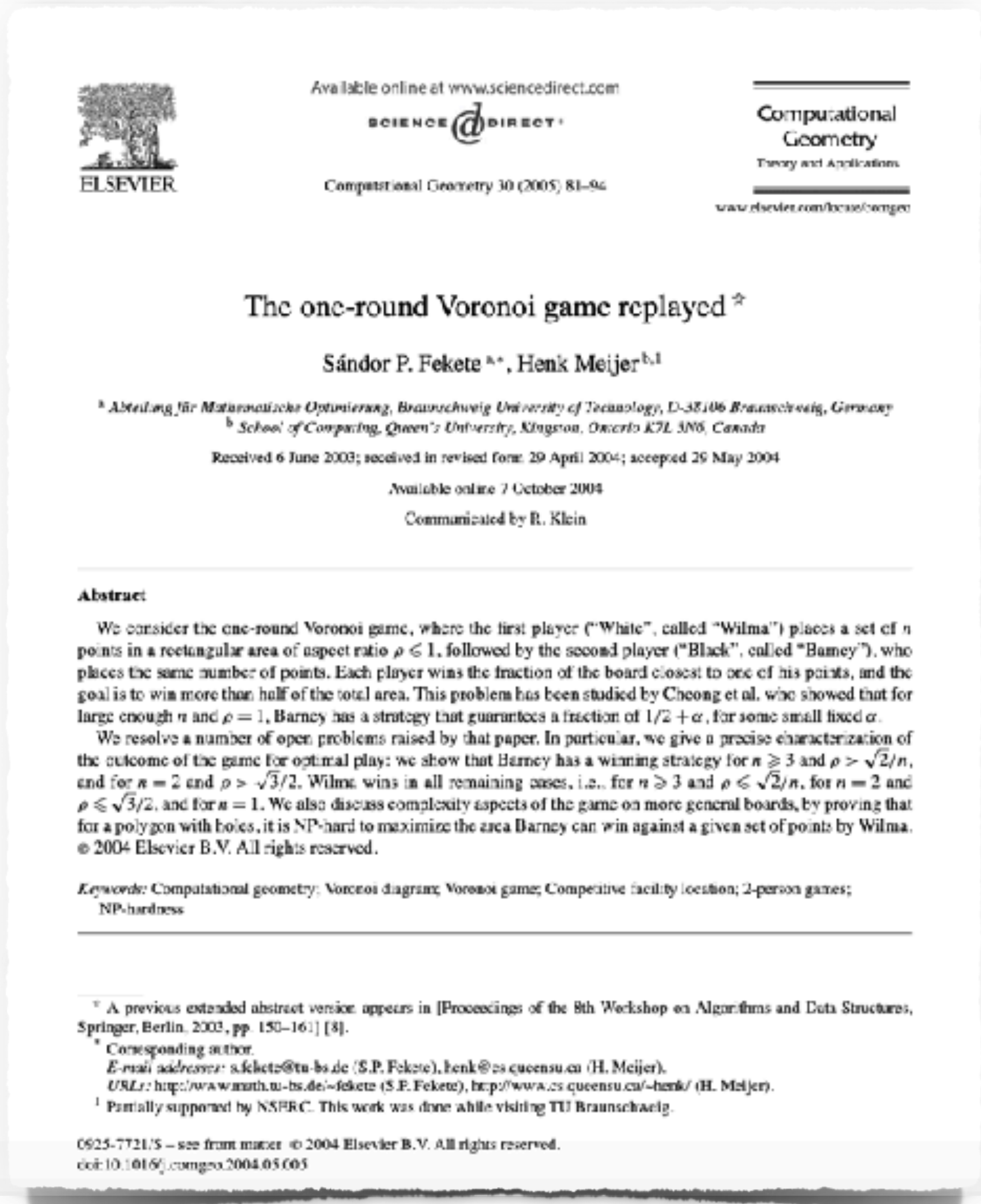
[1] Partially supported by NSERC. This work was done while visiting TU Braunschweig.

0925-7721/$ – see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.comgeo.2004.05.005

Insights:

- **Consider rectangle**
- **Outcome depends on aspect ratio**
- **Game flips at small *n***

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]



Available online at www.sciencedirect.com
SCIENCE DIRECT
Computational Geometry 30 (2005) 81–96
Computational Geometry Theory and Applications
www.elsevier.com/locate/comgeo

The one-round Voronoi game replayed ☆

Sándor P. Fekete [a,*], Henk Meijer [b,1]

[a] *Abteilung für Mathematische Optimierung, Braunschweig University of Technology, D-38106 Braunschweig, Germany*
[b] *School of Computing, Queen's University, Kingston, Ontario K7L 3N6, Canada*

Received 6 June 2003; received in revised form 29 April 2004; accepted 29 May 2004

Available online 7 October 2004

Communicated by R. Klein

Abstract

We consider the one-round Voronoi game, where the first player ("White", called "Wilma") places a set of n points in a rectangular area of aspect ratio $\rho \leqslant 1$, followed by the second player ("Black", called "Barney"), who places the same number of points. Each player wins the fraction of the board closest to one of his points, and the goal is to win more than half of the total area. This problem has been studied by Cheong et al. who showed that for large enough n and $\rho = 1$, Barney has a strategy that guarantees a fraction of $1/2 + \alpha$, for some small fixed α.

We resolve a number of open problems raised by that paper. In particular, we give a precise characterization of the outcome of the game for optimal play: we show that Barney has a winning strategy for $n \geqslant 3$ and $\rho > \sqrt{2}/n$, and for $n = 2$ and $\rho > \sqrt{3}/2$. Wilma wins in all remaining cases, i.e., for $n \geqslant 3$ and $\rho \leqslant \sqrt{2}/n$, for $n = 2$ and $\rho \leqslant \sqrt{3}/2$, and for $n = 1$. We also discuss complexity aspects of the game on more general boards, by proving that for a polygon with holes, it is NP-hard to maximize the area Barney can win against a given set of points by Wilma.

Keywords: Computational geometry; Voronoi diagram; Voronoi game; Competitive facility location; 2-person games; NP-hardness

☆ A previous extended abstract version appears in [Proceedings of the 8th Workshop on Algorithms and Data Structures, Springer, Berlin, 2003, pp. 150–161] [8].

* Corresponding author.
E-mail addresses: s.fekete@tu-bs.de (S.P. Fekete), henk@cs.queensu.ca (H. Meijer).
URLs: http://www.math.tu-bs.de/~fekete (S.P. Fekete), http://www.cs.queensu.ca/~henk/ (H. Meijer).

1 Partially supported by NSERC. This work was done while visiting TU Braunschweig.

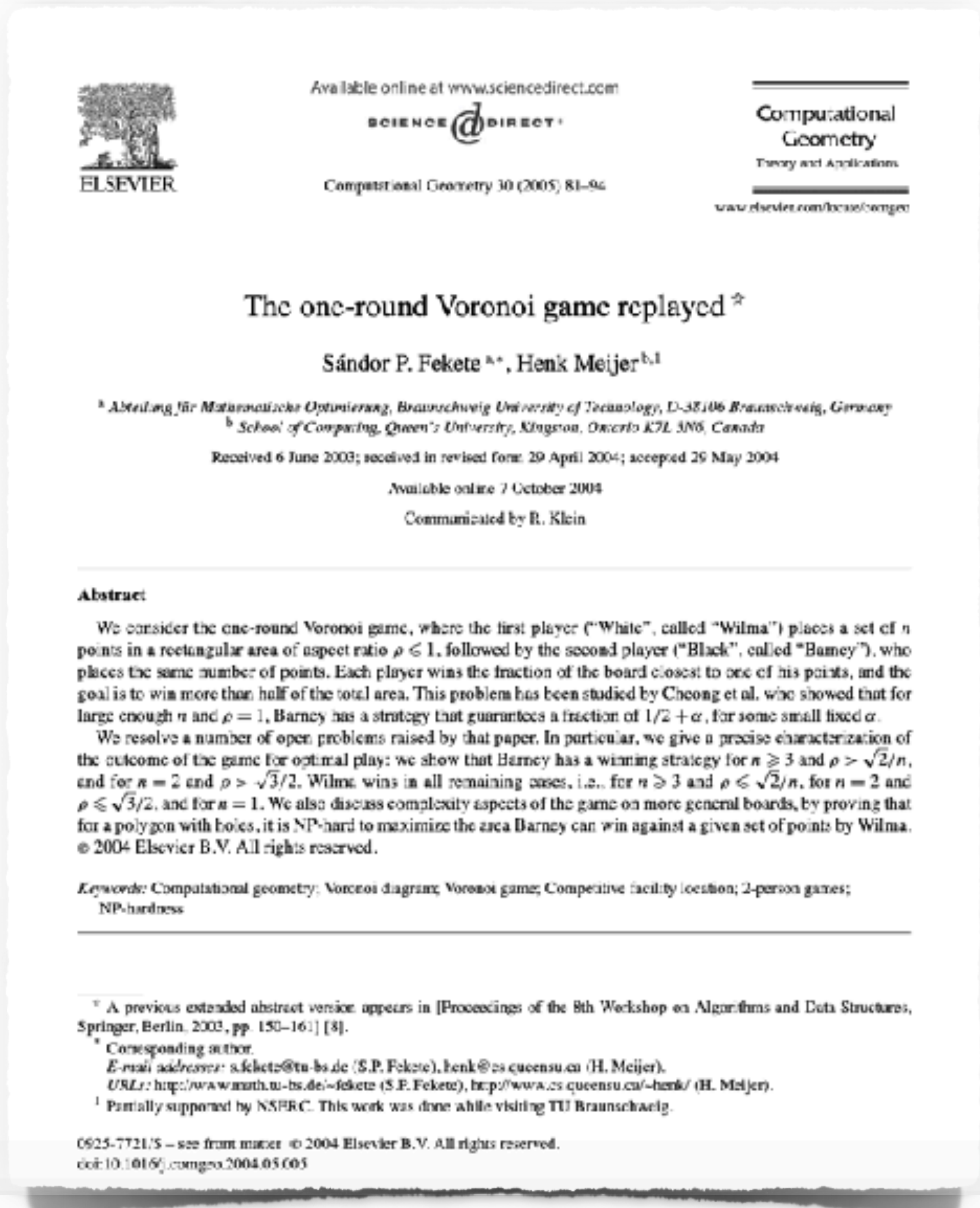
0925-7721/$ – see front matter
doi:10.1016/j.comgeo.2004.05.005

Insights:

- **Consider rectangle**
- **Outcome depends on aspect ratio**
- **Game flips at small n**
- **Simple deterministic strategy**

Technische Universität Braunschweig

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]



Available online at www.sciencedirect.com

SCIENCE DIRECT

Computational Geometry 30 (2005) 81–96

Computational Geometry
Theory and Applications

www.elsevier.com/locate/comgeo

The one-round Voronoi game replayed ☆

Sándor P. Fekete [a,*], Henk Meijer [b,1]

[a] Abteilung für Mathematische Optimierung, Braunschweig University of Technology, D-38106 Braunschweig, Germany
[b] School of Computing, Queen's University, Kingston, Ontario K7L 3N6, Canada

Received 6 June 2003; received in revised form 29 April 2004; accepted 29 May 2004

Available online 7 October 2004

Communicated by R. Klein

Abstract

We consider the one-round Voronoi game, where the first player ("White", called "Wilma") places a set of n points in a rectangular area of aspect ratio $\rho \leqslant 1$, followed by the second player ("Black", called "Barney"), who places the same number of points. Each player wins the fraction of the board closest to one of his points, and the goal is to win more than half of the total area. This problem has been studied by Cheong et al. who showed that for large enough n and $\rho = 1$, Barney has a strategy that guarantees a fraction of $1/2 + \alpha$, for some small fixed α.

We resolve a number of open problems raised by that paper. In particular, we give a precise characterization of the outcome of the game for optimal play: we show that Barney has a winning strategy for $n \geqslant 3$ and $\rho > \sqrt{2}/n$, and for $n = 2$ and $\rho > \sqrt{3}/2$. Wilma wins in all remaining cases, i.e., for $n \geqslant 3$ and $\rho \leqslant \sqrt{2}/n$, for $n = 2$ and $\rho \leqslant \sqrt{3}/2$, and for $n = 1$. We also discuss complexity aspects of the game on more general boards, by proving that for a polygon with holes, it is NP-hard to maximize the area Barney can win against a given set of points by Wilma.

Keywords: Computational geometry; Voronoi diagram; Voronoi game; Competitive facility location; 2-person games; NP-hardness

☆ A previous extended abstract version appears in [Proceedings of the 8th Workshop on Algorithms and Data Structures, Springer, Berlin, 2003, pp. 150–161] [8].

* Corresponding author.
E-mail addresses: s.fekete@tu-bs.de (S.P. Fekete), henk@cs.queensu.ca (H. Meijer).
URLs: http://www.math.tu-bs.de/~fekete (S.P. Fekete), http://www.cs.queensu.ca/~henk/ (H. Meijer).

[1] Partially supported by NSERC. This work was done while visiting TU Braunschweig.

0925-7721/$ – see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.comgeo.2004.05.005

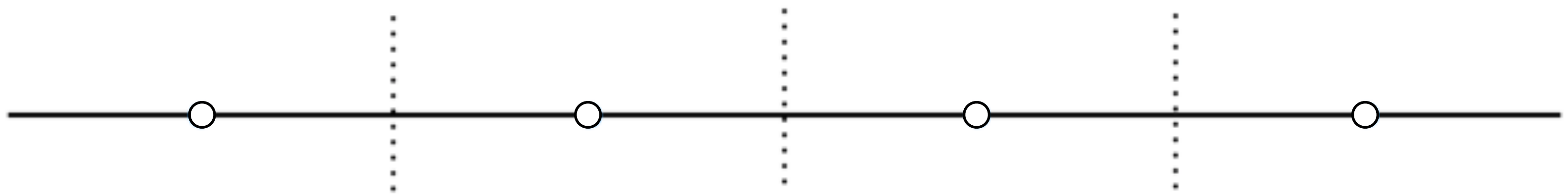
Insights:

- Consider rectangle
- Outcome depends on aspect ratio
- Game flips at small n
- Simple deterministic strategy
- Players „Wilma“ and „Barney“

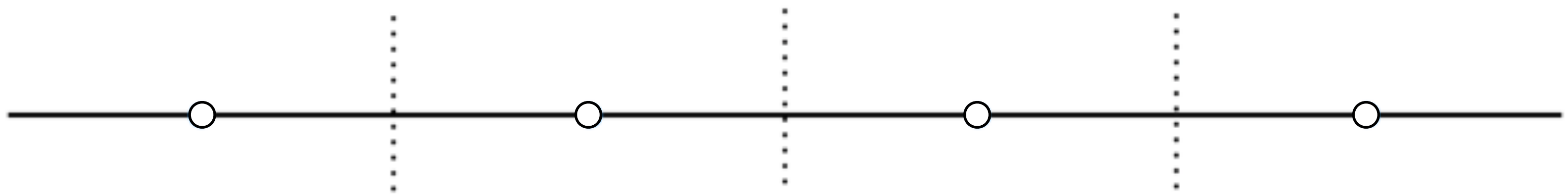
Technische Universität Braunschweig

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

- **1D: Wilma wins by creating uniform cells**

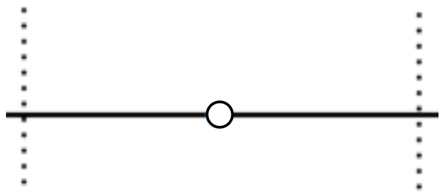


- **1D: Wilma wins by creating uniform cells**

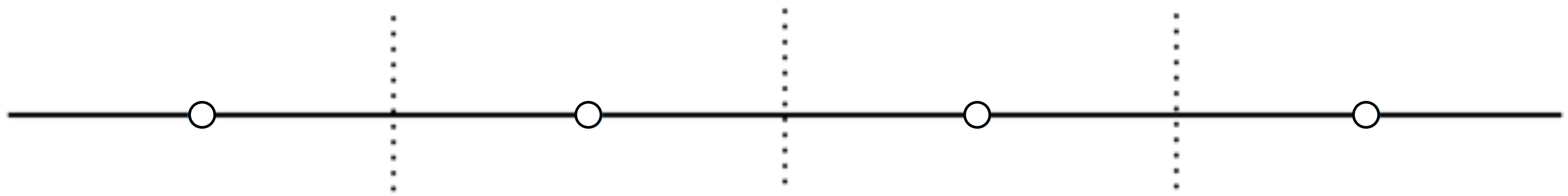


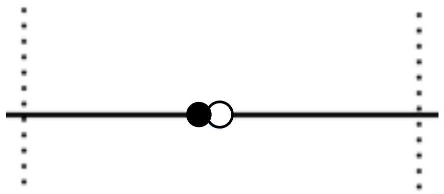
- 1D: Wilma wins by creating uniform cells
- Barney can always claim arbitrarily close to half a cell

- 1D: Wilma wins by creating uniform cells
- Barney can always claim arbitrarily close to half a cell



The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

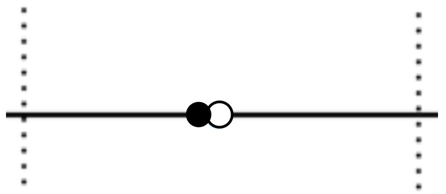


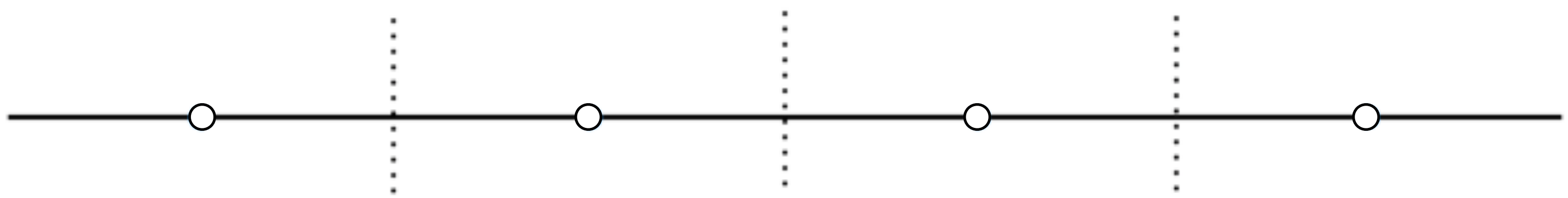


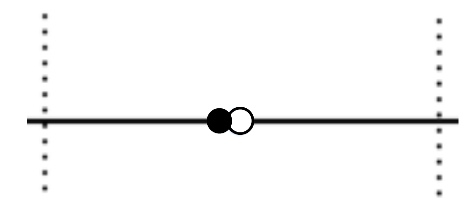
- 1D: Wilma wins by creating uniform cells
- Barney can always claim arbitrarily close to half a cell

Technische Universität Braunschweig

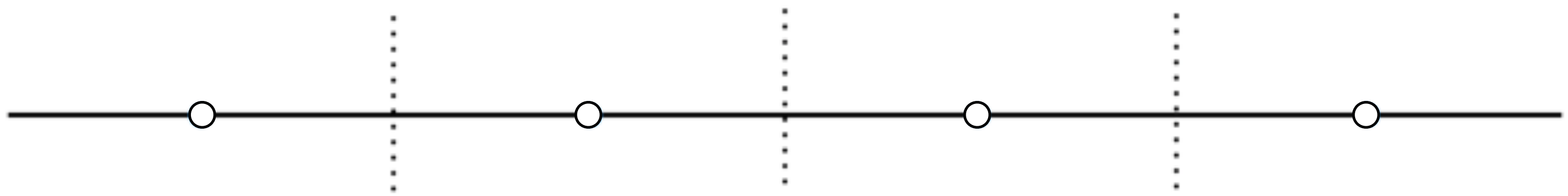
- 1D: Wilma wins by creating uniform cells
- Barney can always claim arbitrarily close to half a cell
- So Barney can win as soon as he gets one cell above average

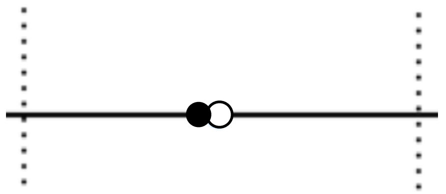




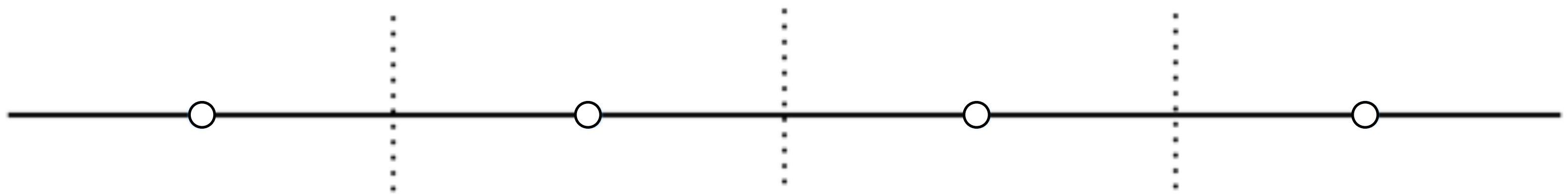


- 1D: Wilma wins by creating uniform cells
- Barney can always claim arbitrarily close to half a cell
- So Barney can win as soon as he gets one cell above average
- Barney wins if Wilma creates uneven cells:

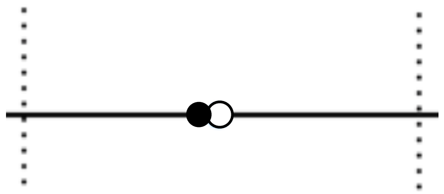


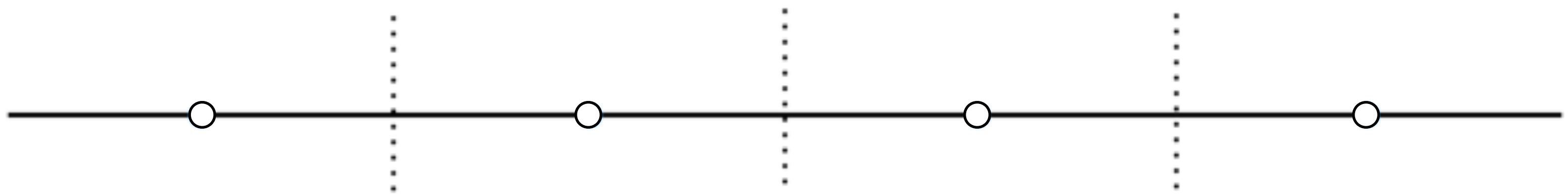


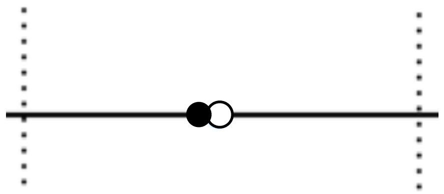
- 1D: Wilma wins by creating uniform cells
- Barney can always claim arbitrarily close to half a cell
- So Barney can win as soon as he gets one cell above average
- Barney wins if Wilma creates uneven cells:



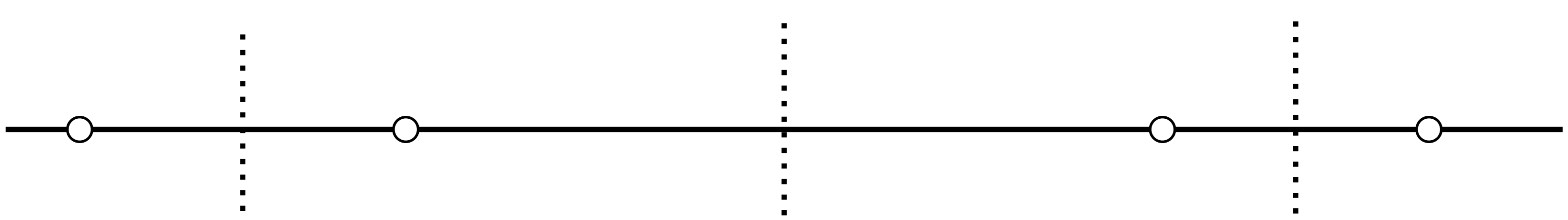
- 1D: Wilma wins by creating uniform cells
- Barney can always claim arbitrarily close to half a cell
- So Barney can win as soon as he gets one cell above average
- Barney wins if Wilma creates uneven cells:

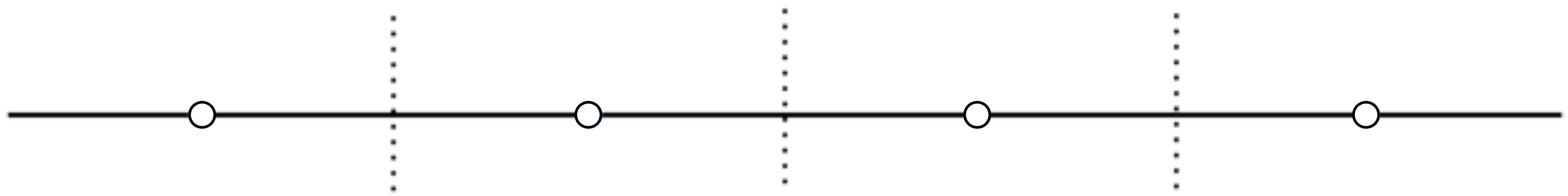


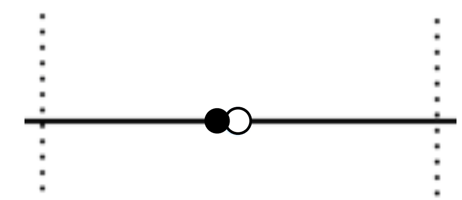




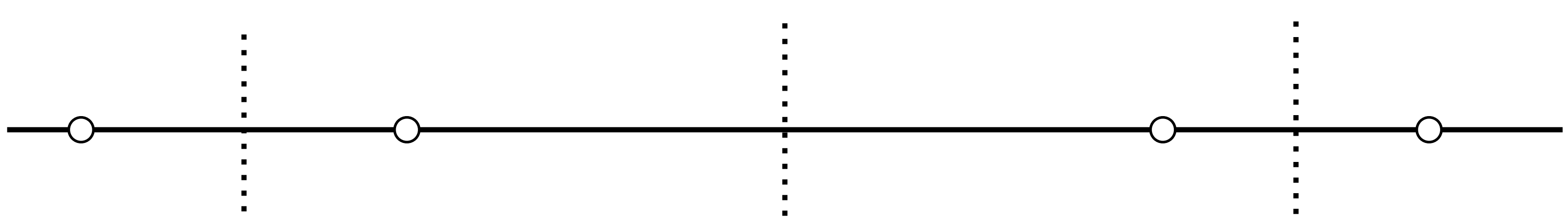
- **1D: Wilma wins by creating uniform cells**
- **Barney can always claim arbitrarily close to half a cell**
- **So Barney can win as soon as he gets one cell above average**
- **Barney wins if Wilma creates uneven cells:**

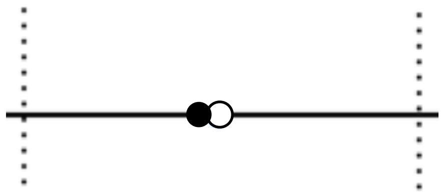




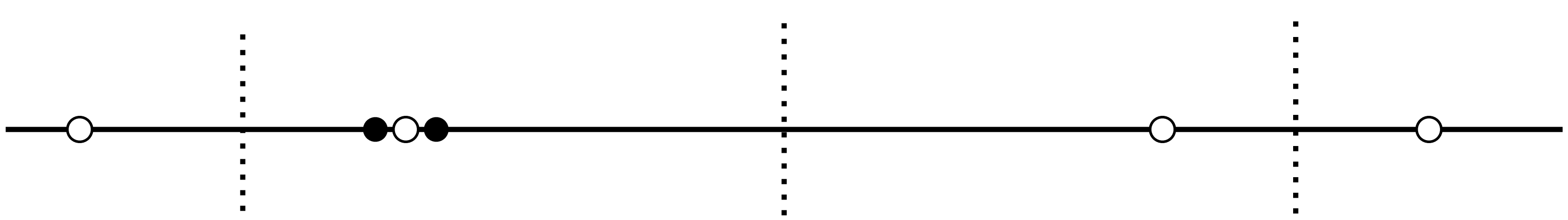


- 1D: Wilma wins by creating uniform cells
- Barney can always claim arbitrarily close to half a cell
- So Barney can win as soon as he gets one cell above average
- Barney wins if Wilma creates uneven cells:
  - Uneven size

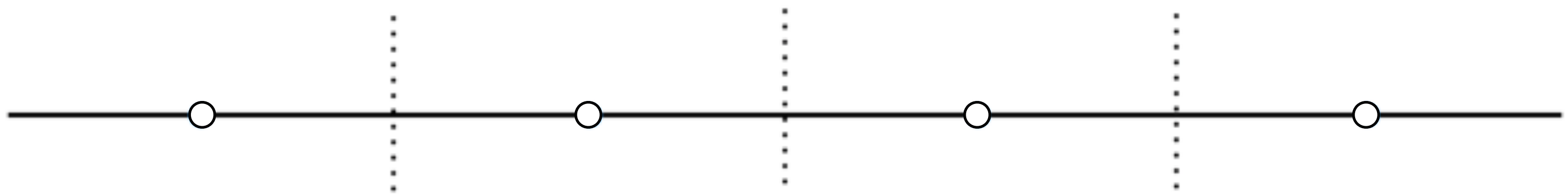


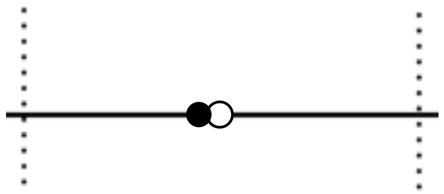


- 1D: Wilma wins by creating uniform cells
- Barney can always claim arbitrarily close to half a cell
- So Barney can win as soon as he gets one cell above average
- Barney wins if Wilma creates uneven cells:
    - Uneven size

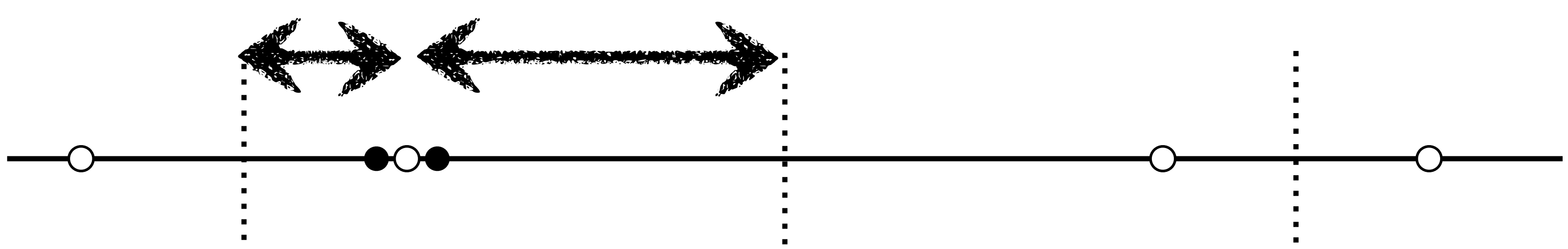


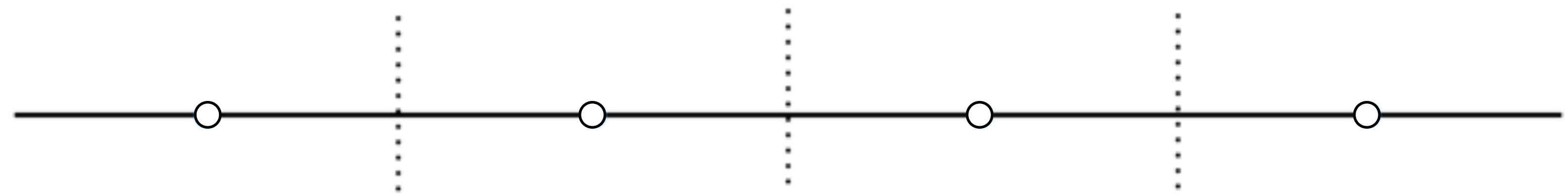
Technische Universität Braunschweig

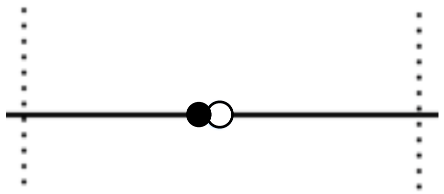




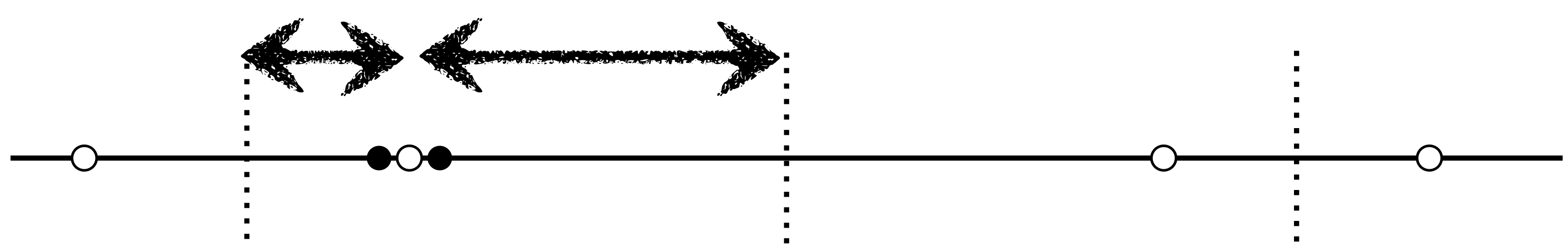
- 1D: Wilma wins by creating uniform cells
- Barney can always claim arbitrarily close to half a cell
- So Barney can win as soon as he gets one cell above average
- Barney wins if Wilma creates uneven cells:
    - Uneven size

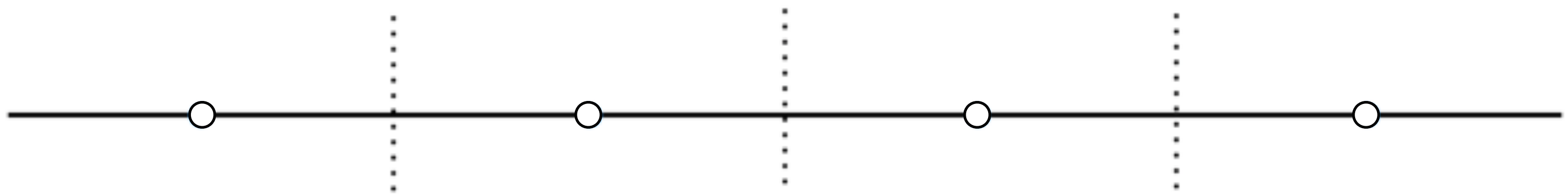




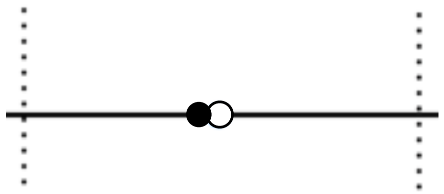


- 1D: Wilma wins by creating uniform cells
- Barney can always claim arbitrarily close to half a cell
- So Barney can win as soon as he gets one cell above average
- Barney wins if Wilma creates uneven cells:
  - Uneven size
  - Uneven position

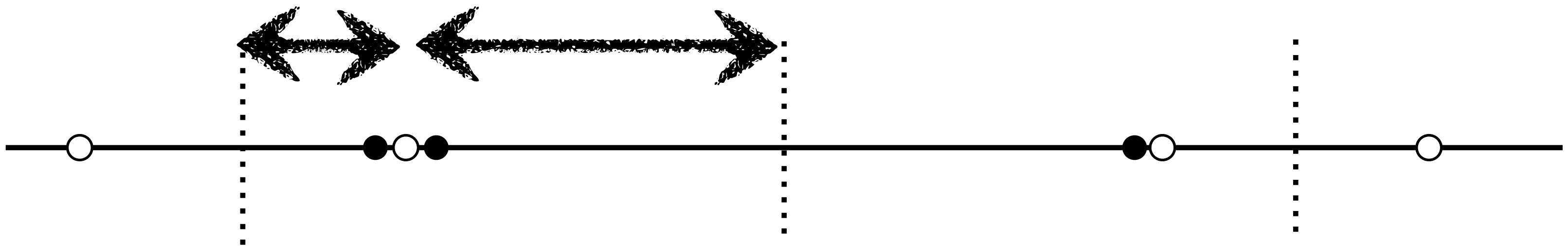




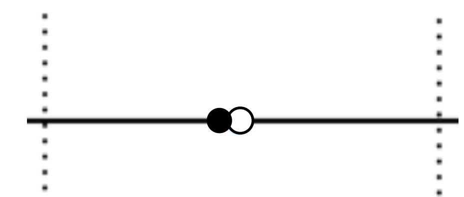
- 1D: Wilma wins by creating uniform cells
- Barney can always claim arbitrarily close to half a cell



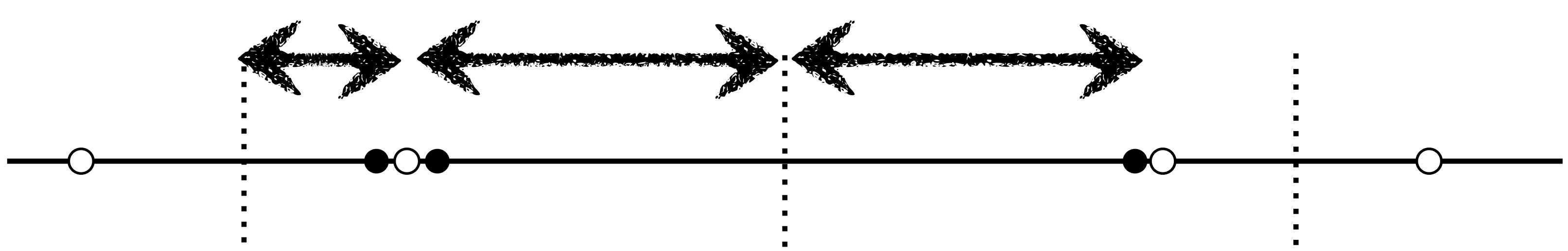
- So Barney can win as soon as he gets one cell above average
- Barney wins if Wilma creates uneven cells:
  - Uneven size
  - Uneven position



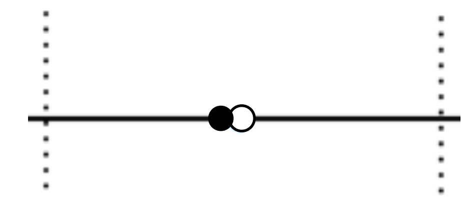
Technische Universität Braunschweig



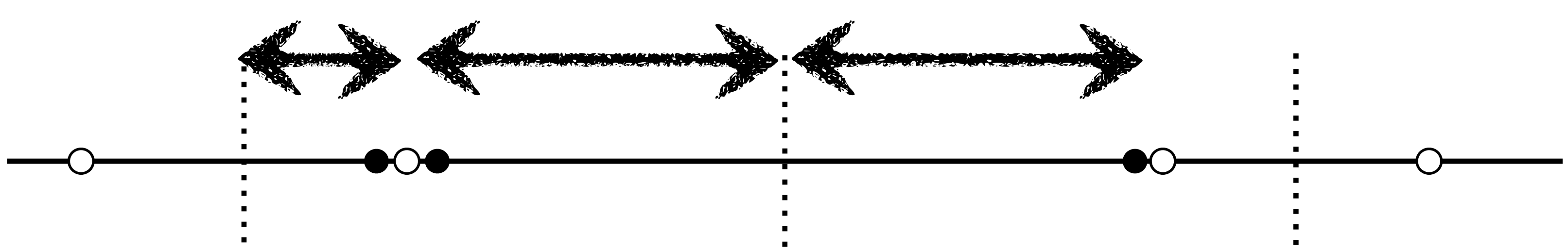
- 1D: Wilma wins by creating uniform cells
- Barney can always claim arbitrarily close to half a cell
- So Barney can win as soon as he gets one cell above average
- Barney wins if Wilma creates uneven cells:
  - Uneven size
  - Uneven position



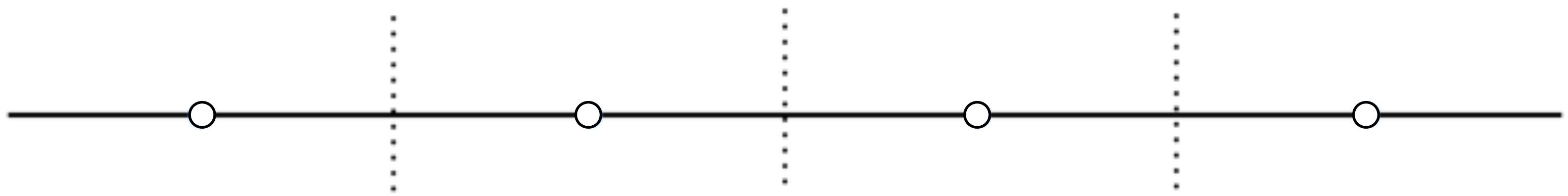
The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

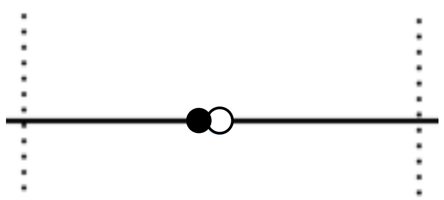


- 1D: Wilma wins by creating uniform cells
- Barney can always claim arbitrarily close to half a cell
- So Barney can win as soon as he gets one cell above average
- Barney wins if Wilma creates uneven cells:
    - Uneven size
    - Uneven position
- So Wilma *must* create uniform cells with centered sites

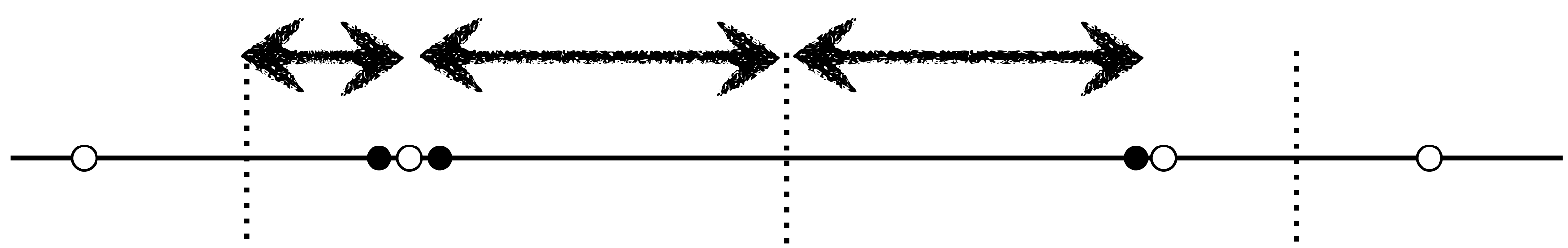


Technische Universität Braunschweig





- 1D: Wilma wins by creating uniform cells
- Barney can always claim arbitrarily close to half a cell
- So Barney can win as soon as he gets one cell above average
- Barney wins if Wilma creates uneven cells:
  - Uneven size
  - Uneven position
- So Wilma *must* create uniform cells with centered sites
- How can we exploit this in 2D?



The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Technische Universität Braunschweig

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

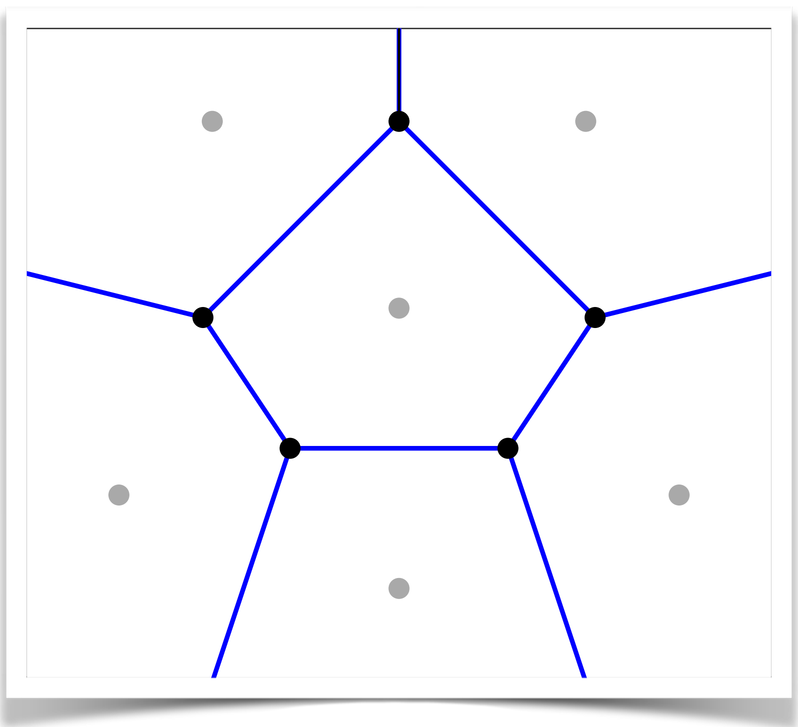
Lemma 3. *Barney wins, if and only if he can place a point* p *that steals an area strictly larger than* $|Q|/(2n)$ *from* W.

Lemma 3. *Barney wins, if and only if he can place a point* p *that steals an area strictly larger than* $|Q|/(2n)$ *from* W.

2D:

Lemma 3. *Barney wins, if and only if he can place a point* p *that steals an area strictly larger than* $|Q|/(2n)$ *from* W.

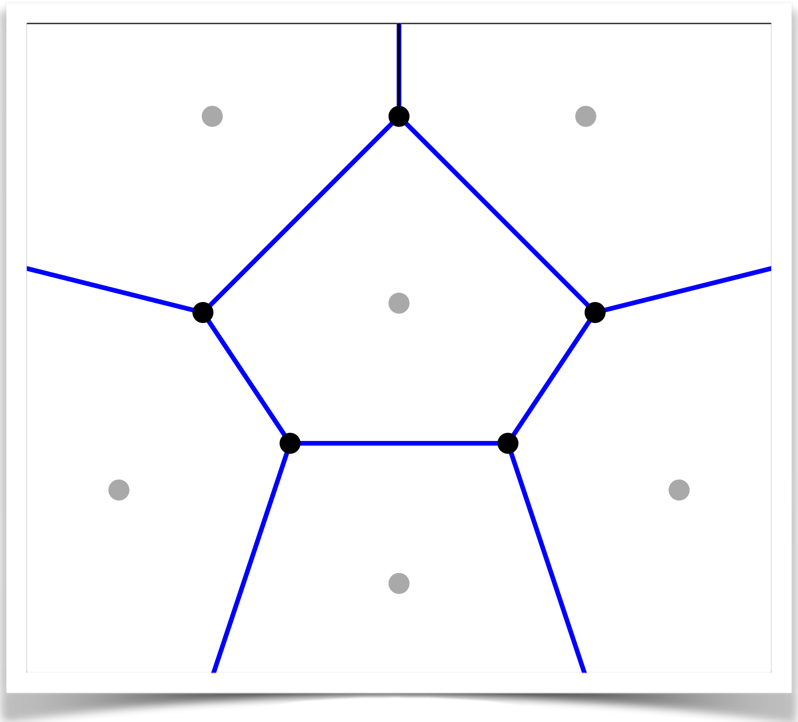
2D:



Lemma 3. *Barney wins, if and only if he can place a point p that steals an area strictly larger than $|Q|/(2n)$ from W.*

2D:

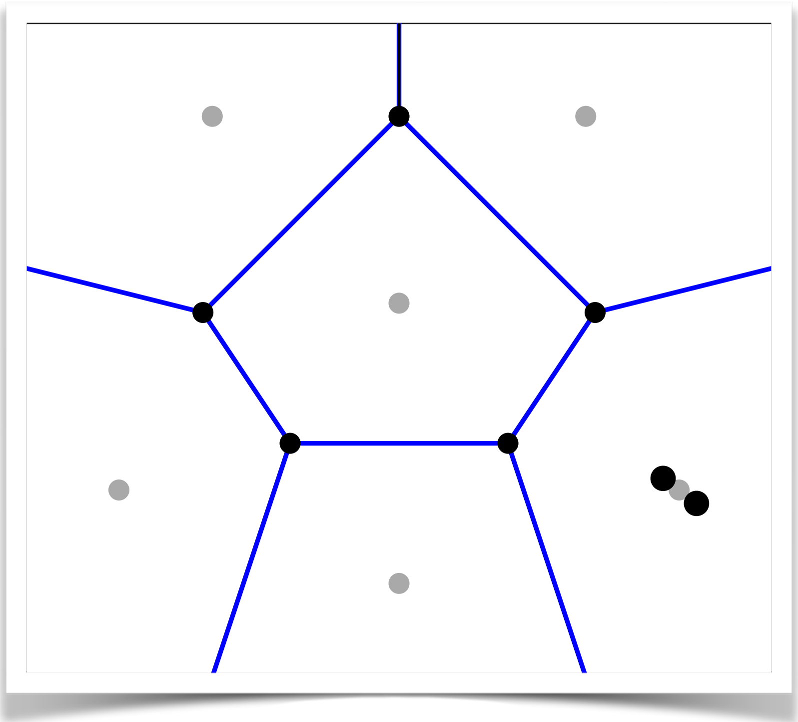
- **All cells must have the same area**



Lemma 3. *Barney wins, if and only if he can place a point p that steals an area strictly larger than* $|Q|/(2n)$ *from* W.

2D:

- **All cells must have the same area**

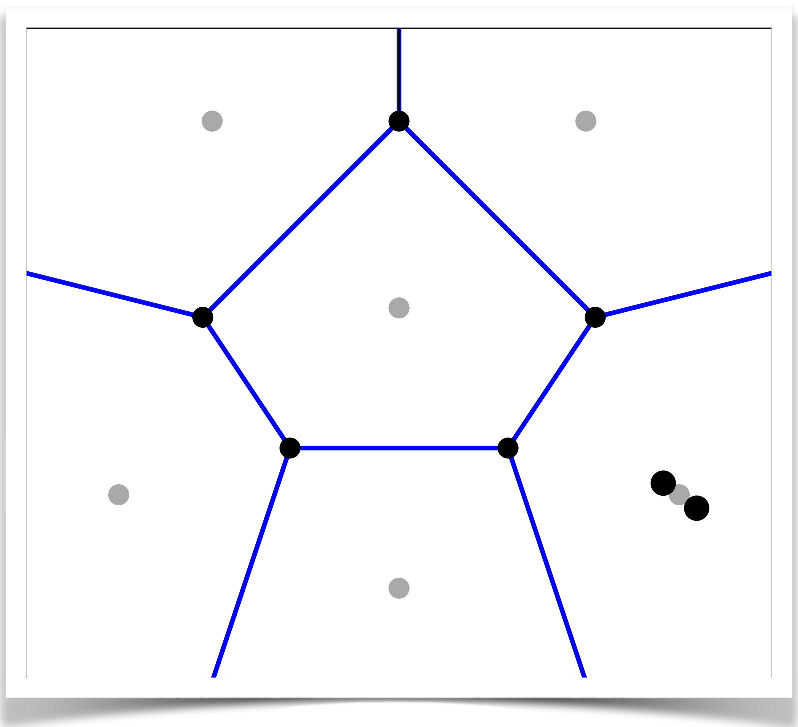


The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Lemma 3. *Barney wins, if and only if he can place a point* p *that steals an area strictly larger than* $|Q|/(2n)$ *from* W.

2D:

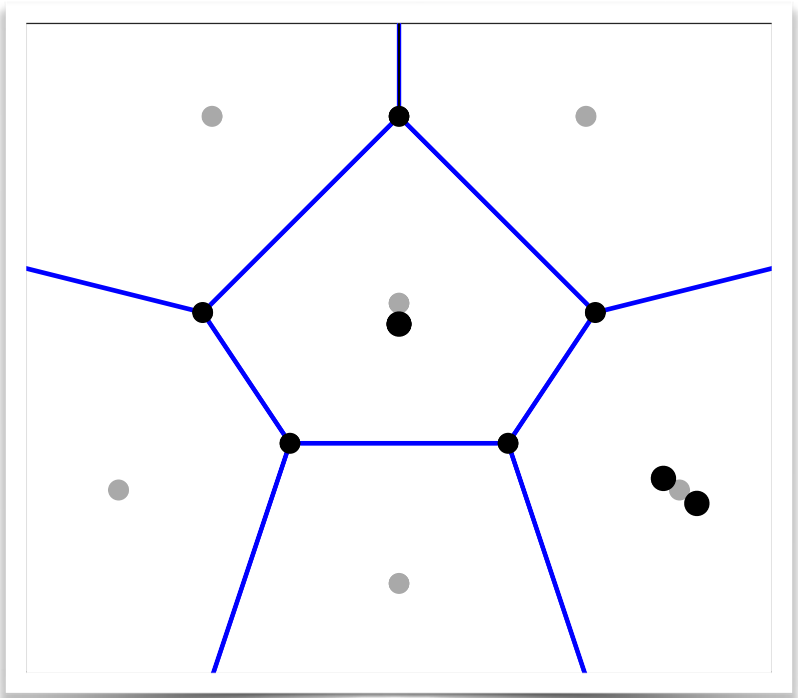
- **All cells must have the same area**
- **The sites must be centrally symmetric in each cell**



Lemma 3. *Barney wins, if and only if he can place a point* p *that steals an area strictly larger than* $|Q|/(2n)$ *from* W.

2D:

- **All cells must have the same area**
- **The sites must be centrally symmetric in each cell**

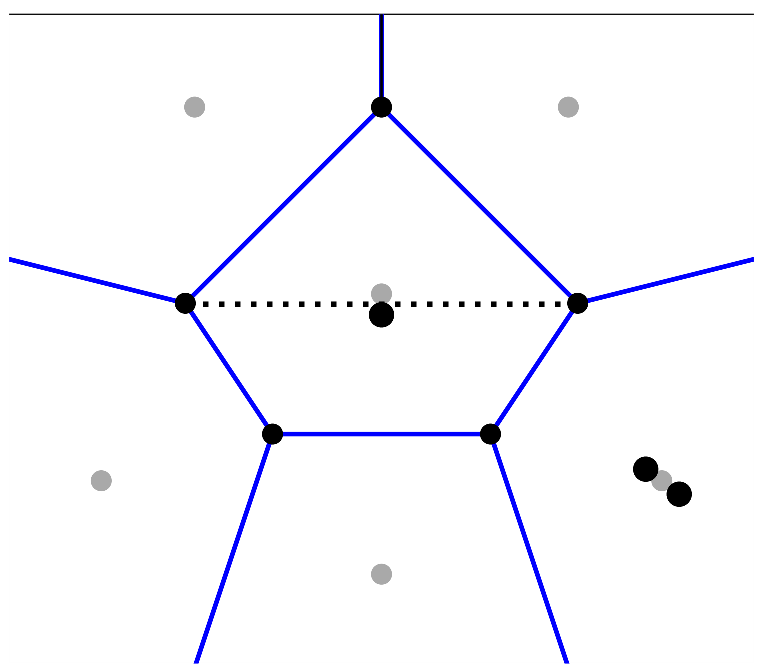


Technische Universität Braunschweig

Lemma 3. *Barney wins, if and only if he can place a point* p *that steals an area strictly larger than* $|Q|/(2n)$ *from* W.

2D:

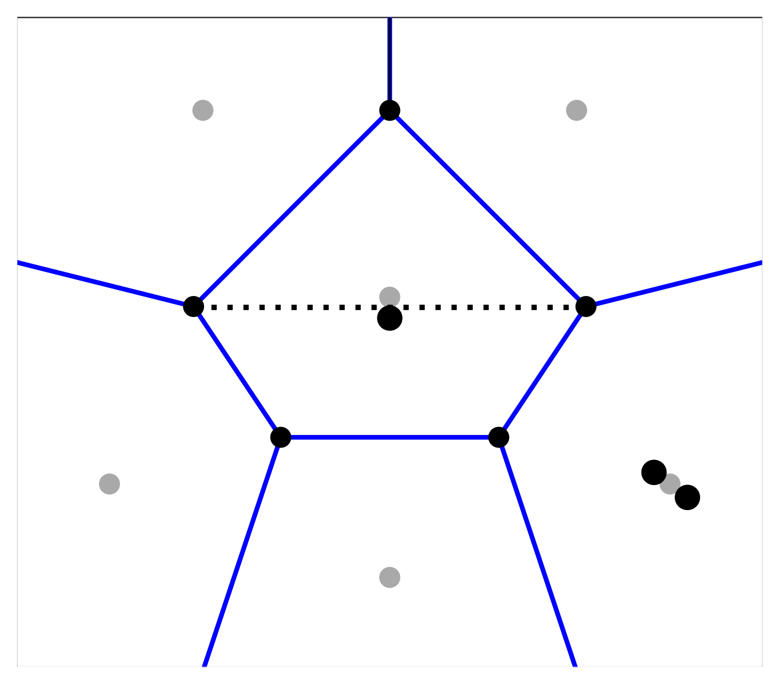
- **All cells must have the same area**
- **The sites must be centrally symmetric in each cell**



Lemma 3. *Barney wins, if and only if he can place a point p that steals an area strictly larger than $|Q|/(2n)$ from W.*

2D:

- **All cells must have the same area**
- **The sites must be centrally symmetric in each cell**



Lemma 1. *If $V(W)$ contains a cell that is not point symmetric, then Barney wins.*

Technische Universität Braunschweig

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Theorem 2. *If the board is a rectangle and if V(W) is not a regular grid, then Barney wins.*

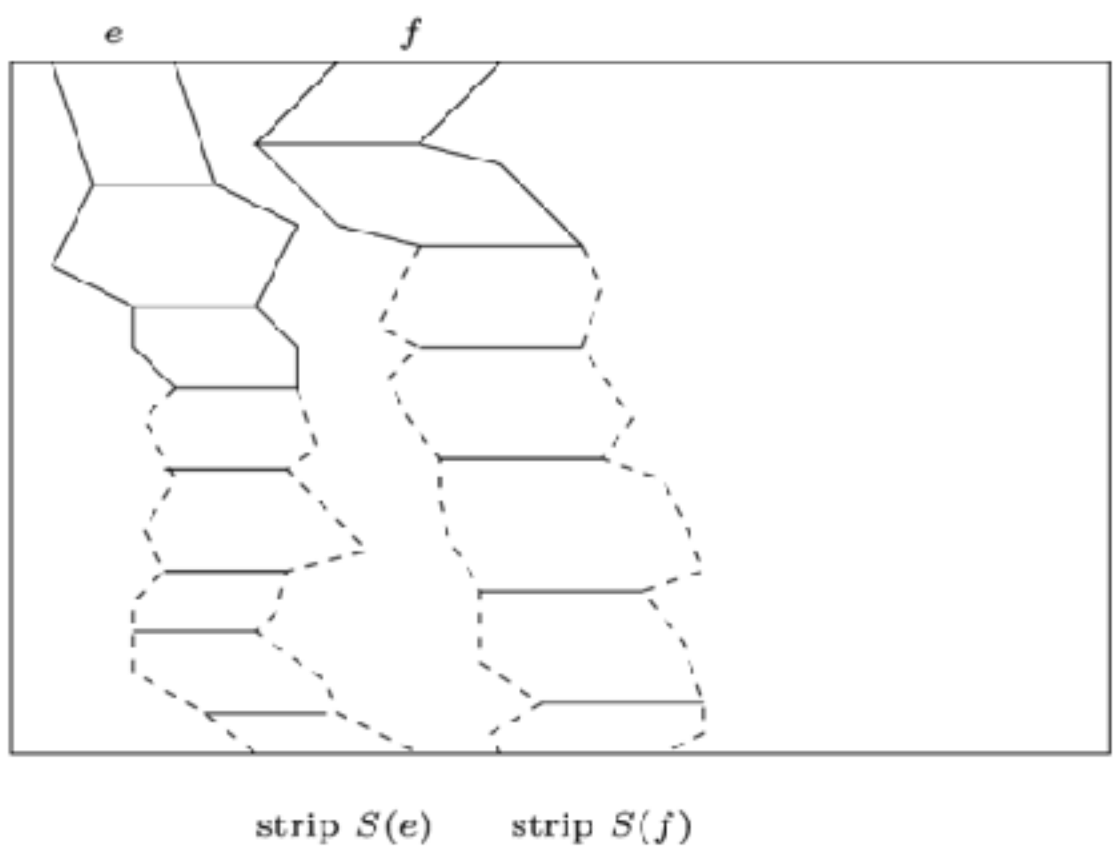


Fig. 1. Playing board with two strips.

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

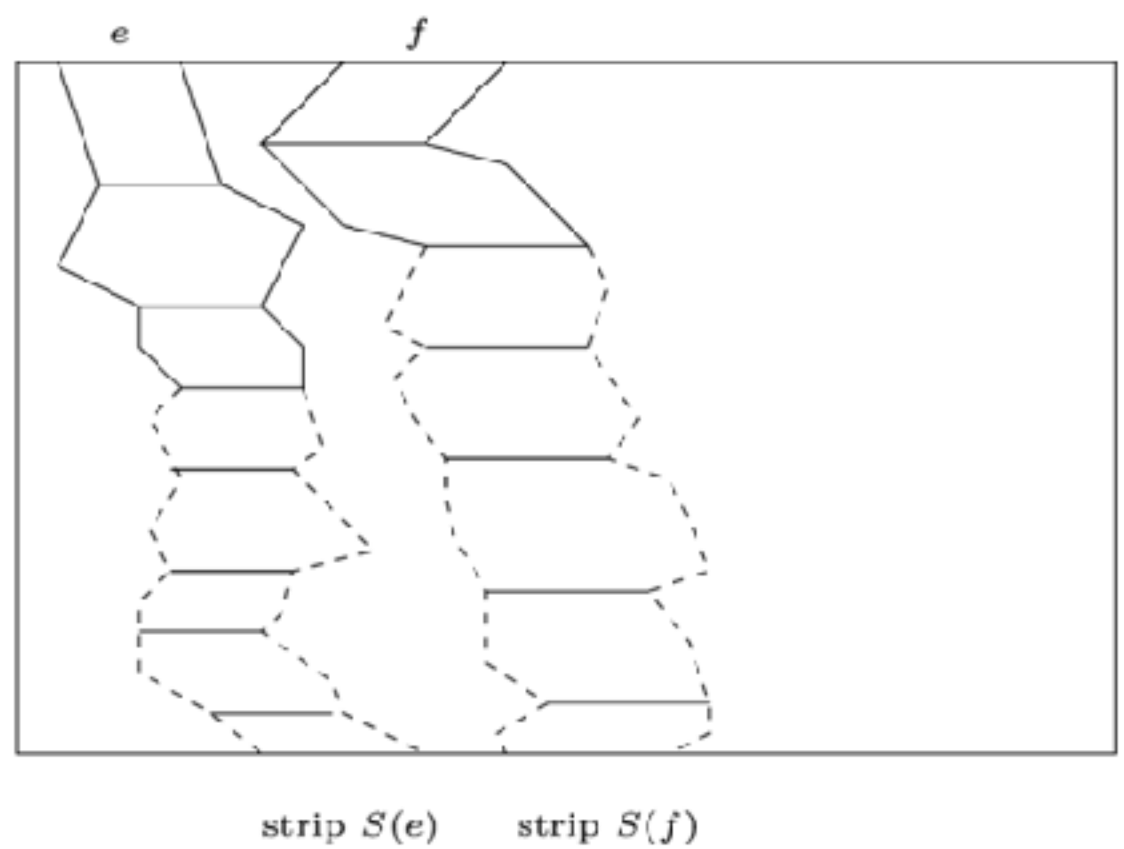


Fig. 1. Playing board with two strips.

- **Consider the top edge of a cell.**

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

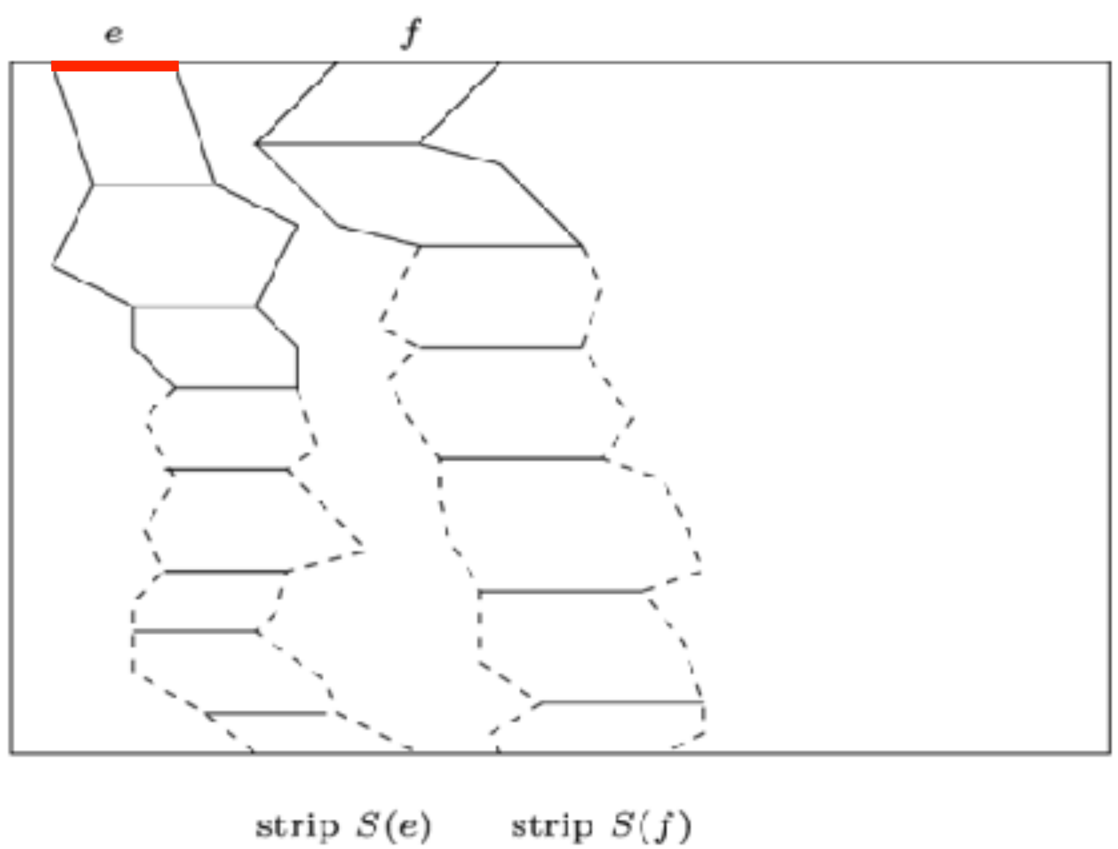


Fig. 1. Playing board with two strips.

- **Consider the top edge of a cell.**

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

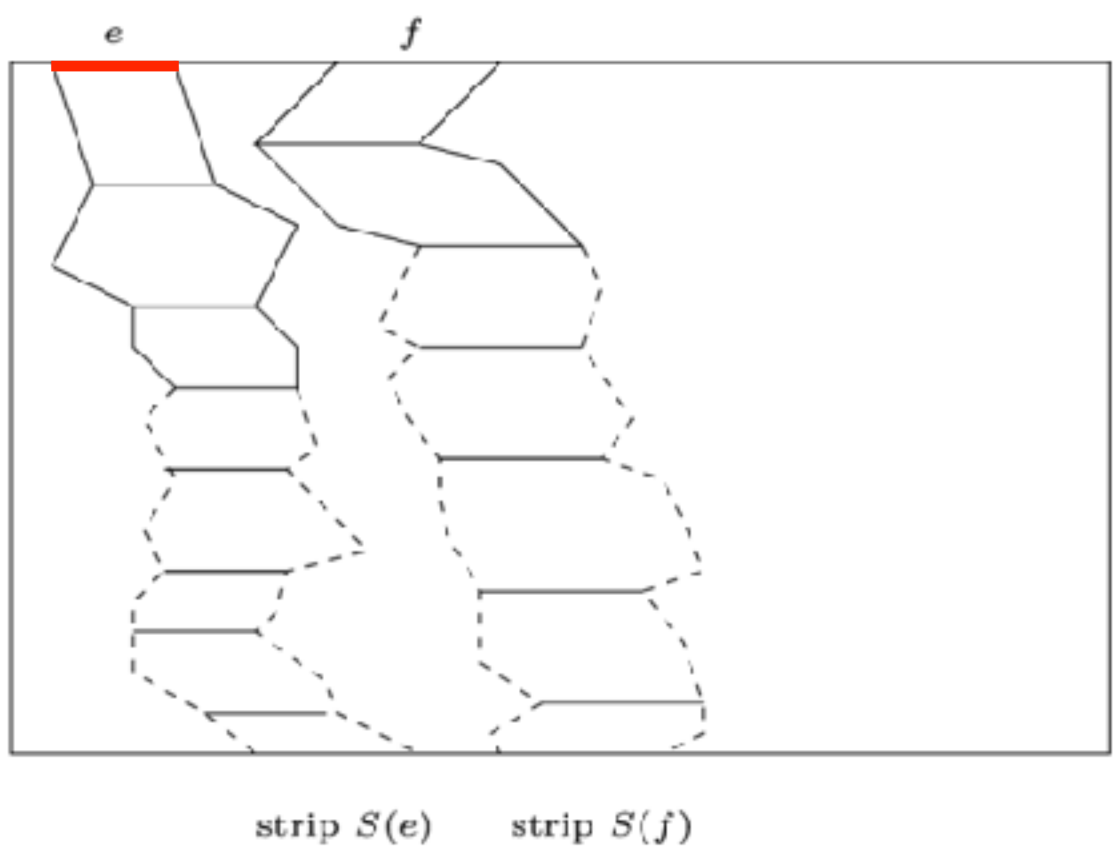


Fig. 1. Playing board with two strips.

- **Consider the top edge of a cell.**
- **By point symmetry, there is a corresponding bottom edge.**

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

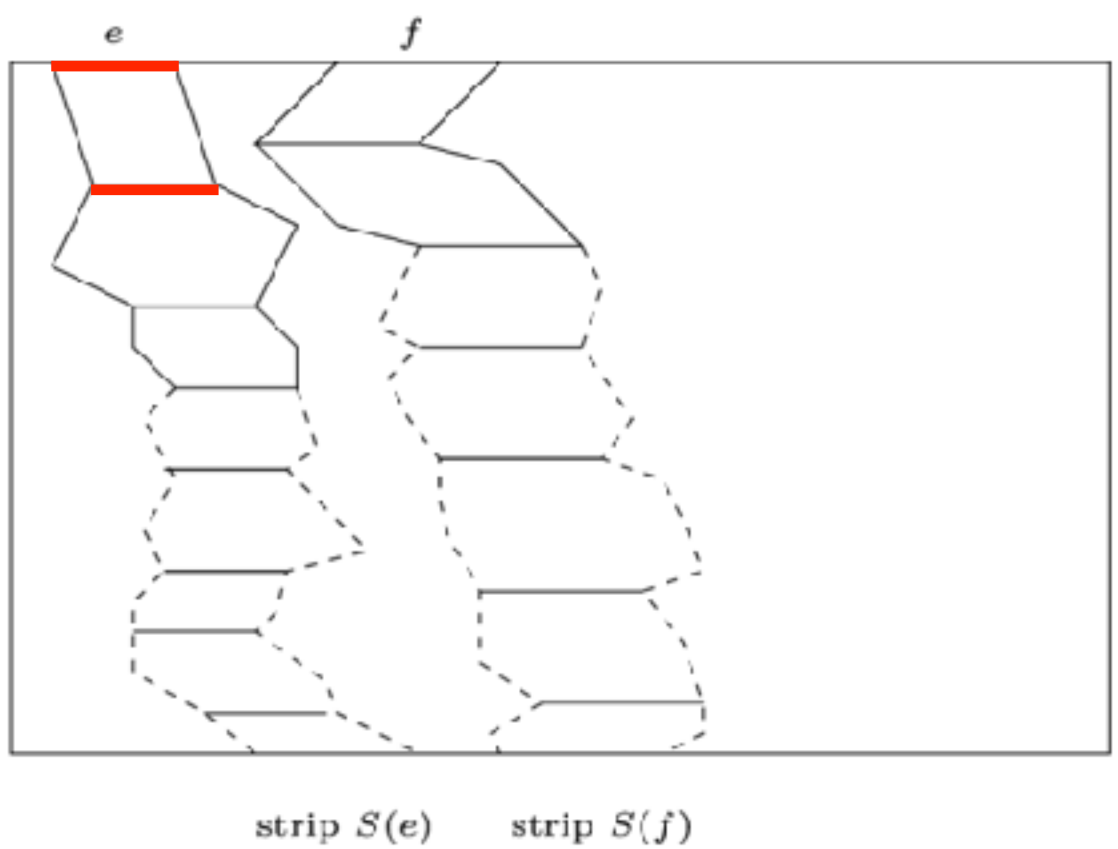


Fig. 1. Playing board with two strips.

- Consider the top edge of a cell.
- By point symmetry, there is a corresponding bottom edge.

Technische Universität Braunschweig

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

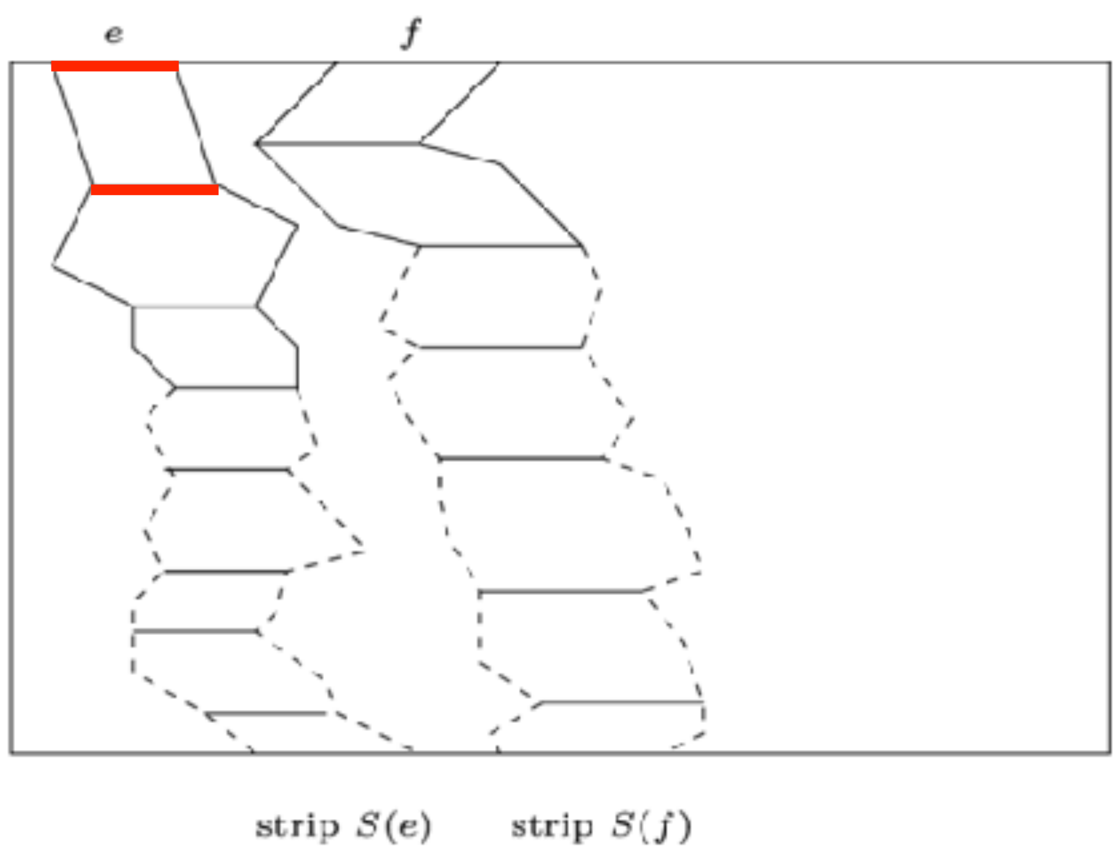


Fig. 1. Playing board with two strips.

- Consider the top edge of a cell.
- By point symmetry, there is a corresponding bottom edge.
- This induces a „strip“ sequence of cells.

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

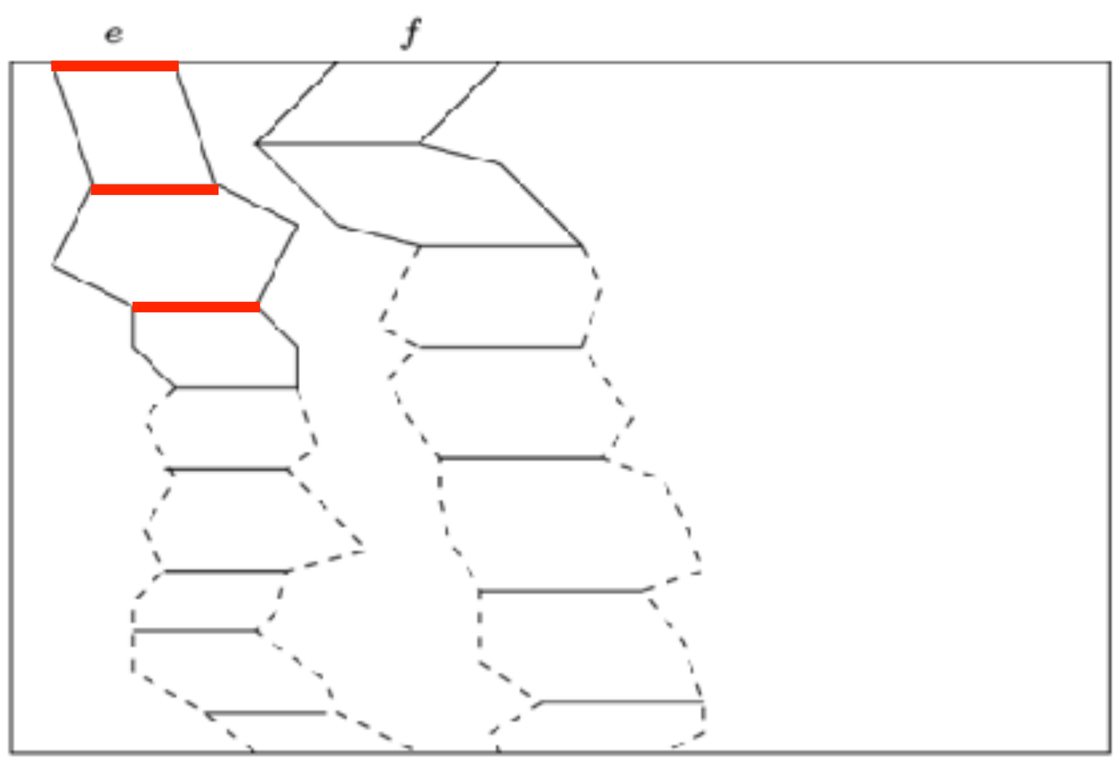


Fig. 1. Playing board with two strips.

- Consider the top edge of a cell.
- By point symmetry, there is a corresponding bottom edge.
- This induces a „strip“ sequence of cells.

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

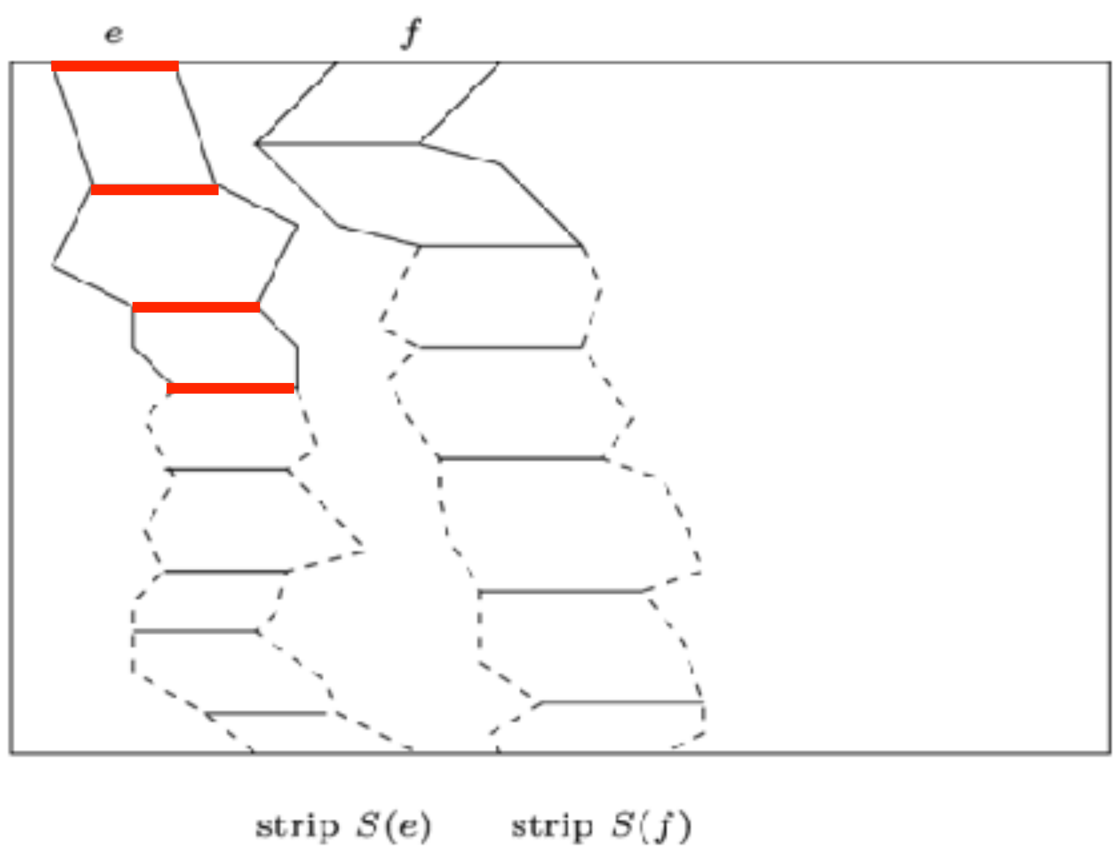


Fig. 1. Playing board with two strips.

- Consider the top edge of a cell.
- By point symmetry, there is a corresponding bottom edge.
- This induces a „strip“ sequence of cells.

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

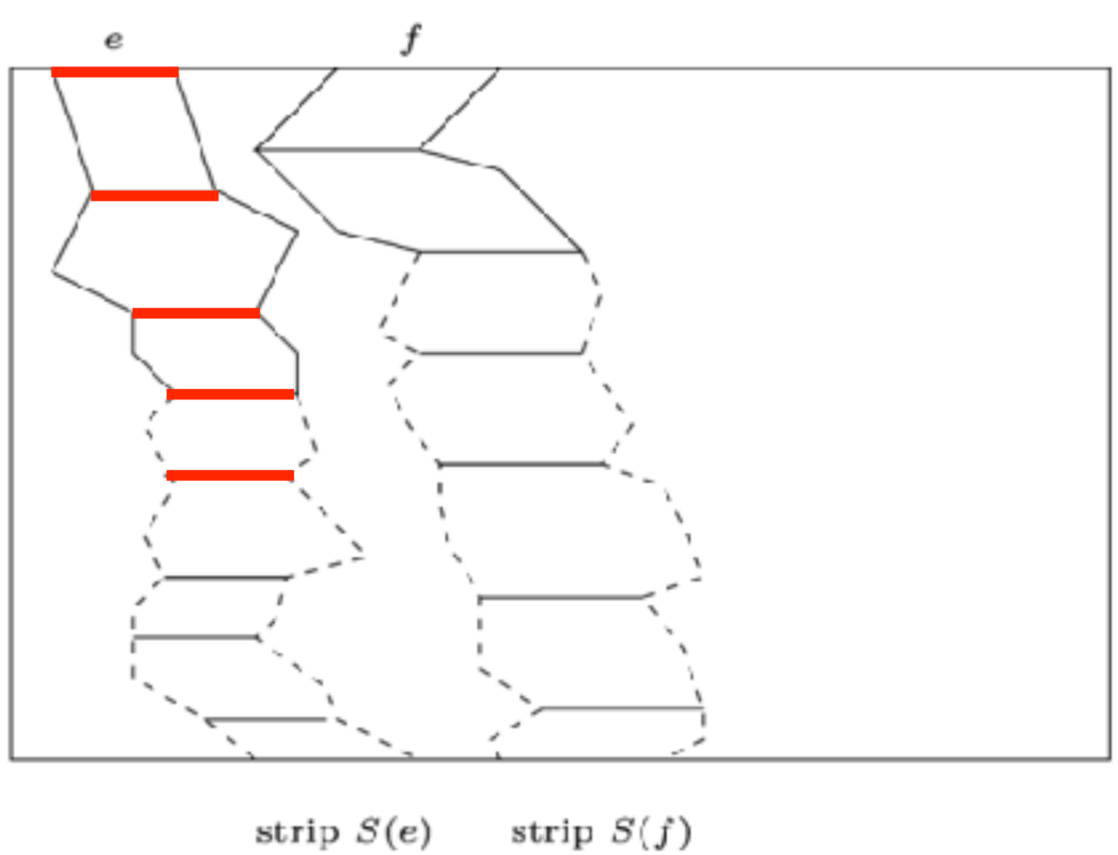


Fig. 1. Playing board with two strips.

- **Consider the top edge of a cell.**
- **By point symmetry, there is a corresponding bottom edge.**
- **This induces a „strip“ sequence of cells.**

Technische Universität Braunschweig

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

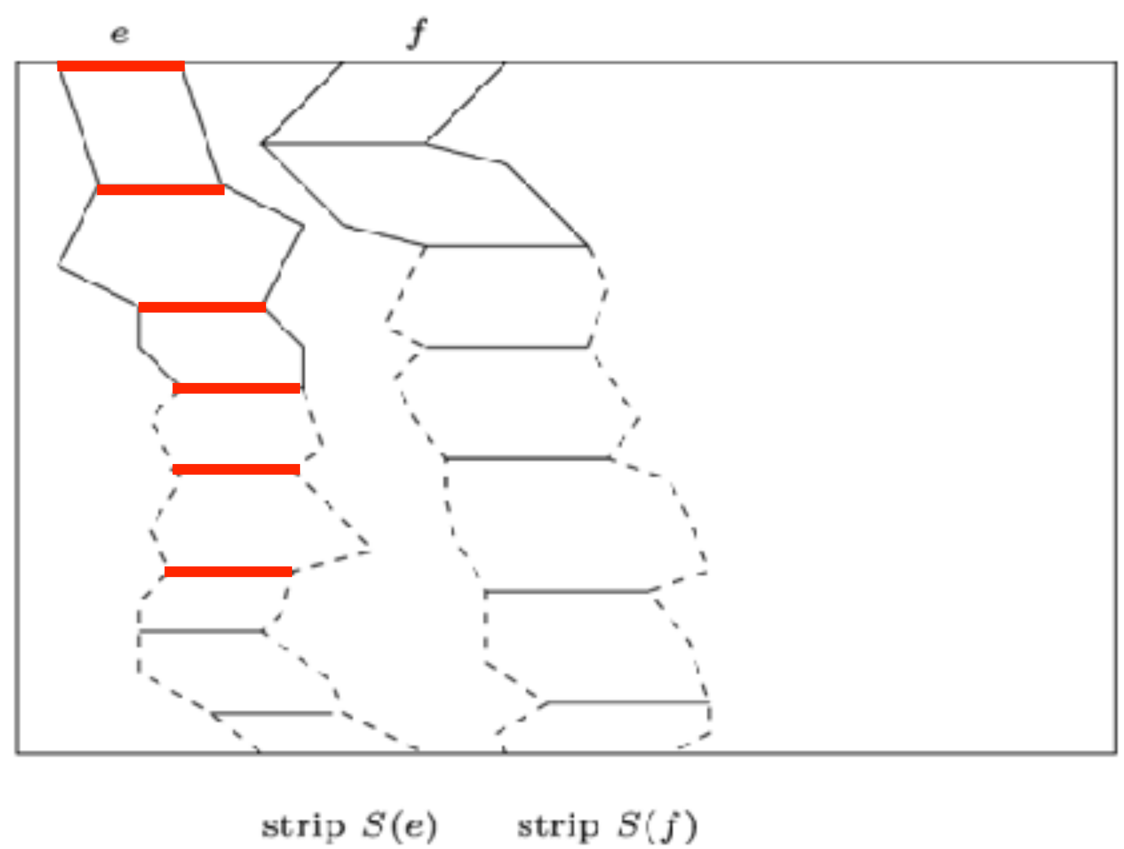
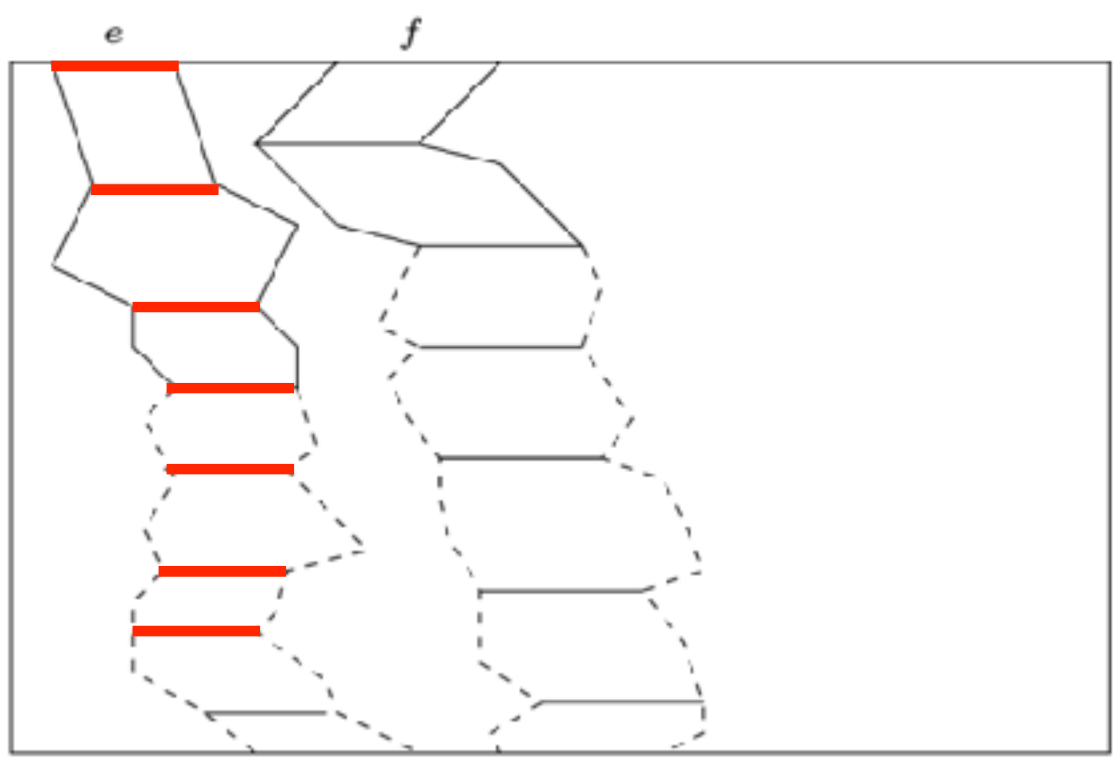


Fig. 1. Playing board with two strips.

- Consider the top edge of a cell.
- By point symmetry, there is a corresponding bottom edge.
- This induces a „strip“ sequence of cells.

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*



strip $S(e)$ strip $S(f)$

Fig. 1. Playing board with two strips.

- **Consider the top edge of a cell.**
- **By point symmetry, there is a corresponding bottom edge.**
- **This induces a „strip“ sequence of cells.**

Technische Universität Braunschweig

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

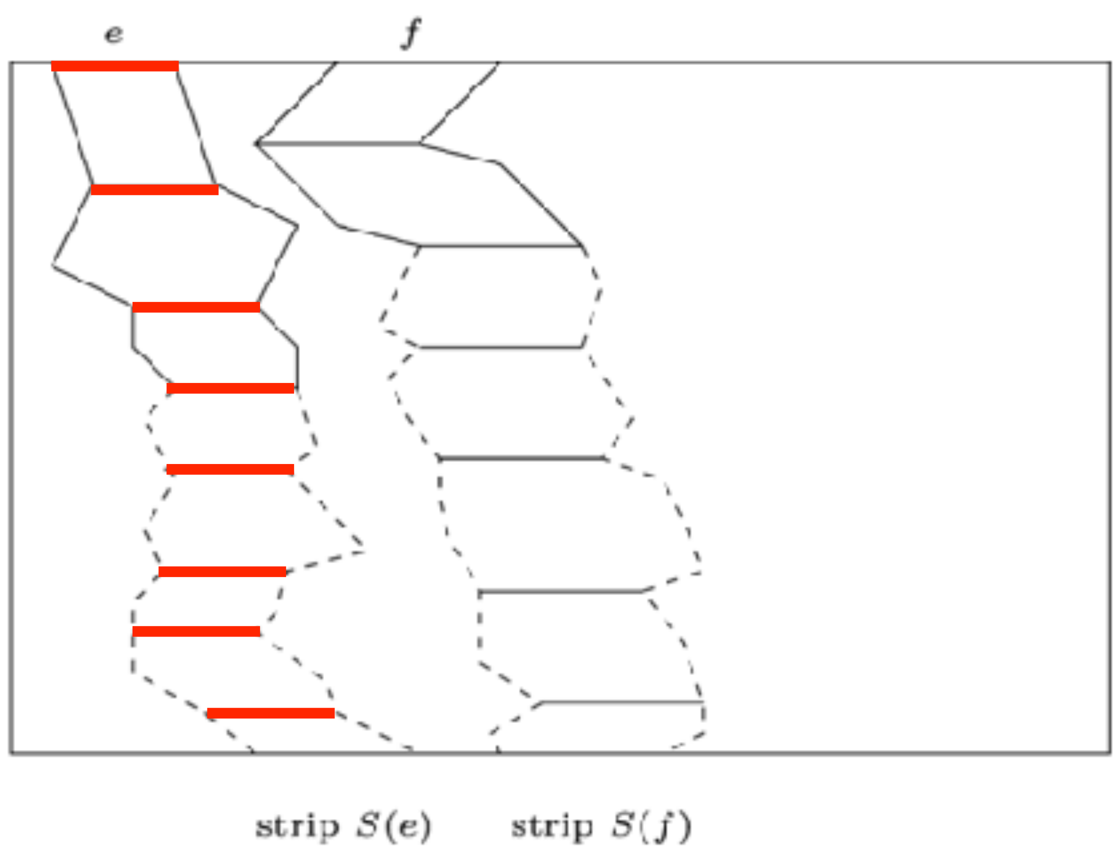


Fig. 1. Playing board with two strips.

- **Consider the top edge of a cell.**
- **By point symmetry, there is a corresponding bottom edge.**
- **This induces a „strip“ sequence of cells.**

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

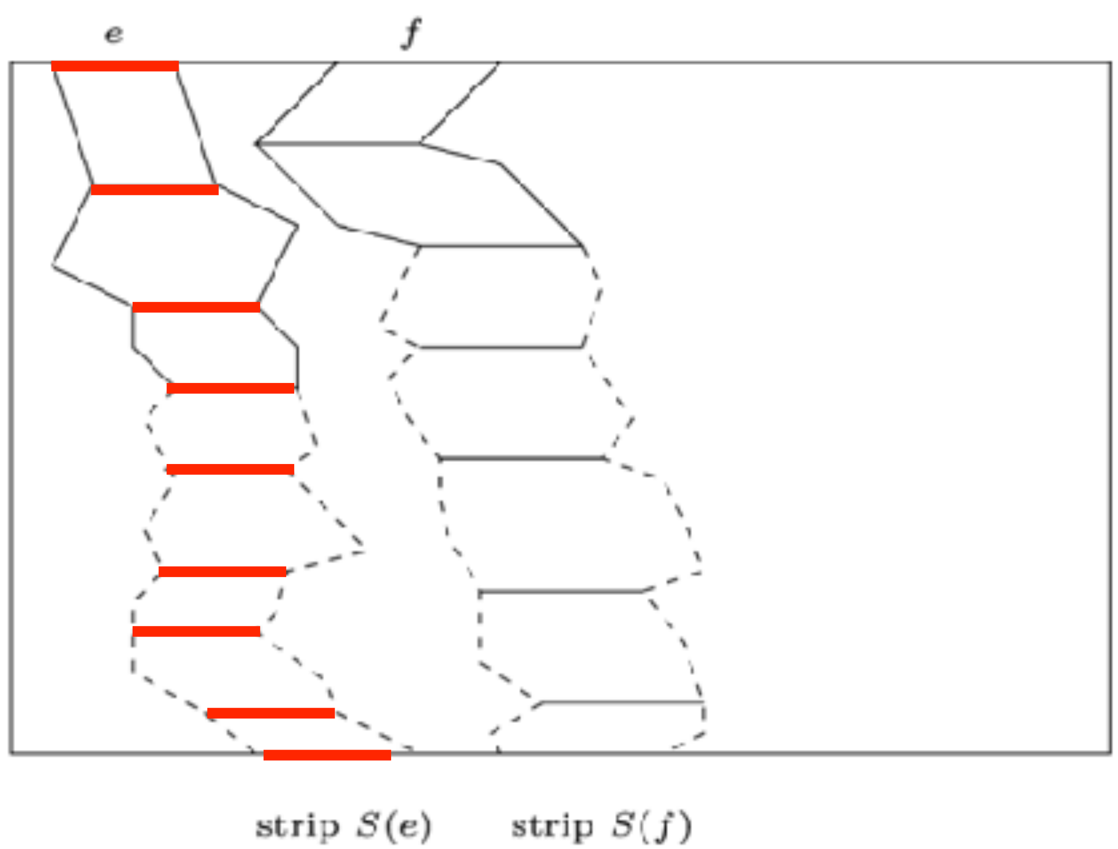


Fig. 1. Playing board with two strips.

- **Consider the top edge of a cell.**
- **By point symmetry, there is a corresponding bottom edge.**
- **This induces a „strip“ sequence of cells.**

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

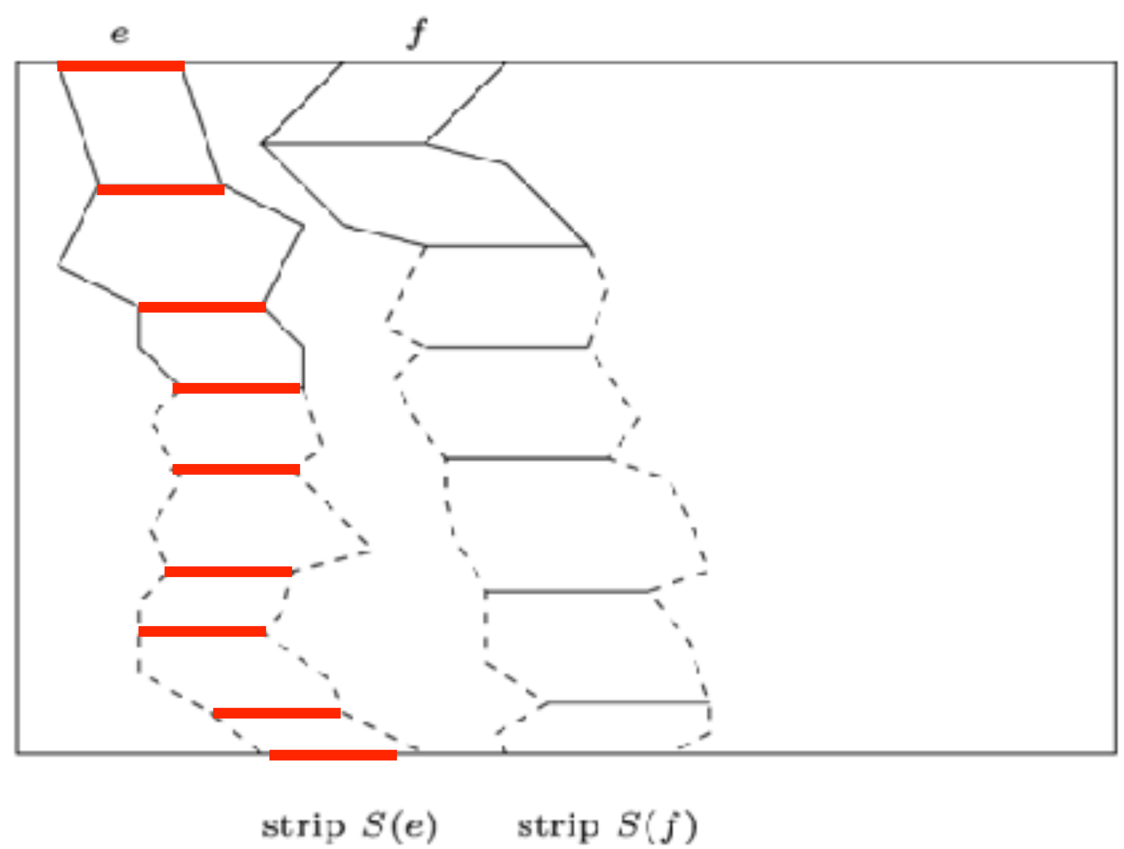


Fig. 1. Playing board with two strips.

- Consider the top edge of a cell.
- By point symmetry, there is a corresponding bottom edge.
- This induces a „strip“ sequence of cells.
- Because Voronoi cells are convex, all cells have minimum width of top edge length.

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

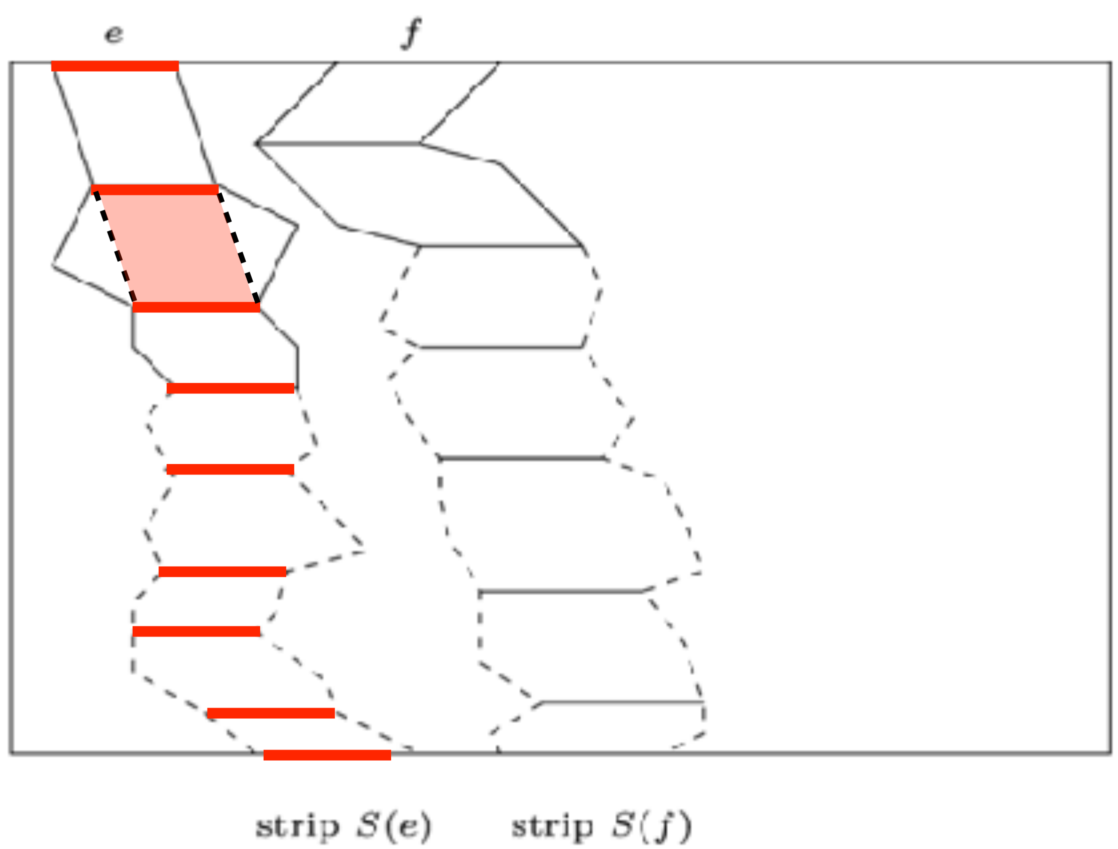


Fig. 1. Playing board with two strips.

- **Consider the top edge of a cell.**
- **By point symmetry, there is a corresponding bottom edge.**
- **This induces a „strip“ sequence of cells.**
- **Because Voronoi cells are convex, all cells have minimum width of top edge length.**

Technische Universität Braunschweig

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

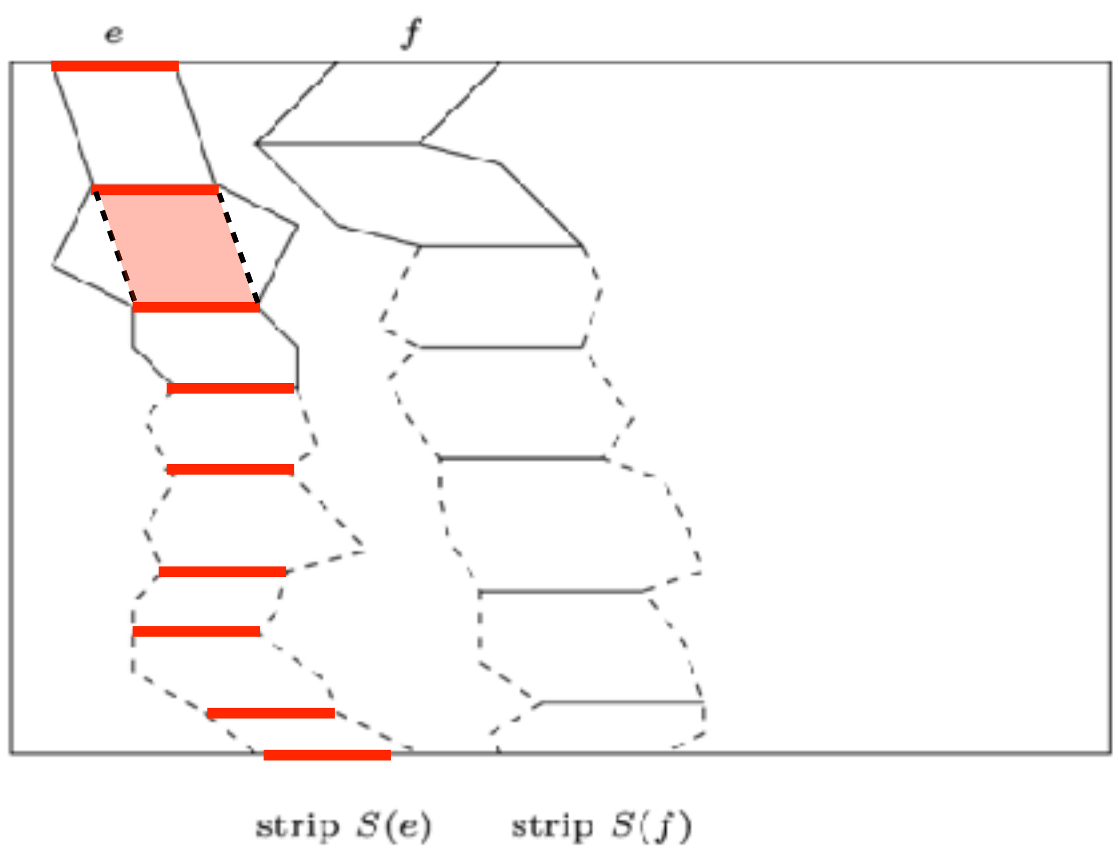


Fig. 1. Playing board with two strips.

- Consider the top edge of a cell.
- By point symmetry, there is a corresponding bottom edge.
- This induces a „strip“ sequence of cells.
- Because Voronoi cells are convex, all cells have minimum width of top edge length.
- Because cells are disjoint, so are different strips.

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

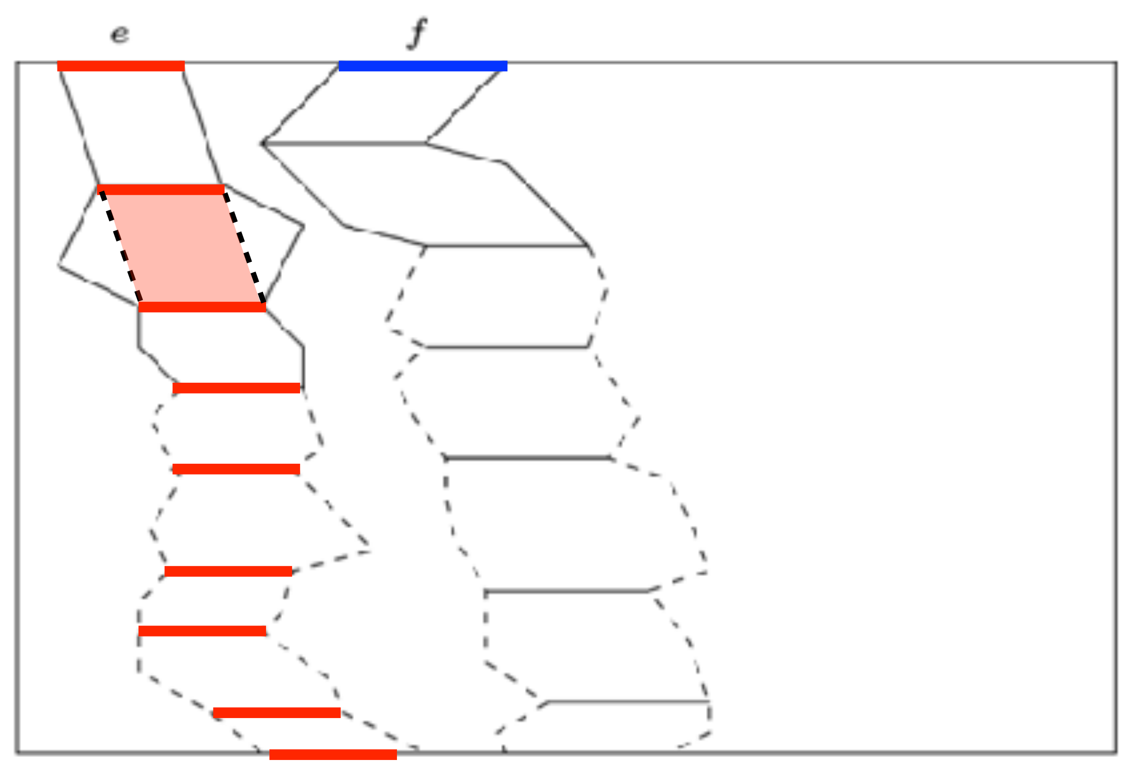


Fig. 1. Playing board with two strips.

- Consider the top edge of a cell.
- By point symmetry, there is a corresponding bottom edge.
- This induces a „strip“ sequence of cells.
- Because Voronoi cells are convex, all cells have minimum width of top edge length.
- Because cells are disjoint, so are different strips.

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

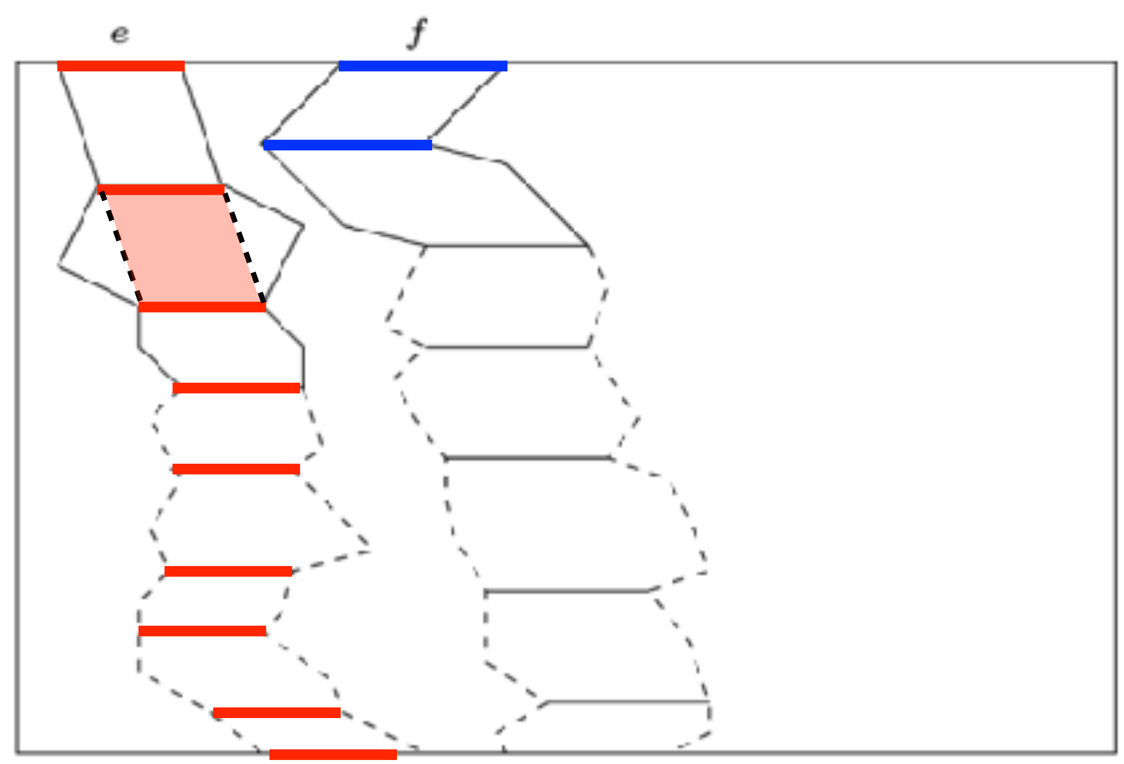


Fig. 1. Playing board with two strips.

- **Consider the top edge of a cell.**
- **By point symmetry, there is a corresponding bottom edge.**
- **This induces a „strip“ sequence of cells.**
- **Because Voronoi cells are convex, all cells have minimum width of top edge length.**
- **Because cells are disjoint, so are different strips.**

Technische Universität Braunschweig

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

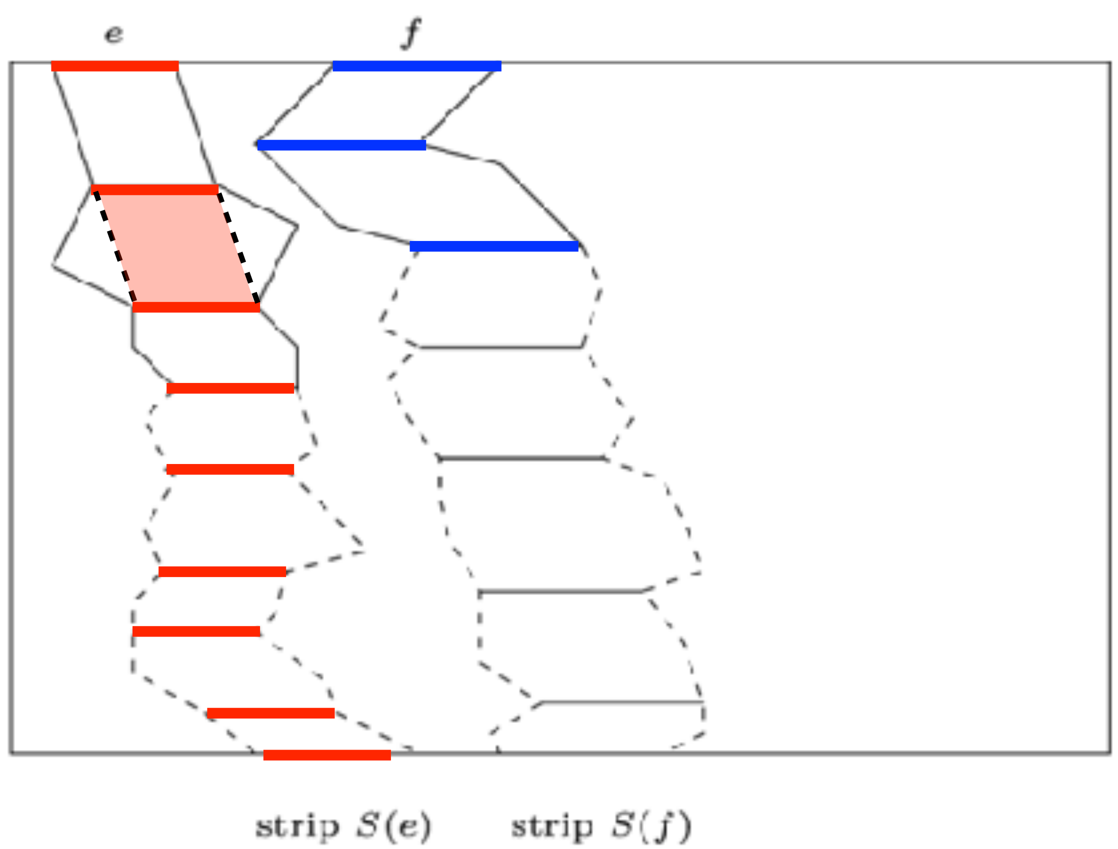


Fig. 1. Playing board with two strips.

- Consider the top edge of a cell.
- By point symmetry, there is a corresponding bottom edge.
- This induces a „strip“ sequence of cells.
- Because Voronoi cells are convex, all cells have minimum width of top edge length.
- Because cells are disjoint, so are different strips.

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

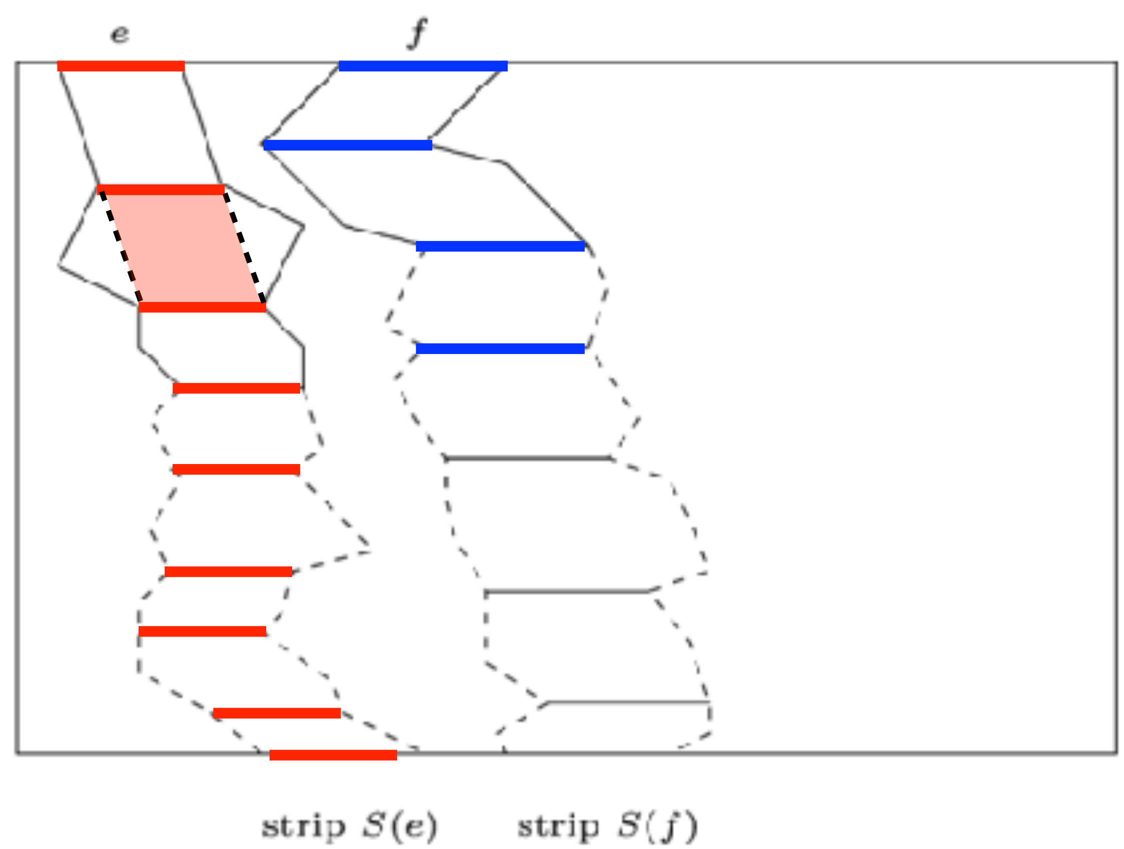


Fig. 1. Playing board with two strips.

- Consider the top edge of a cell.
- By point symmetry, there is a corresponding bottom edge.
- This induces a „strip“ sequence of cells.
- Because Voronoi cells are convex, all cells have minimum width of top edge length.
- Because cells are disjoint, so are different strips.

Technische Universität Braunschweig

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

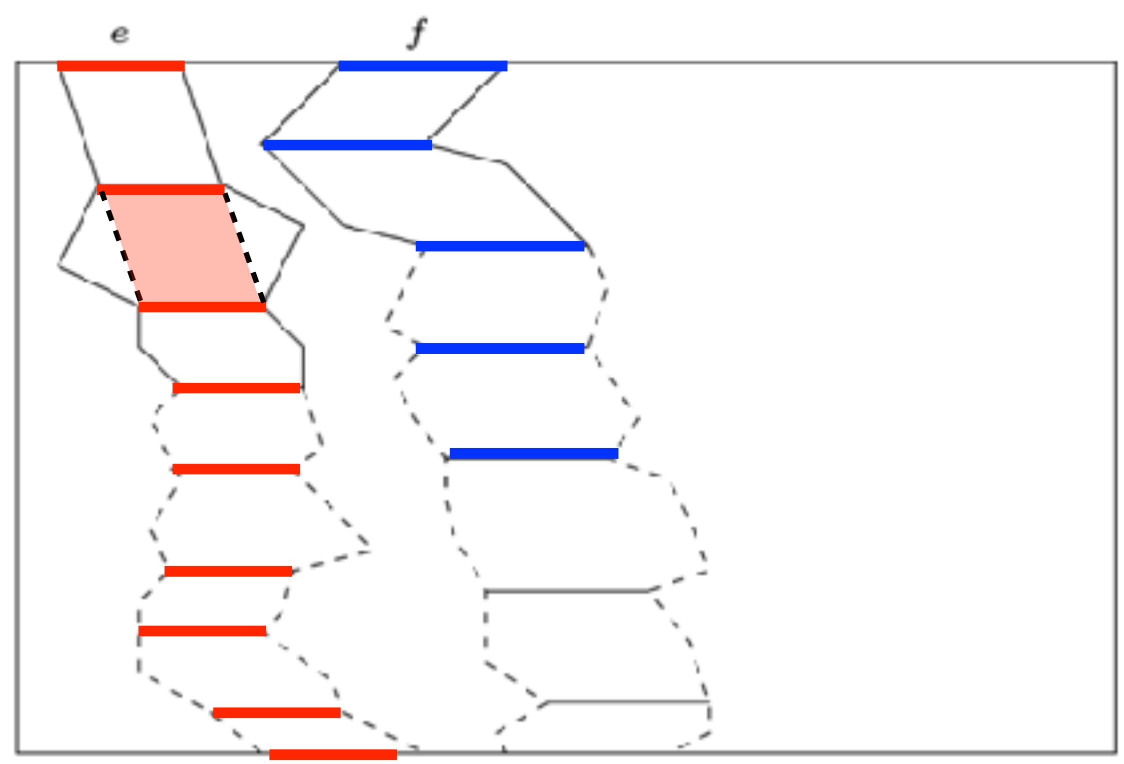


Fig. 1. Playing board with two strips.

- Consider the top edge of a cell.
- By point symmetry, there is a corresponding bottom edge.
- This induces a „strip“ sequence of cells.
- Because Voronoi cells are convex, all cells have minimum width of top edge length.
- Because cells are disjoint, so are different strips.

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Theorem 2. *If the board is a rectangle and if* $V(W)$ *is not a regular grid, then Barney wins.*

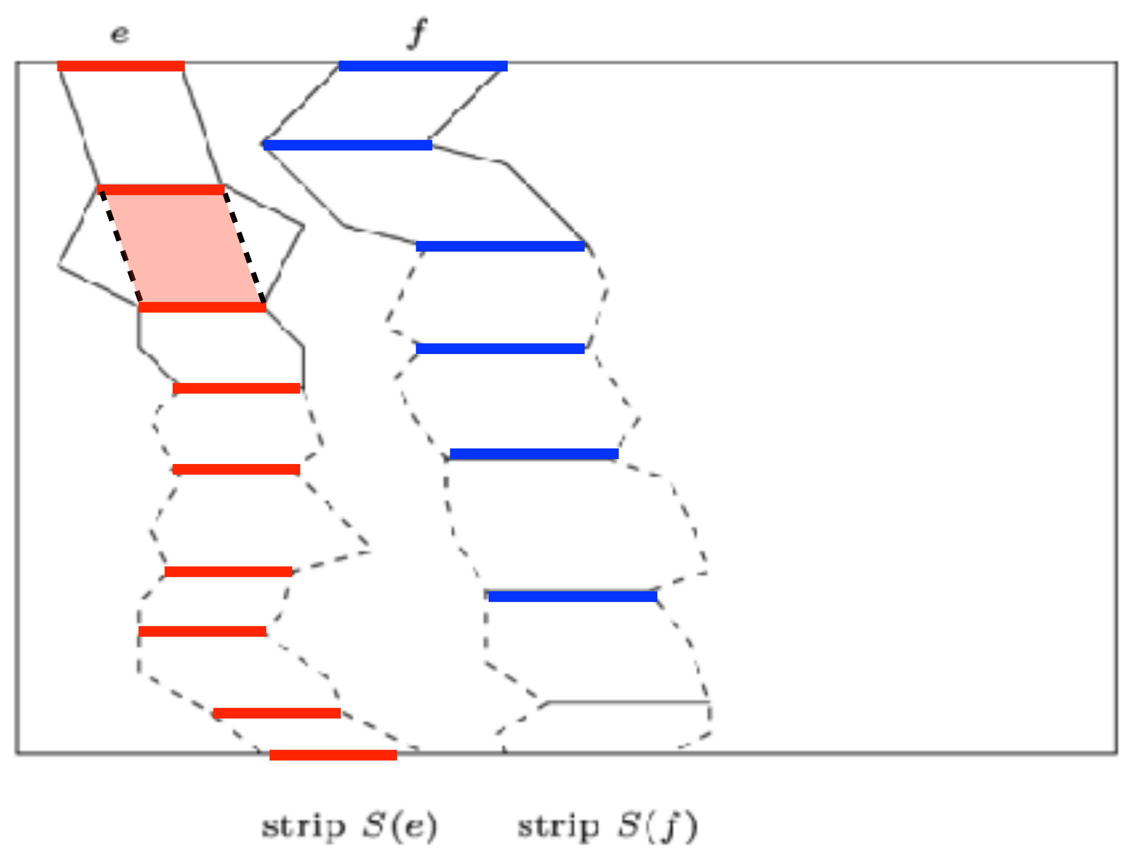


Fig. 1. Playing board with two strips.

- Consider the top edge of a cell.
- By point symmetry, there is a corresponding bottom edge.
- This induces a „strip“ sequence of cells.
- Because Voronoi cells are convex, all cells have minimum width of top edge length.
- Because cells are disjoint, so are different strips.

Technische Universität Braunschweig

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

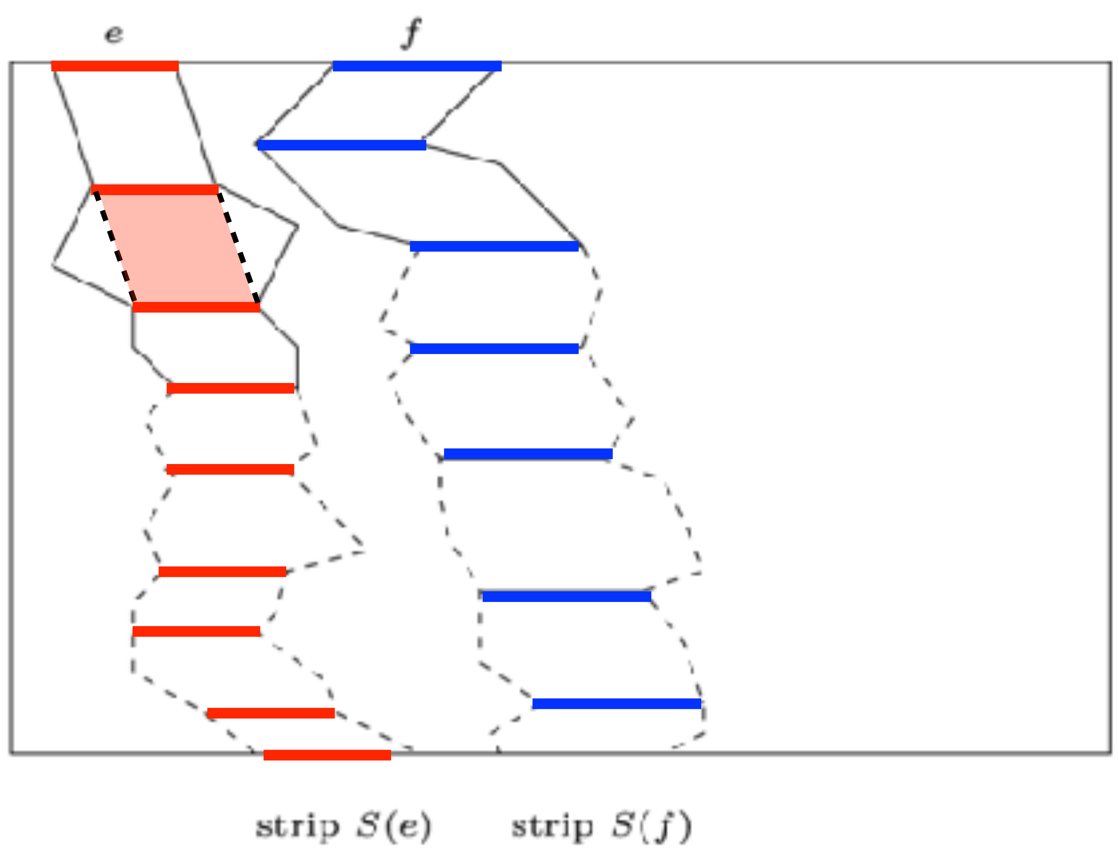


Fig. 1. Playing board with two strips.

- Consider the top edge of a cell.
- By point symmetry, there is a corresponding bottom edge.
- This induces a „strip“ sequence of cells.
- Because Voronoi cells are convex, all cells have minimum width of top edge length.
- Because cells are disjoint, so are different strips.

Theorem 2. *If the board is a rectangle and if V(W) is not a regular grid, then Barney wins.*

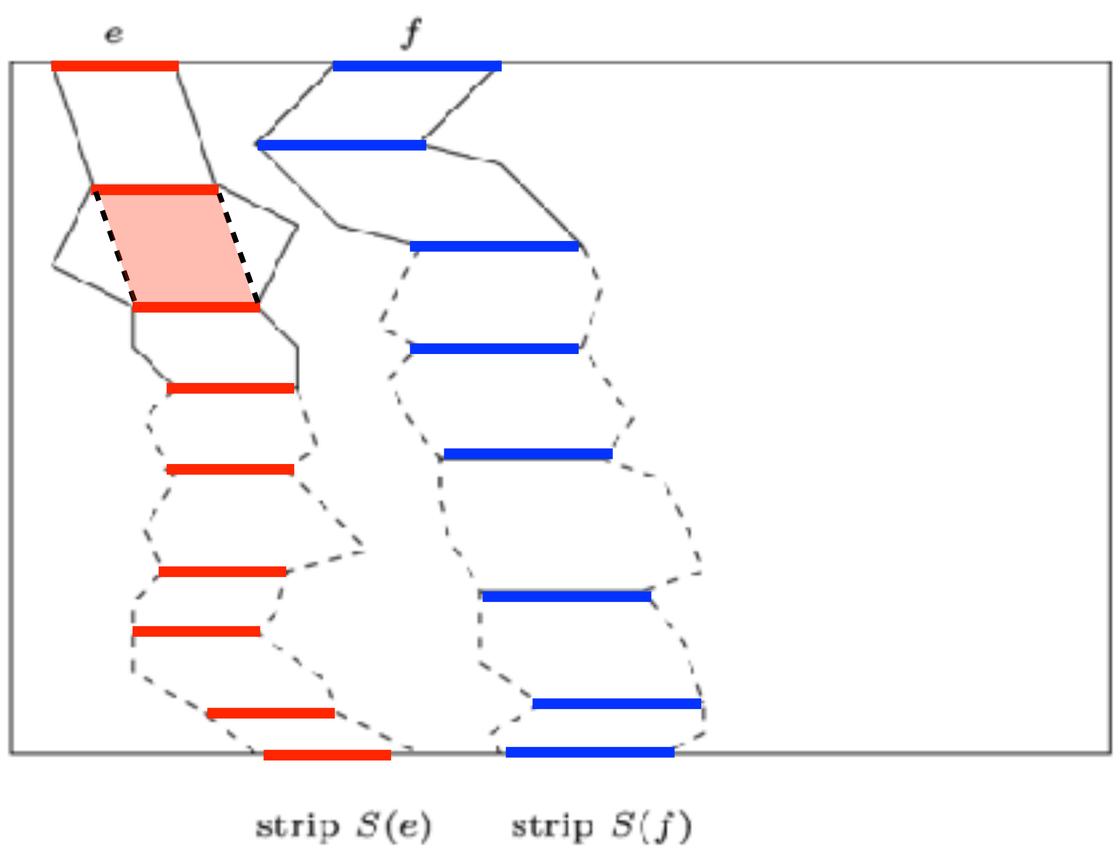


Fig. 1. Playing board with two strips.

- Consider the top edge of a cell.
- By point symmetry, there is a corresponding bottom edge.
- This induces a „strip“ sequence of cells.
- Because Voronoi cells are convex, all cells have minimum width of top edge length.
- Because cells are disjoint, so are different strips.

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

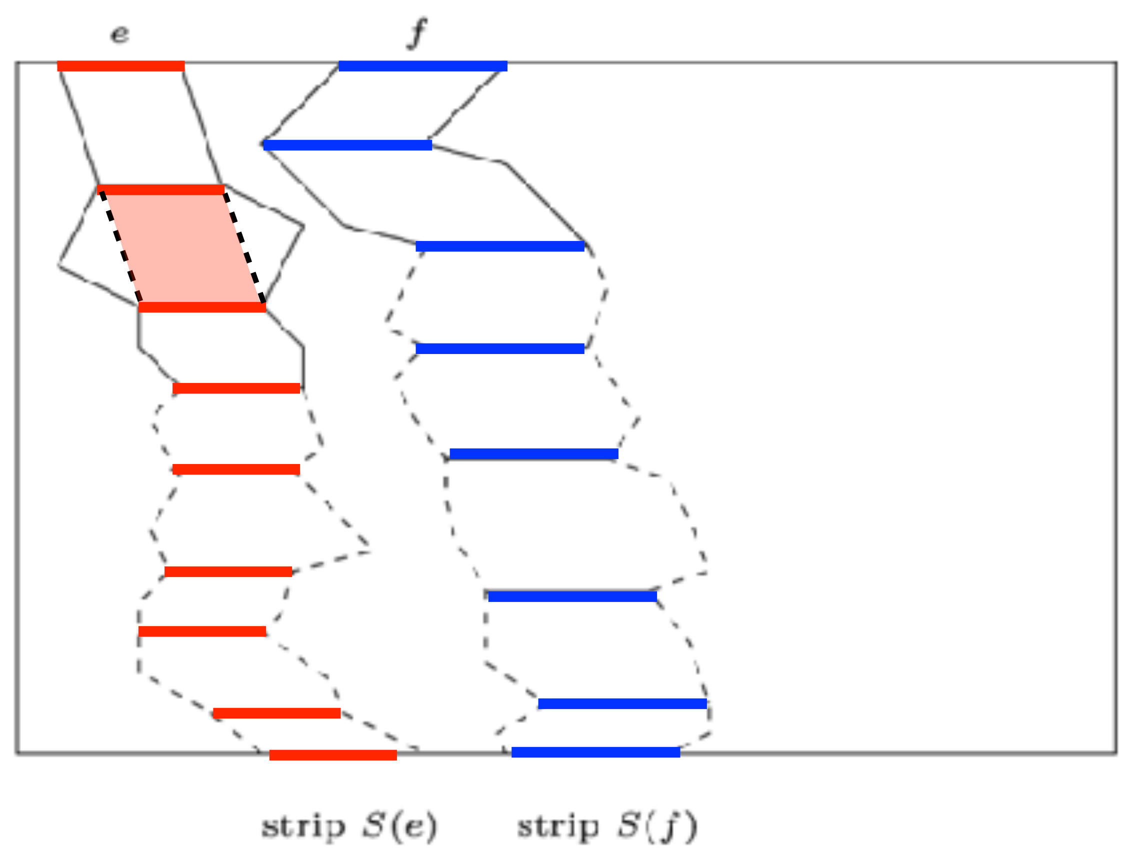


Fig. 1. Playing board with two strips.

- Consider the top edge of a cell.
- By point symmetry, there is a corresponding bottom edge.
- This induces a „strip“ sequence of cells.
- Because Voronoi cells are convex, all cells have minimum width of top edge length.
- Because cells are disjoint, so are different strips.
- Because total horizontal width is constant, strip width must be constant.

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

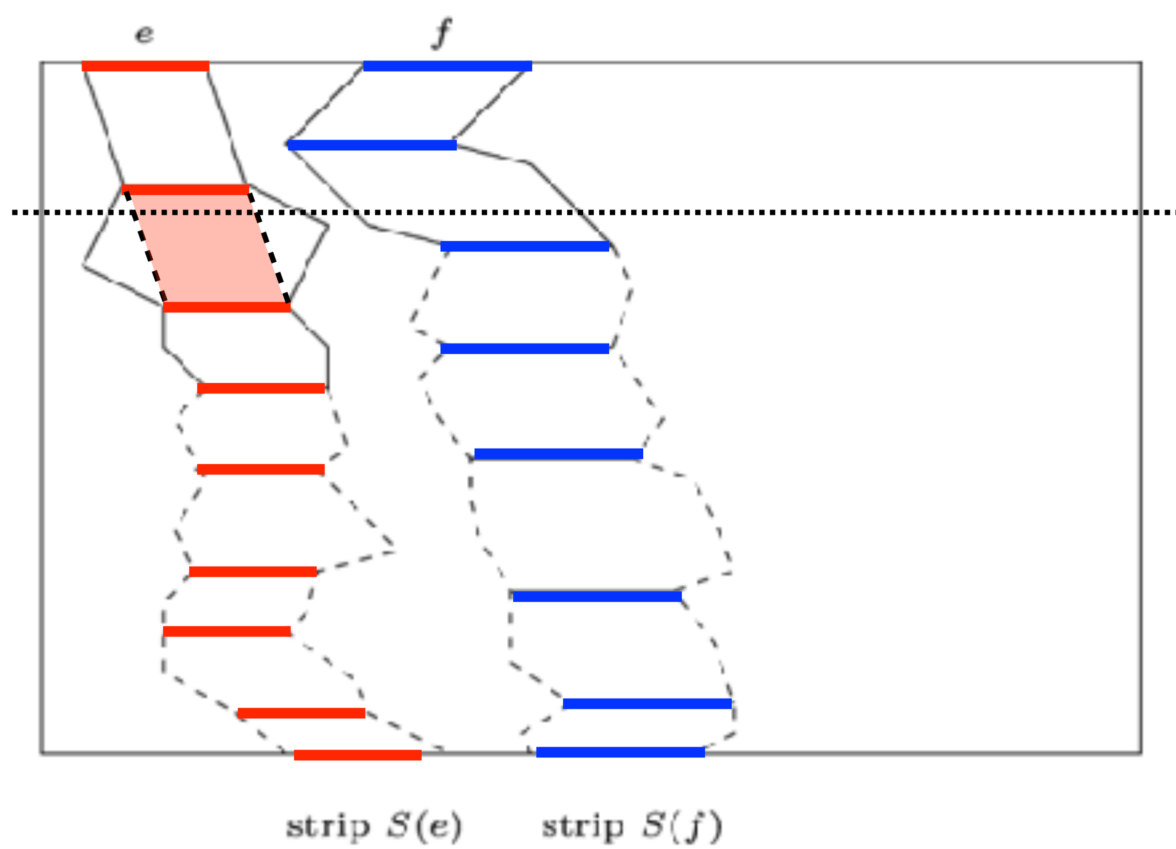


Fig. 1. Playing board with two strips.

- Consider the top edge of a cell.
- By point symmetry, there is a corresponding bottom edge.
- This induces a „strip“ sequence of cells.
- Because Voronoi cells are convex, all cells have minimum width of top edge length.
- Because cells are disjoint, so are different strips.
- Because total horizontal width is constant, strip width must be constant.

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Theorem 2. *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

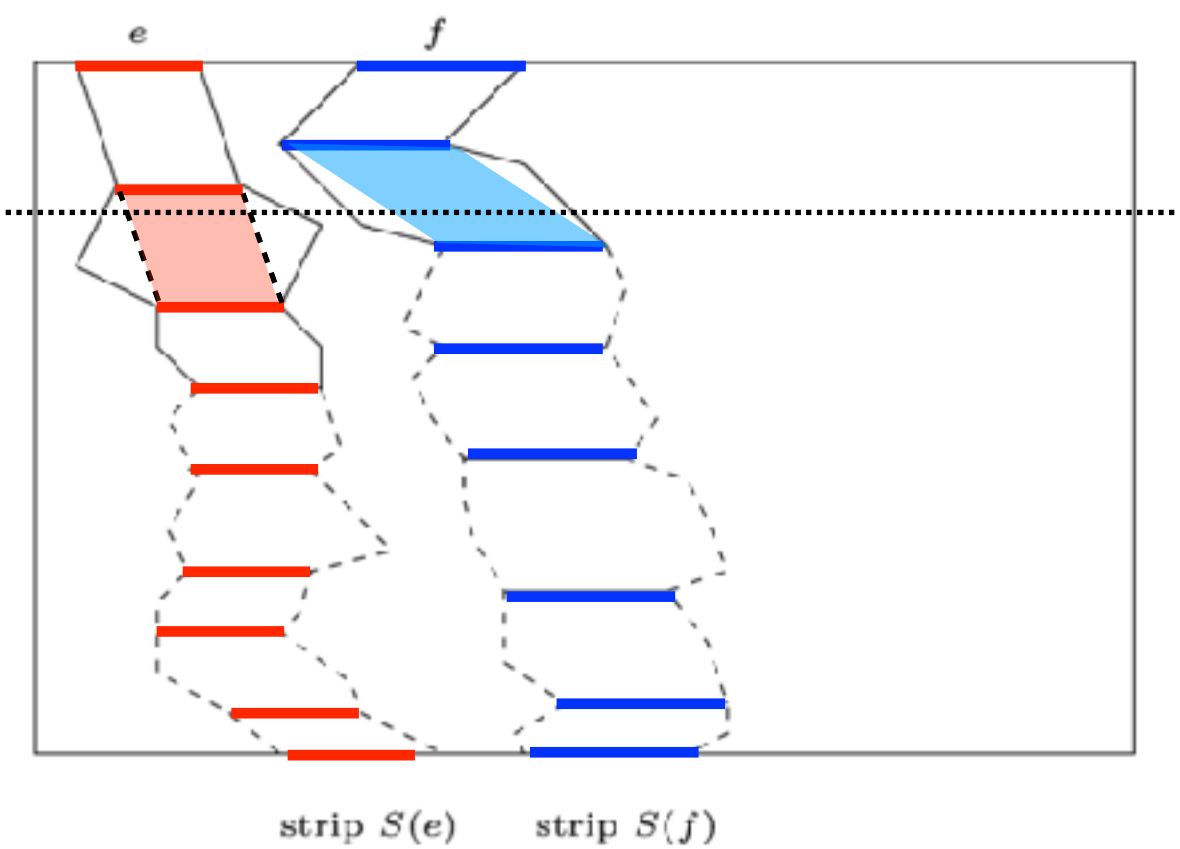


Fig. 1. Playing board with two strips.

- **Consider the top edge of a cell.**
- **By point symmetry, there is a corresponding bottom edge.**
- **This induces a „strip“ sequence of cells.**
- **Because Voronoi cells are convex, all cells have minimum width of top edge length.**
- **Because cells are disjoint, so are different strips.**
- **Because total horizontal width is constant, strip width must be constant.**

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Theorem 2. *If the board is a rectangle and if V(W) is not a regular grid, then Barney wins.*

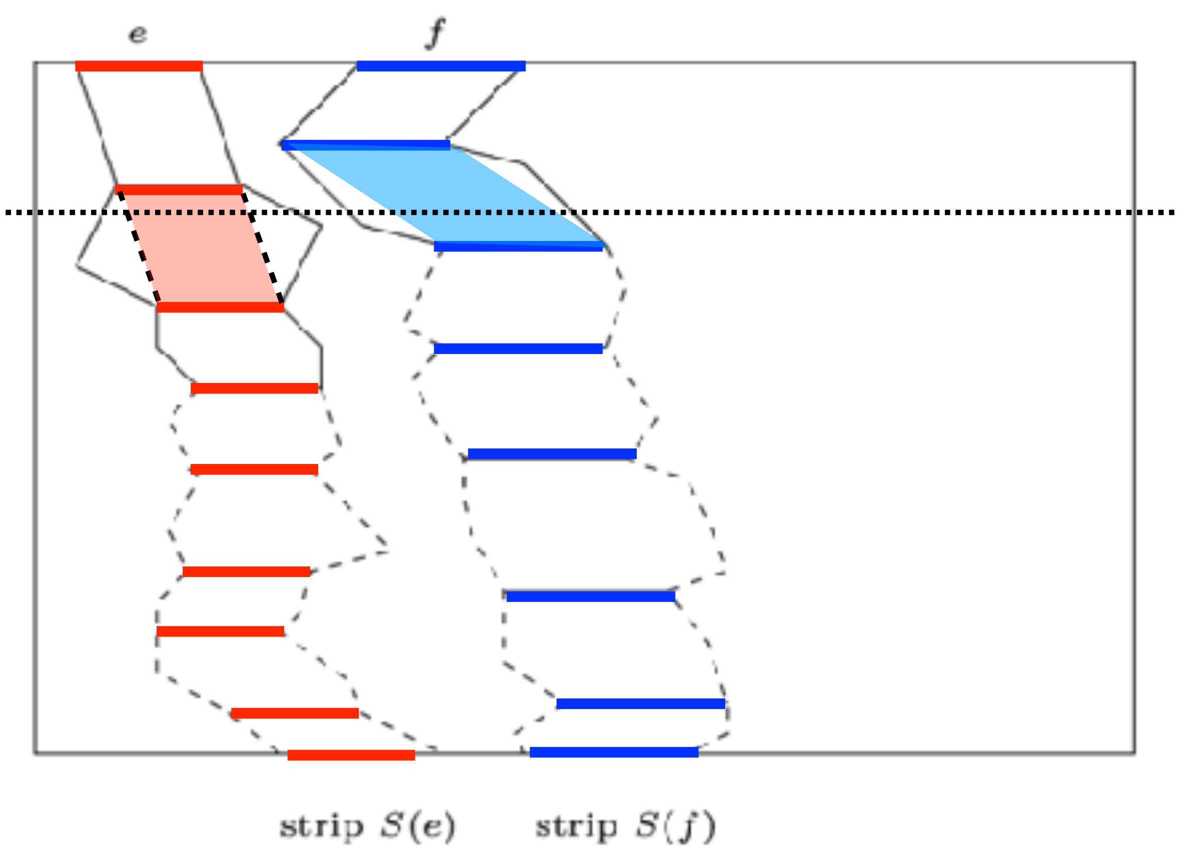


Fig. 1. Playing board with two strips.

- Consider the top edge of a cell.
- By point symmetry, there is a corresponding bottom edge.
- This induces a „strip“ sequence of cells.
- Because Voronoi cells are convex, all cells have minimum width of top edge length.
- Because cells are disjoint, so are different strips.
- Because total horizontal width is constant, strip width must be constant.
- We get a rectangular grid.

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Lemma 4. *If $n = 2$ and $\rho > \sqrt{3}/2$, then Barney wins. If the aspect ratio is smaller, Barney loses.*

Lemma 4. *If $n = 2$ and $\rho > \sqrt{3}/2$, then Barney wins. If the aspect ratio is smaller, Barney loses.*

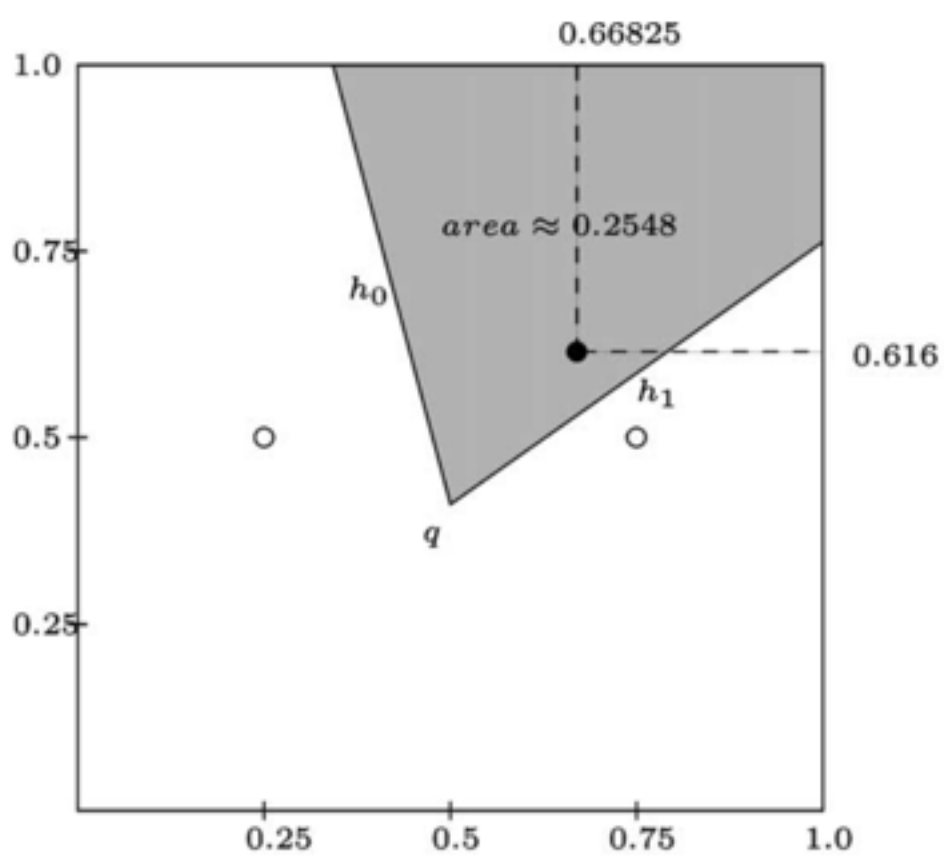
- **If Wilma does not play a regular grid, Barney can win.**

Lemma 4. *If $n = 2$ and $\rho > \sqrt{3}/2$, then Barney wins. If the aspect ratio is smaller, Barney loses.*

- **If Wilma does not play a regular grid, Barney can win.**
- **If Wilma does play a regular grid, Barney can gain area from *both* cells and win.**

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Lemma 4. *If* $n = 2$ *and* $\rho > \sqrt{3}/2$, *then Barney wins. If the aspect ratio is smaller, Barney loses.*



- **If Wilma does not play a regular grid, Barney can win.**
- **If Wilma does play a regular grid, Barney can gain area from *both* cells and win.**

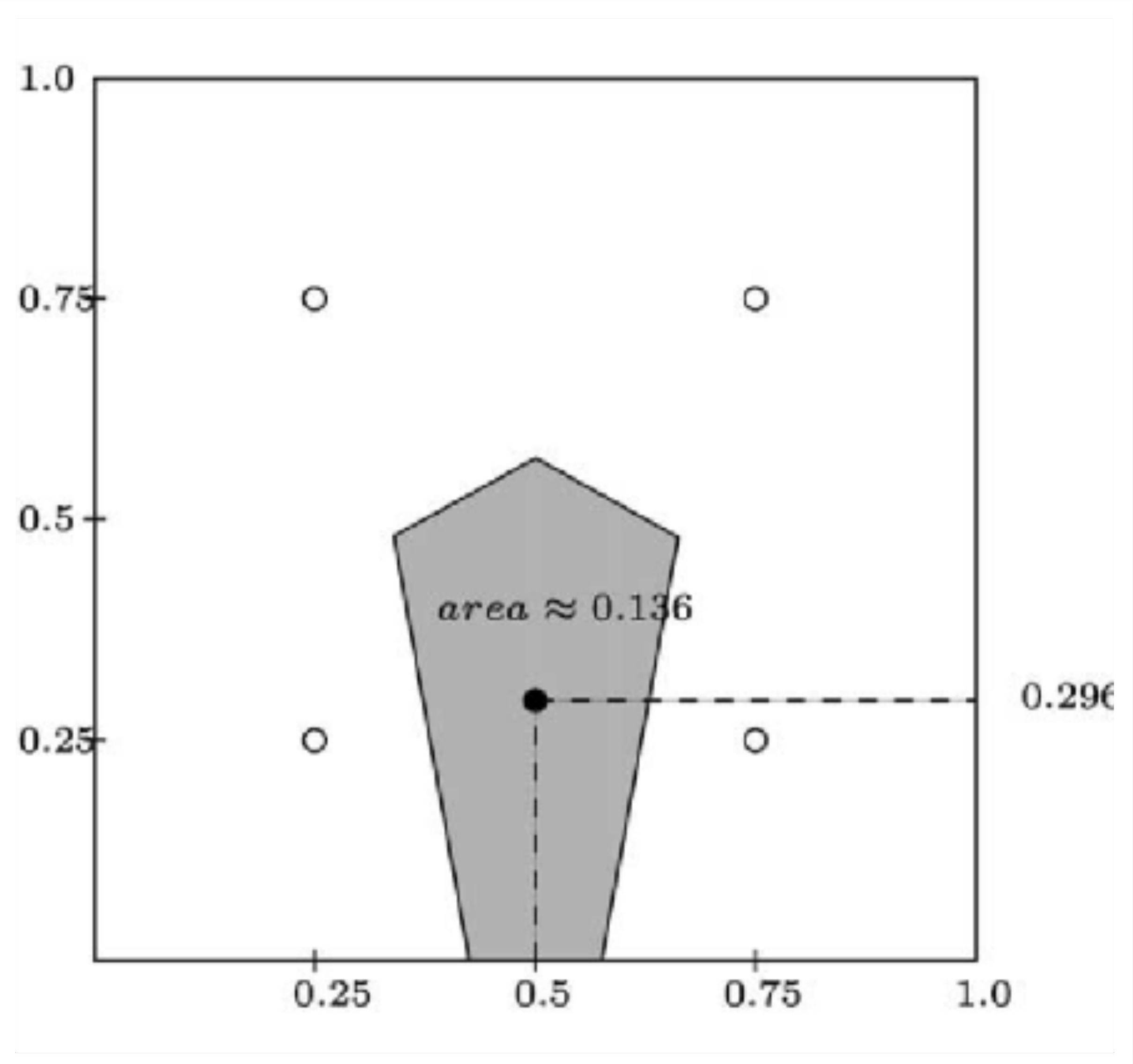
Technische Universität Braunschweig

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Lemma 5. *Suppose that the board is rectangular and that $n = 4$. If Wilma places her points on a regular 2×2 grid, Barney can gain 50.78% of the board.*

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Lemma 5. *Suppose that the board is rectangular and that $n = 4$. If Wilma places her points on a regular 2×2 grid, Barney can gain 50.78% of the board.*

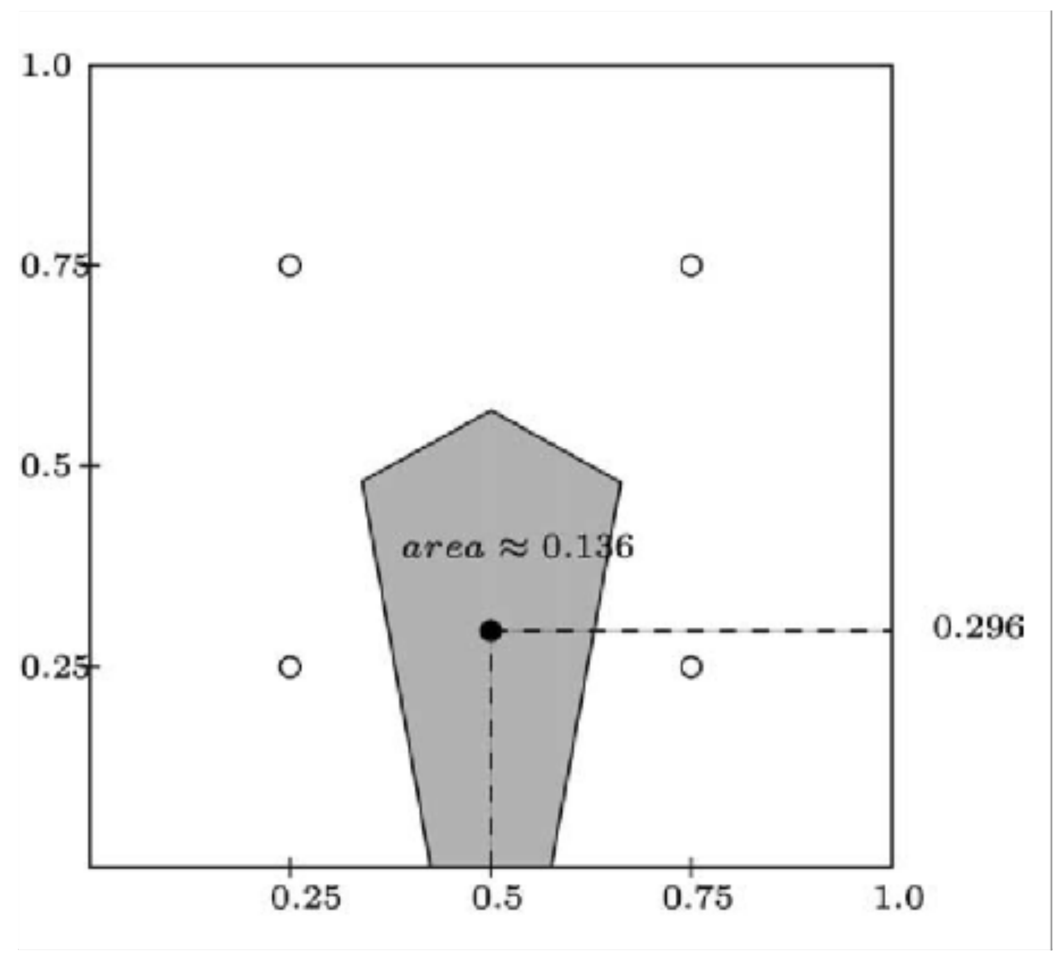


The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

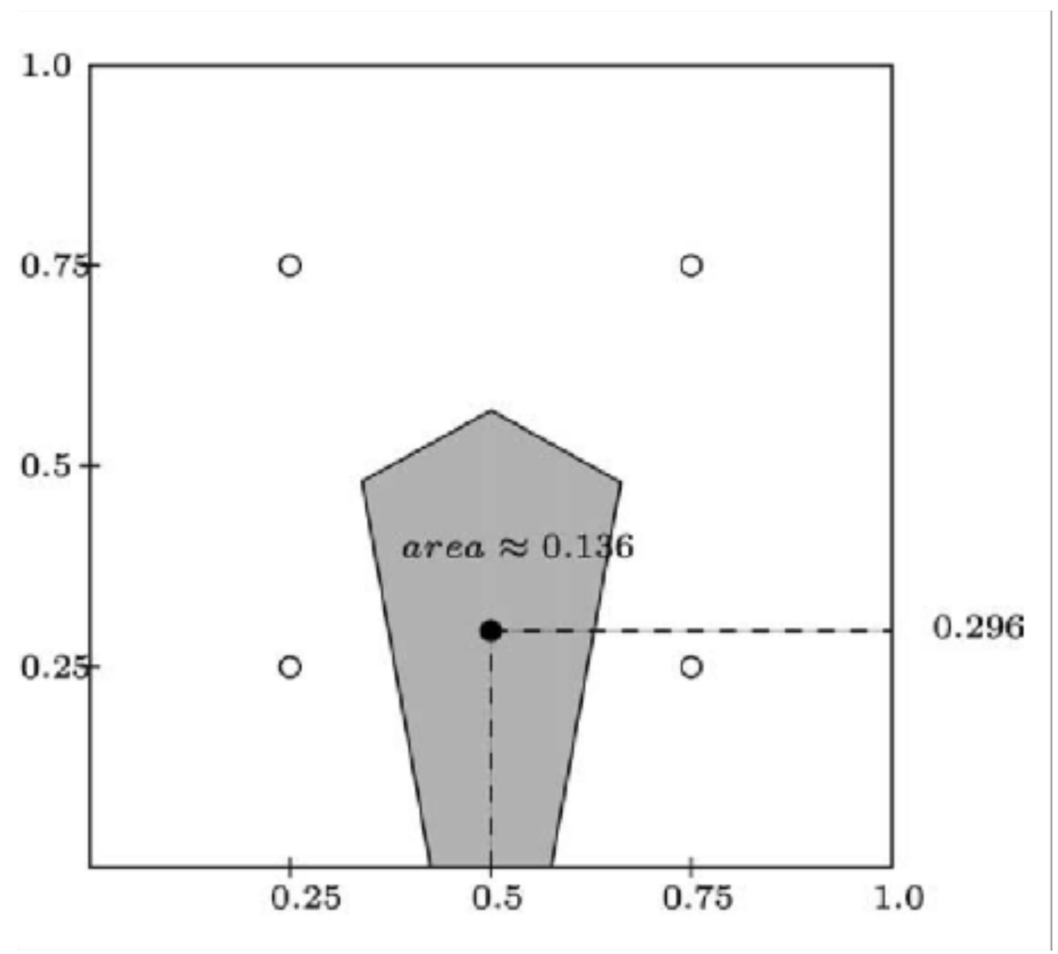
Lemma 5. *Suppose that the board is rectangular and that* $n = 4$. *If Wilma places her points on a regular* 2×2 *grid, Barney can gain* 50.78% *of the board.*

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Lemma 5. *Suppose that the board is rectangular and that* $n = 4$. *If Wilma places her points on a regular* 2×2 *grid, Barney can gain* 50.78% *of the board.*



Lemma 5. *Suppose that the board is rectangular and that $n = 4$. If Wilma places her points on a regular 2×2 grid, Barney can gain 50.78% of the board.*



Corollary 2. *If $n \geqslant 3$, then Wilma can only win by placing her points in a $1 \times n$ grid.*

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Lemma 6. *Let $n \geqslant 3$. Barney can win if $\rho > \sqrt{2}/n$; otherwise, he loses.*

Lemma 6. *Let $n \geqslant 3$. Barney can win if $\rho > \sqrt{2}/n$; otherwise, he loses.*

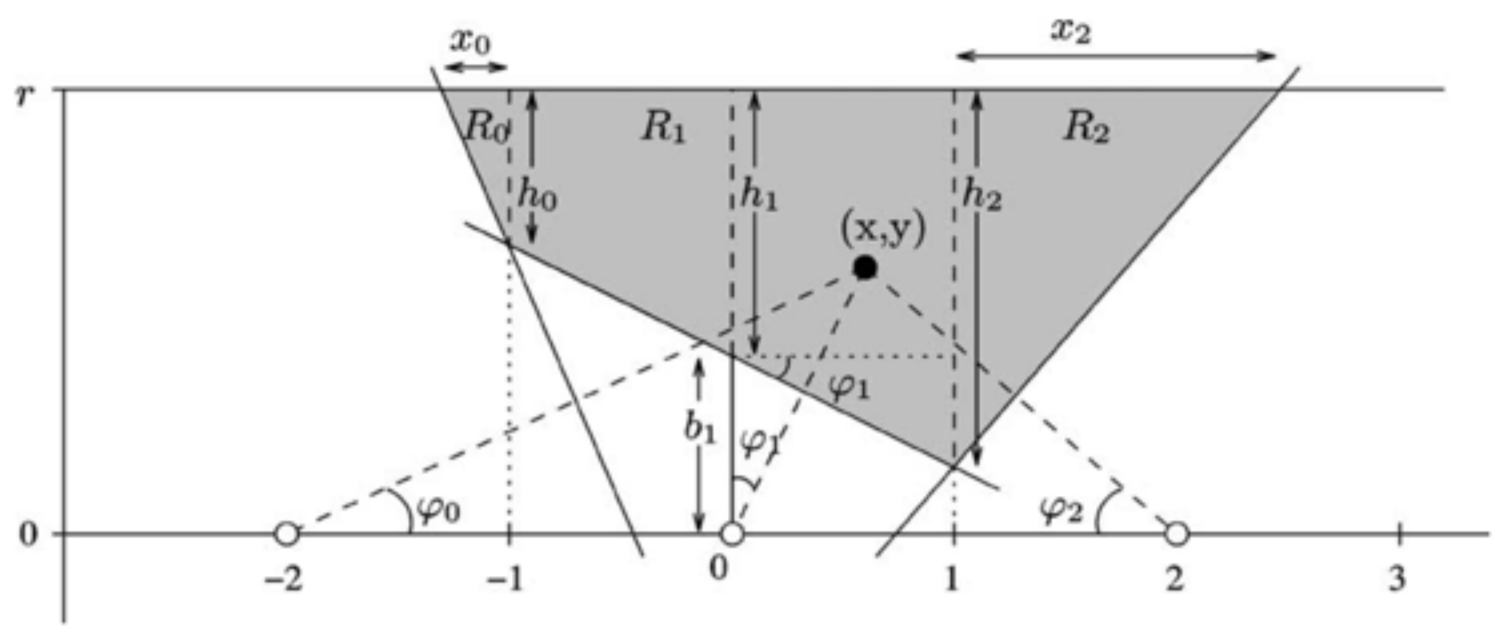


Fig. 3. Wilma has placed at least three points on a line.

Lemma 6. *Let $n \geqslant 3$. Barney can win if $\rho > \sqrt{2}/n$; otherwise, he loses.*

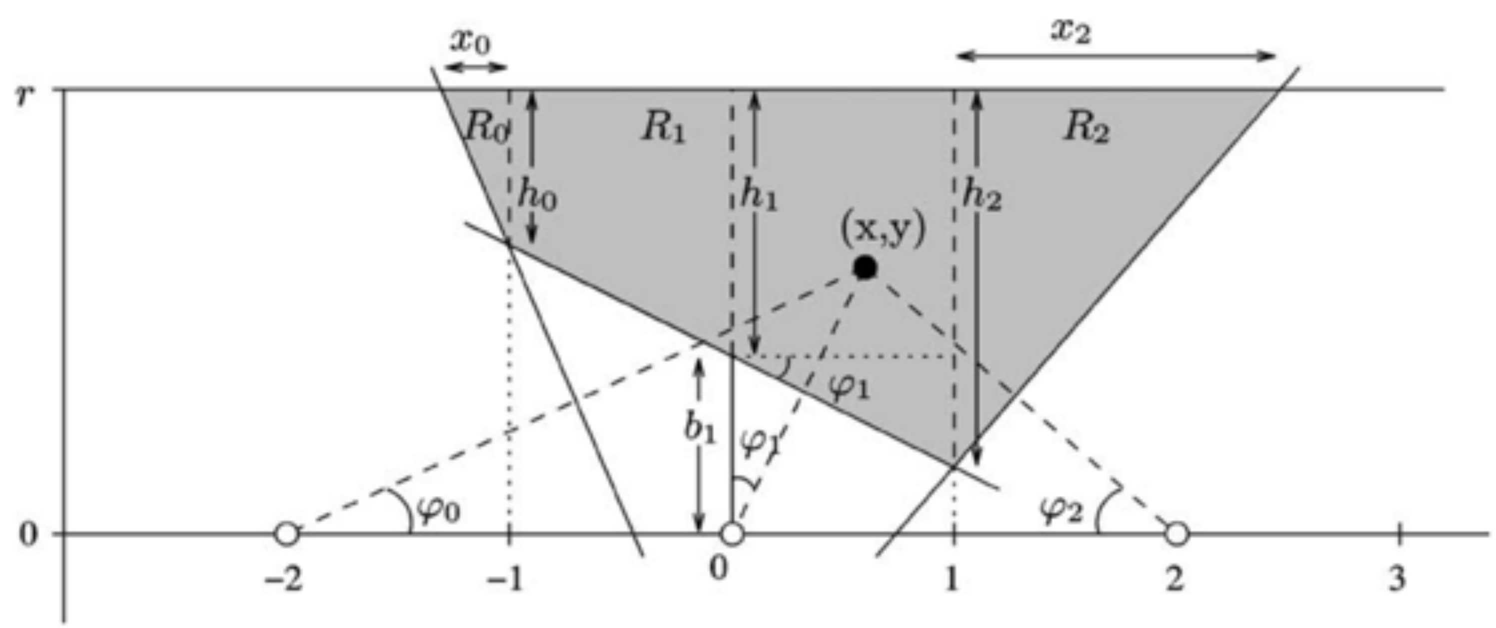


Fig. 3. Wilma has placed at least three points on a line.

Theorem 7. *If $n \geqslant 3$ and $\rho > \sqrt{2}/n$, or $n = 2$ and $\rho > \sqrt{3}/2$, then Barney wins. In all other cases, Wilma wins.*

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

General polygons instead of rectangles?!

General polygons instead of rectangles?!

Theorem 8. *For a polygon with holes, it is NP-hard to maximize the area Barney can claim, even if all of Wilma's points have been placed.*

General polygons instead of rectangles?!

Theorem 8. *For a polygon with holes, it is NP-hard to maximize the area Barney can claim, even if all of Wilma's points have been placed.*

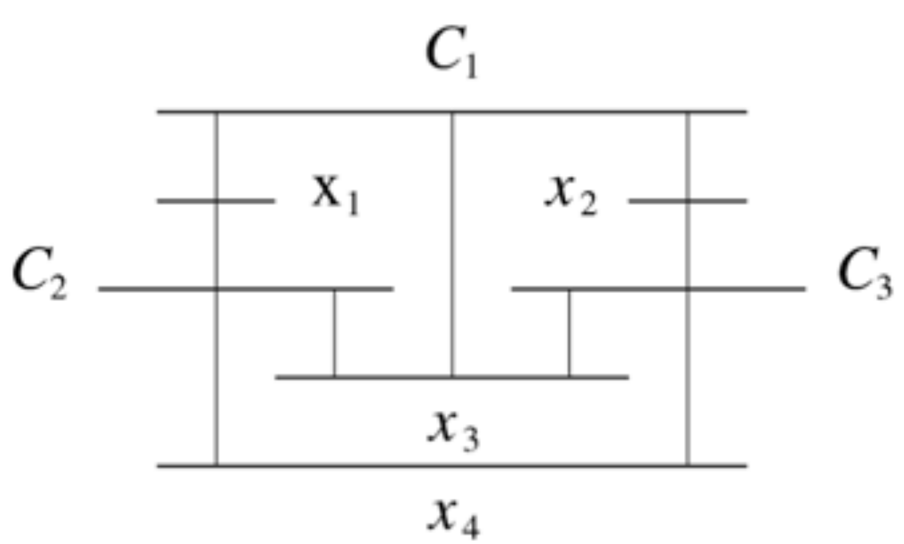
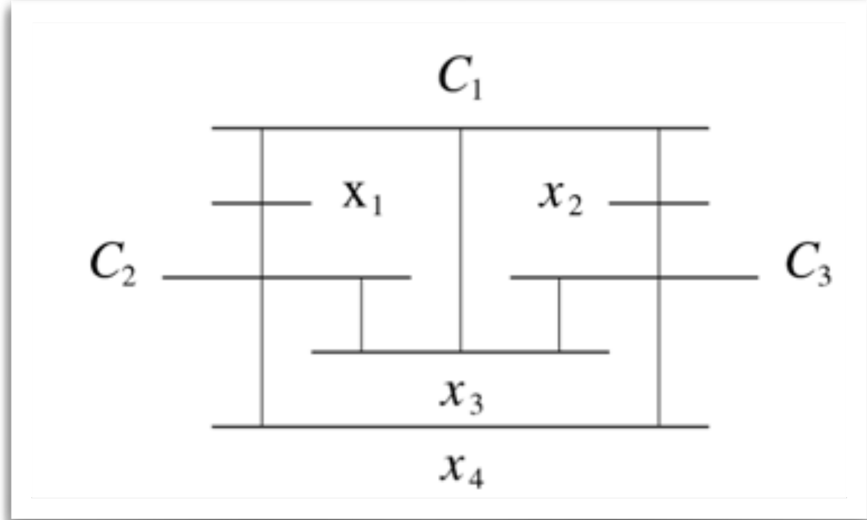
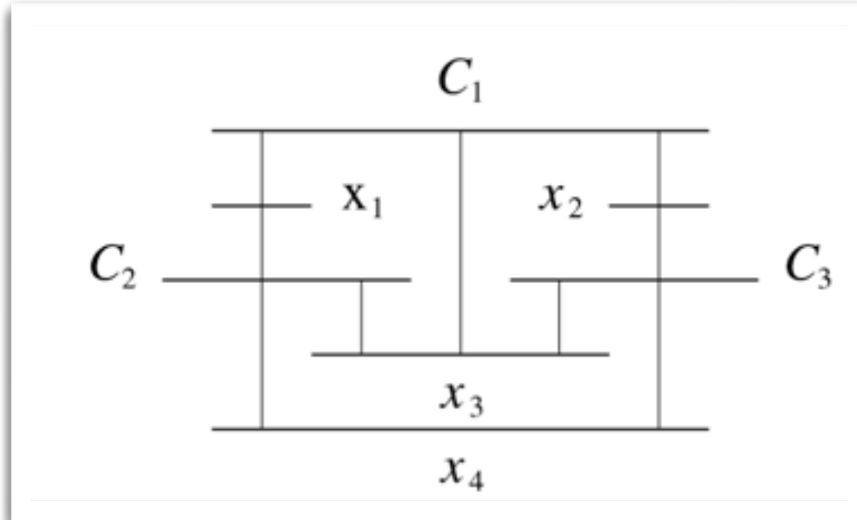


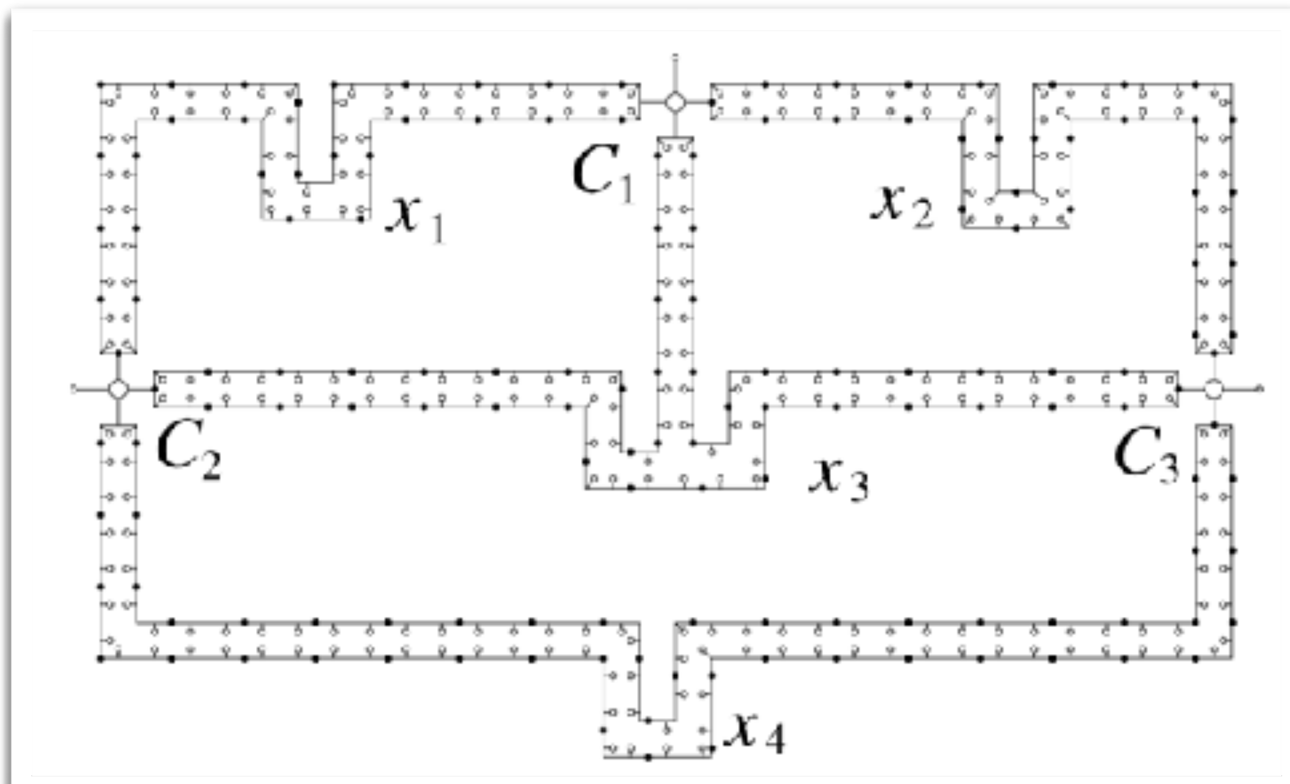
Fig. 4. A geometric representation of the variable-clause incidence graph G_I for the Planar 3SAT instance $I = (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4) \wedge (\bar{x}_2 \vee \bar{x}_3 \vee x_4)$.

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]



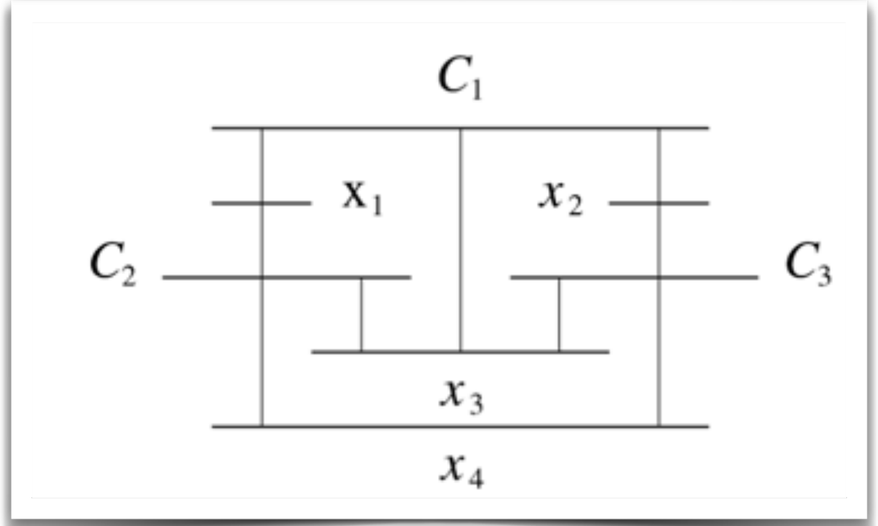
The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

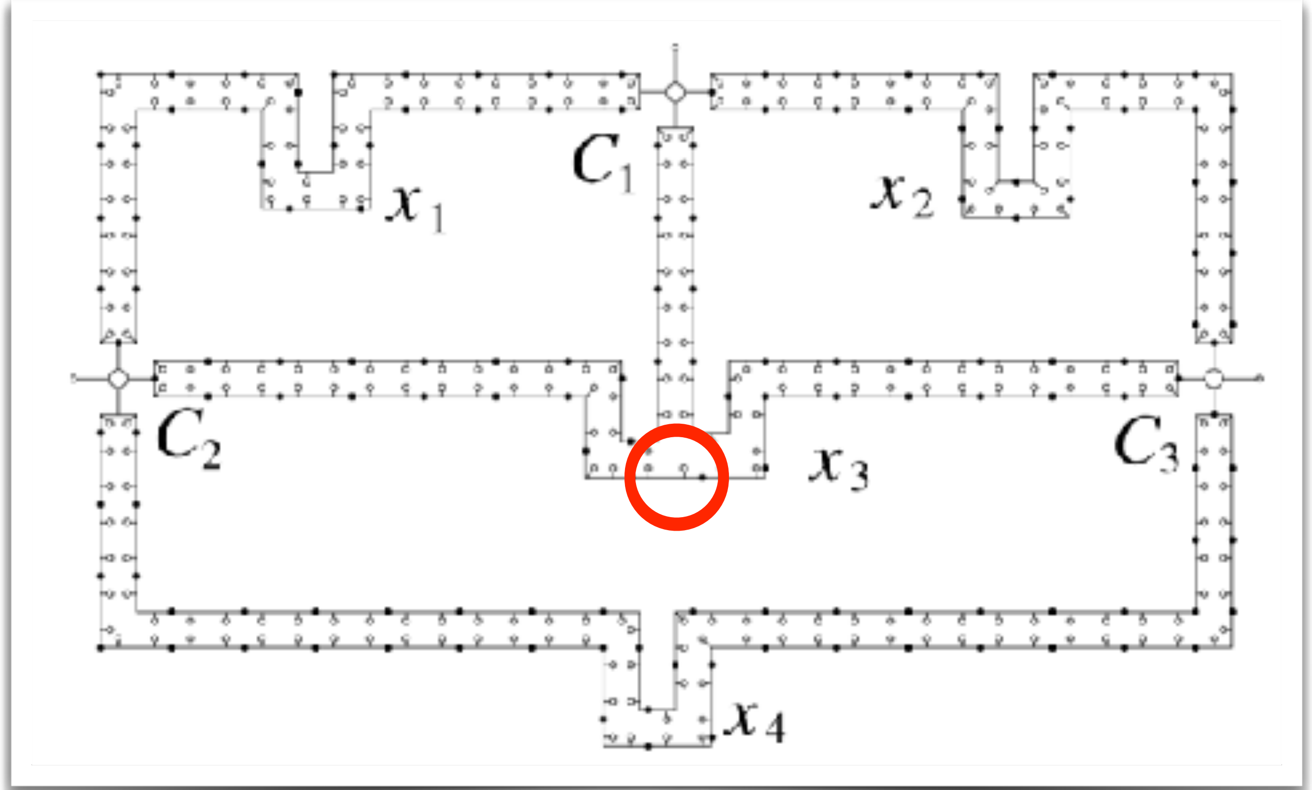




Technische Universität Braunschweig

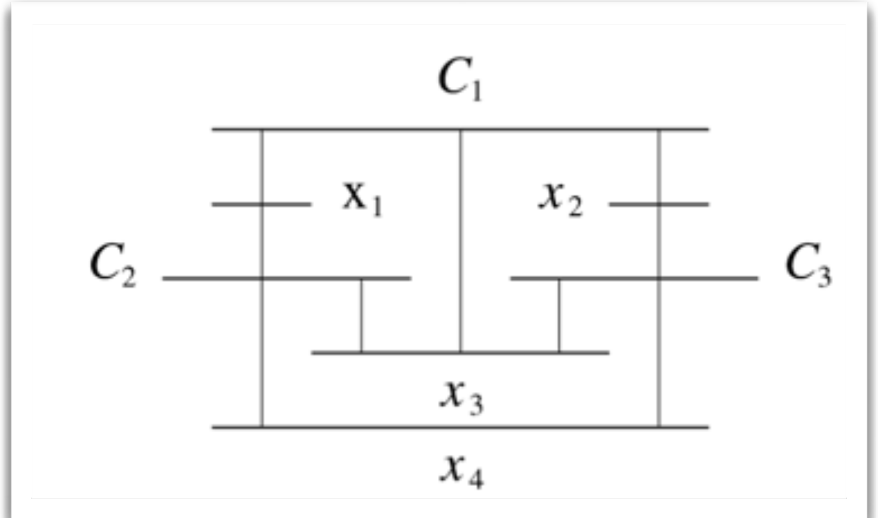
The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

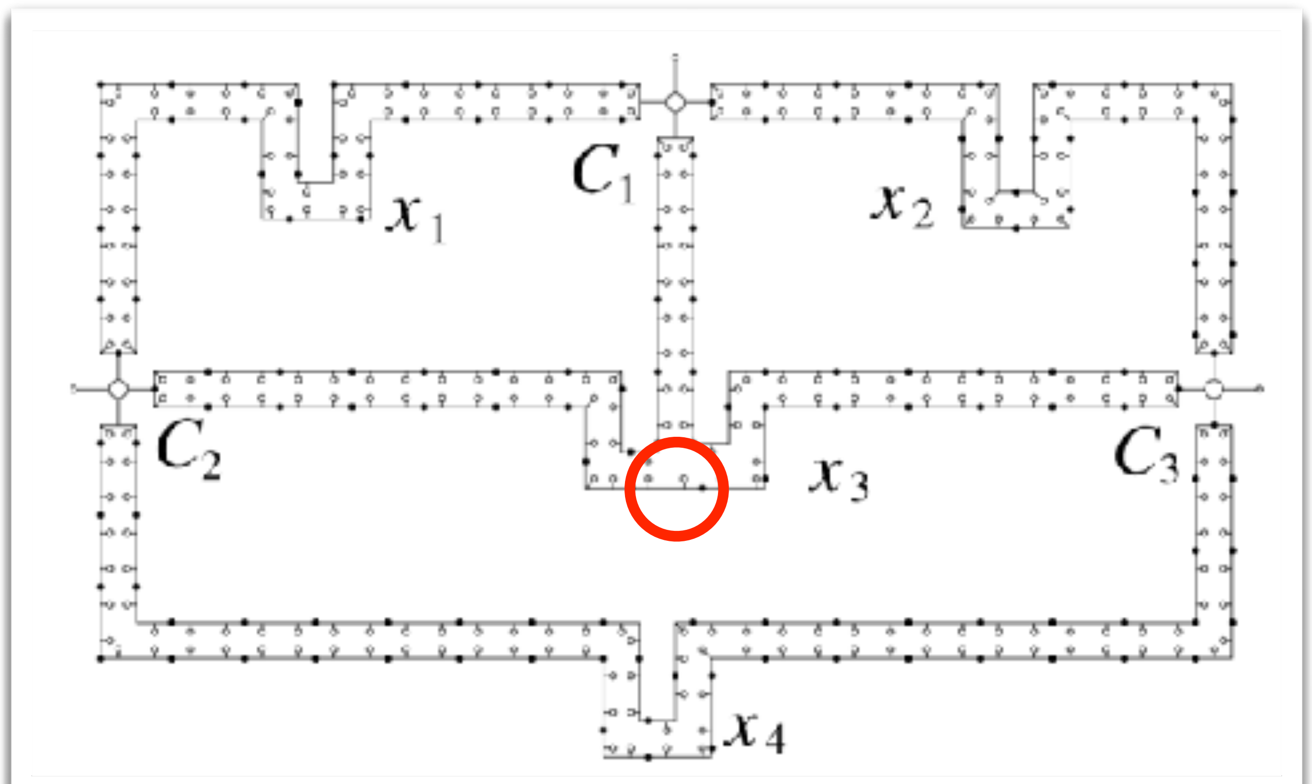


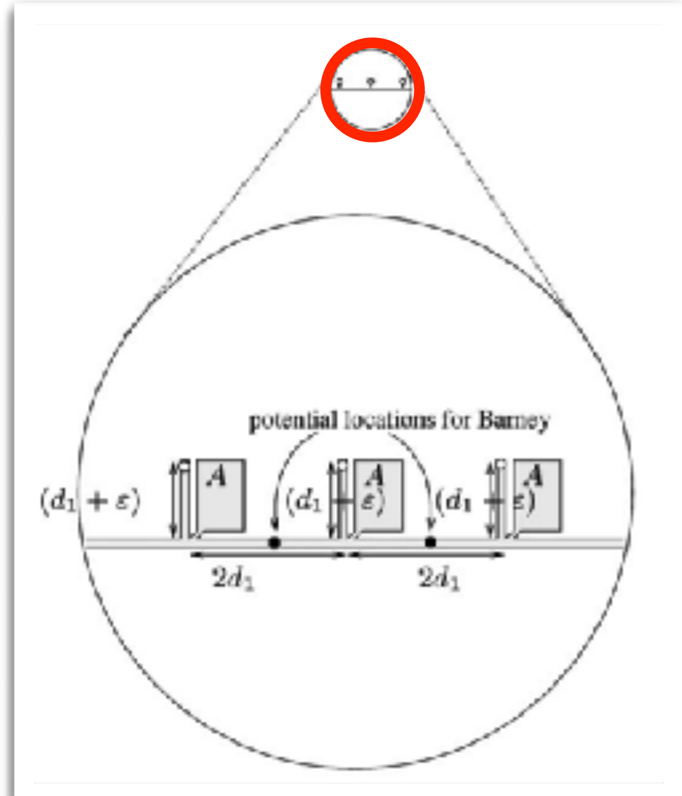


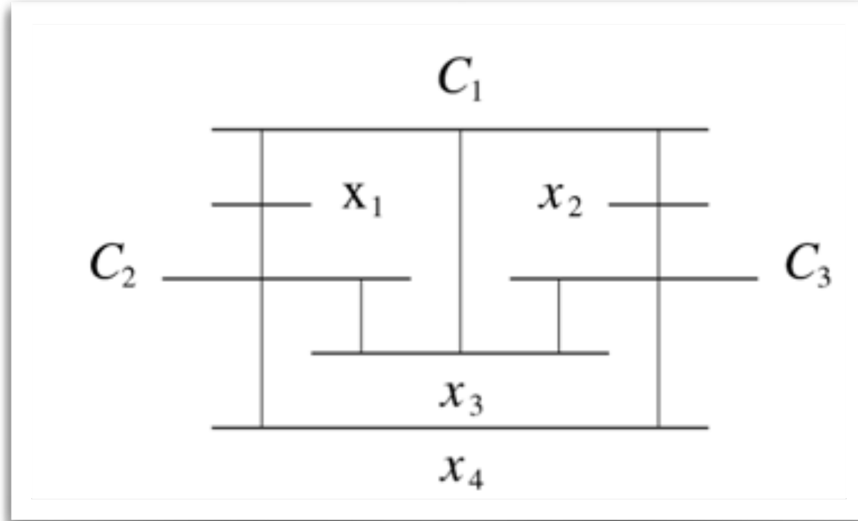
Technische Universität Braunschweig

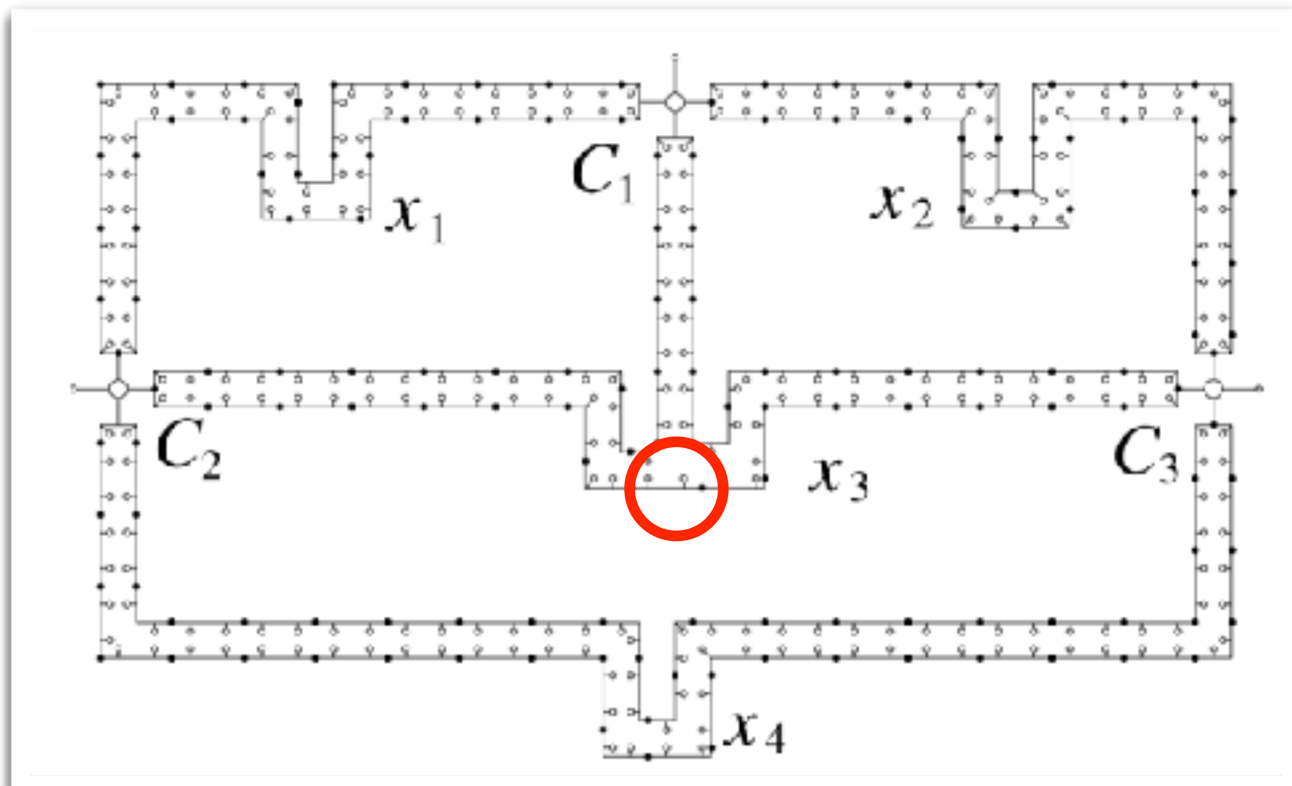
The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

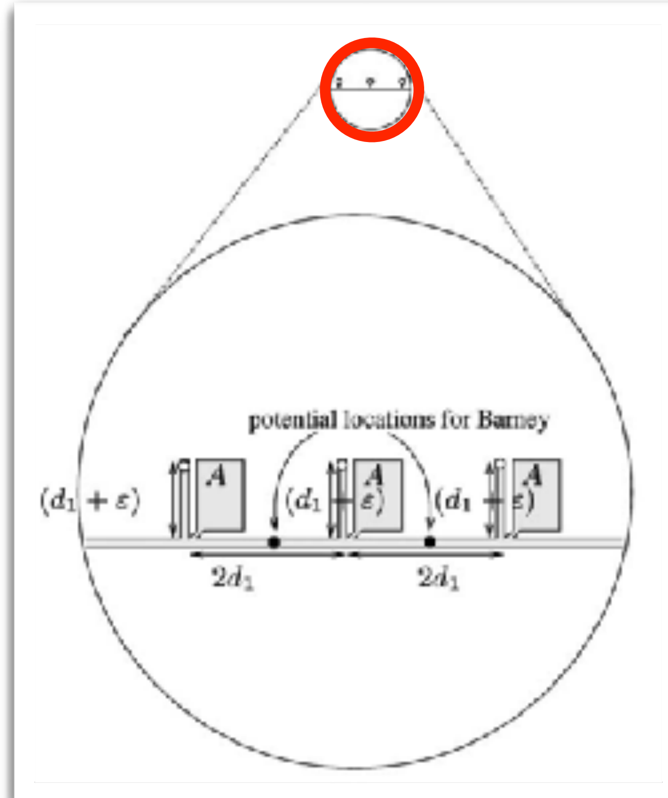






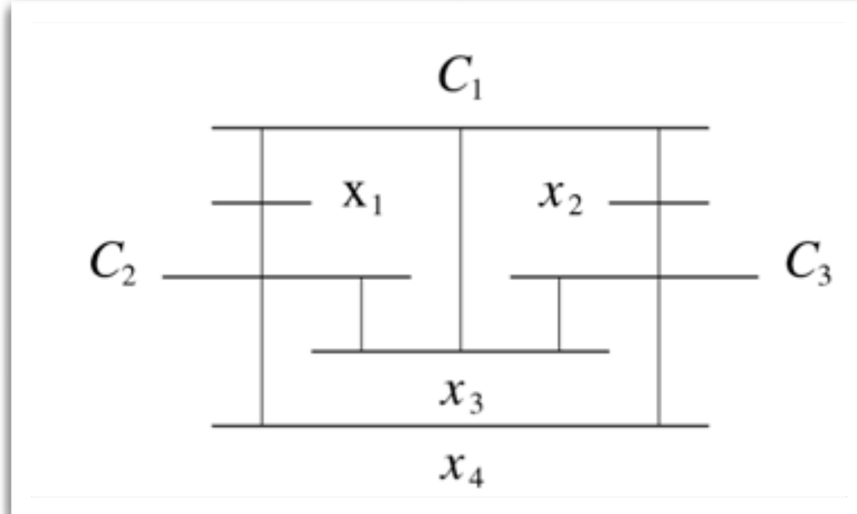


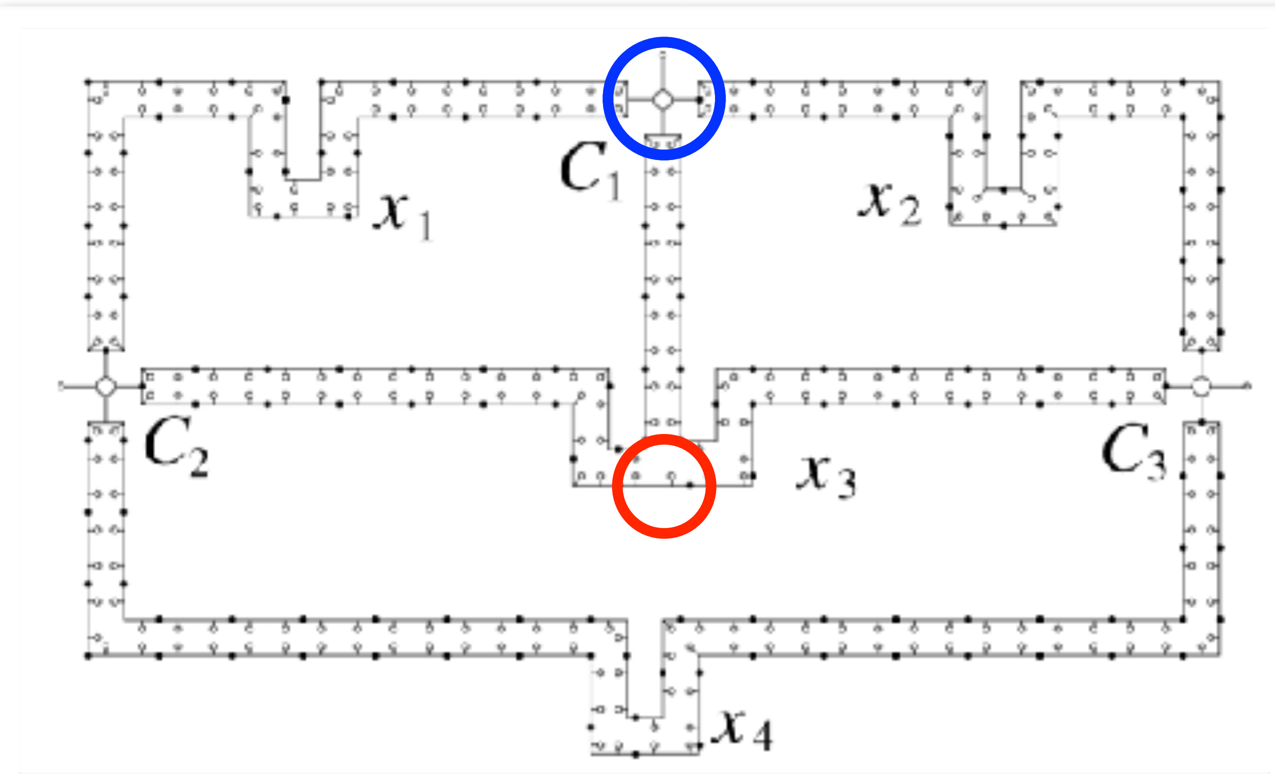


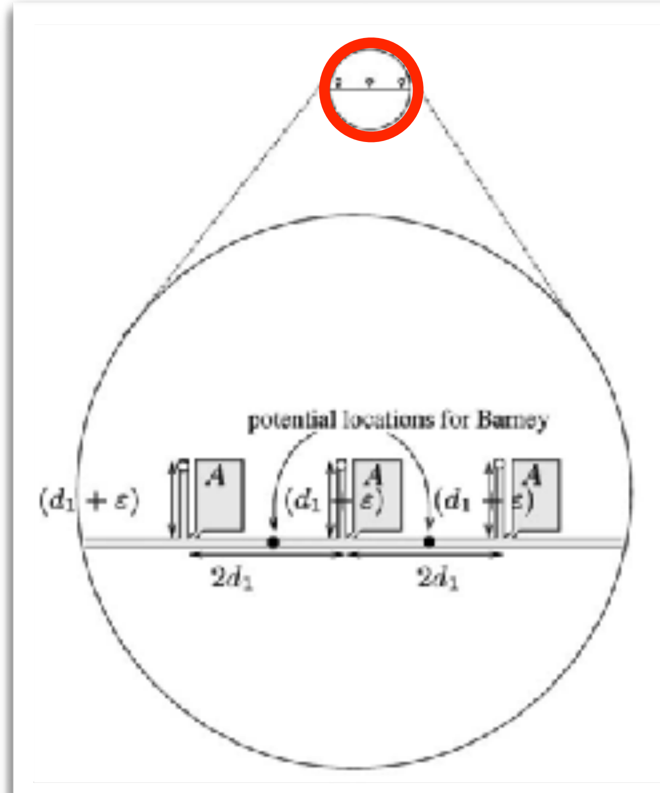


- For each variable, choose `true` or `false` by picking all even or all odd black positions.

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

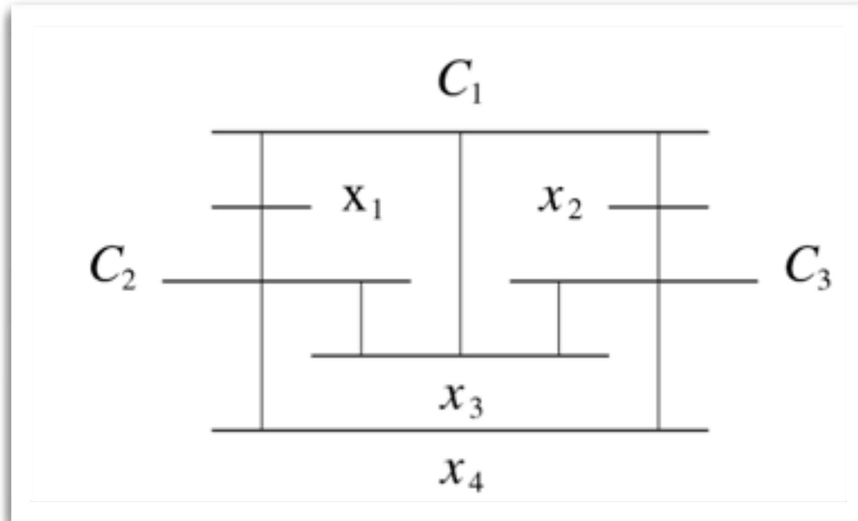


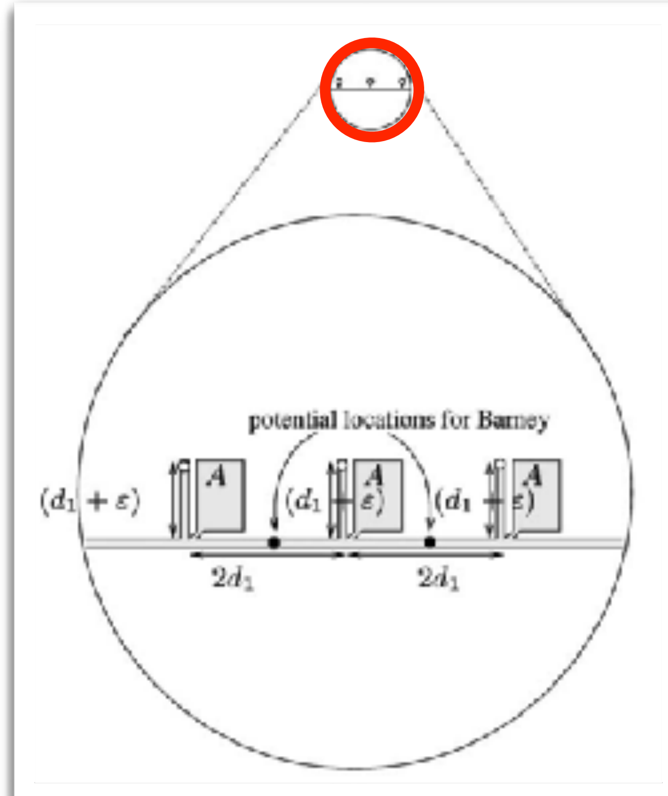


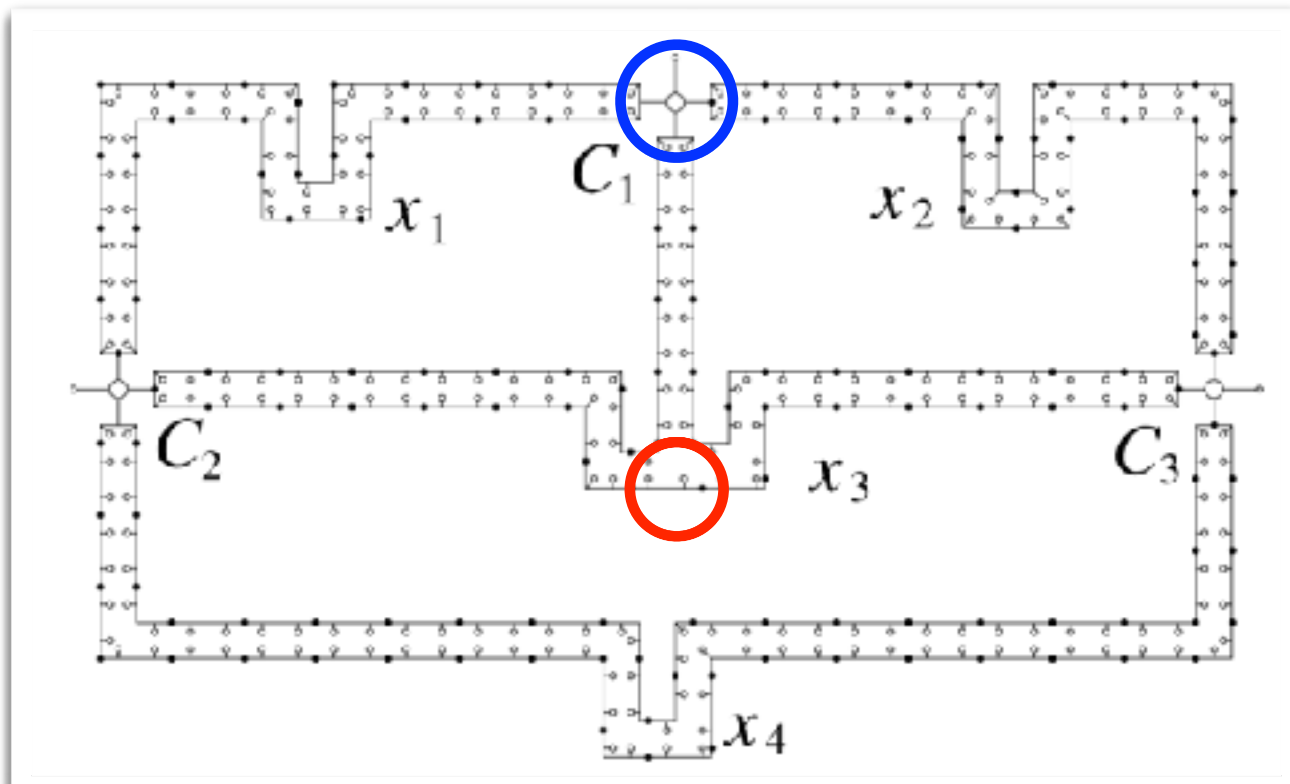


- For each variable, choose `true` or `false` by picking all even or all odd black positions.

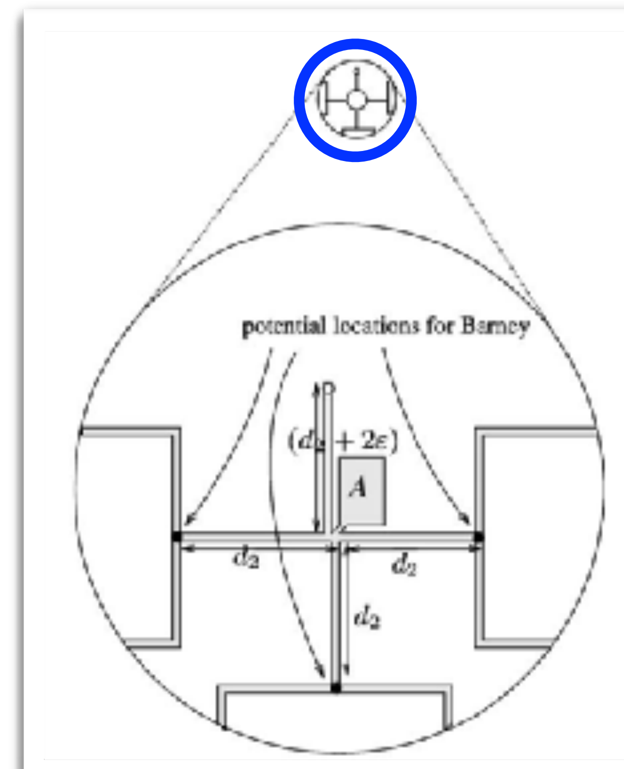
The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]





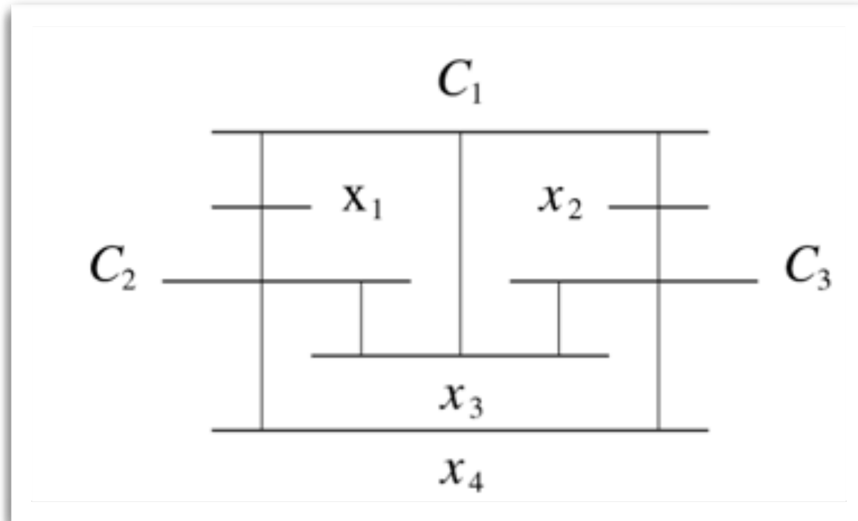


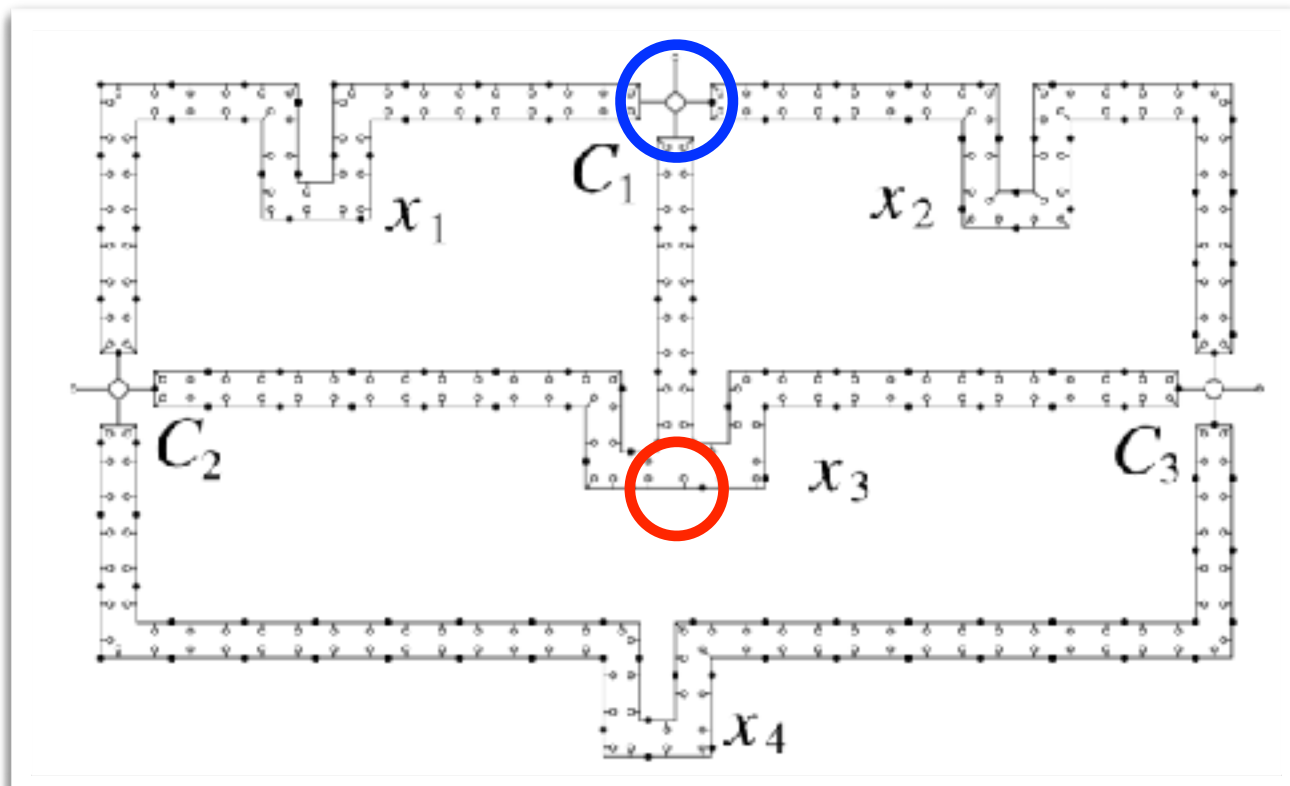
- For each variable, choose `true` or `false` by picking all even or all odd black positions.

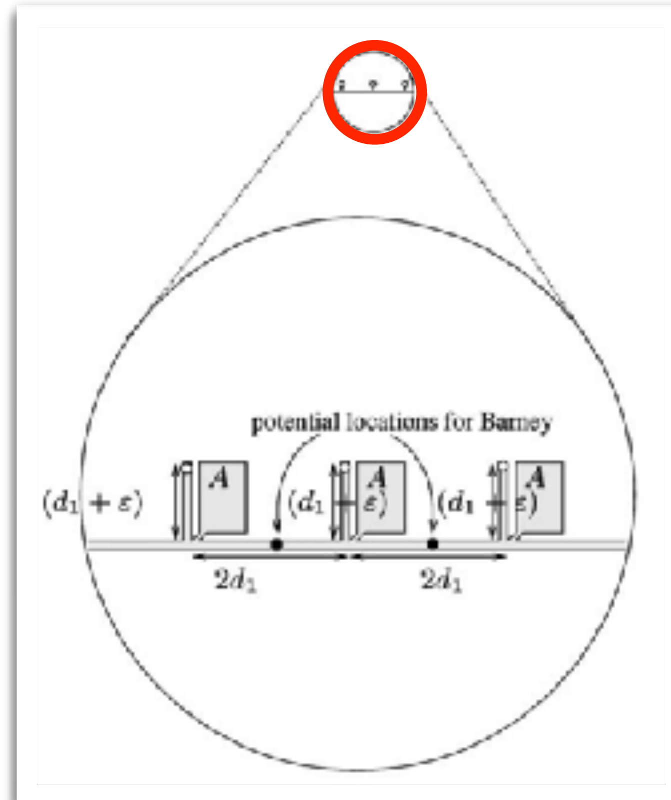


Technische Universität Braunschweig

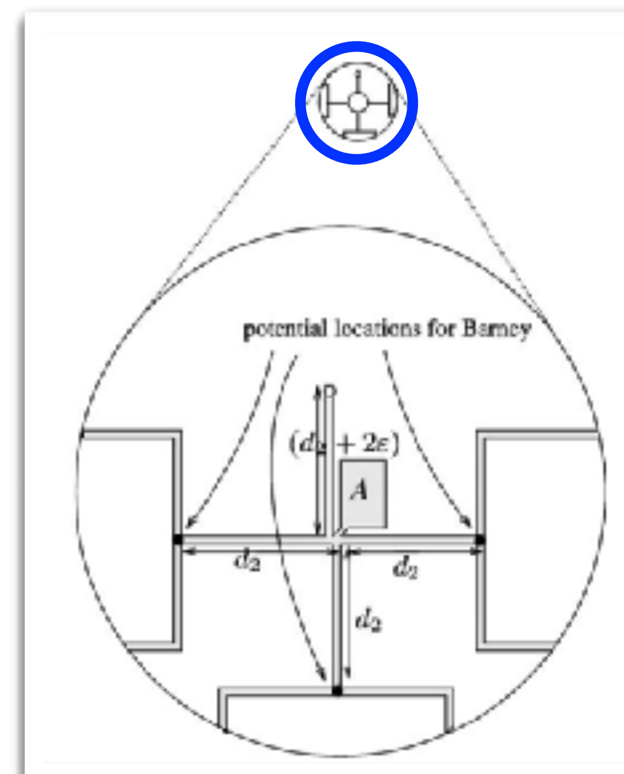
The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]







- For each variable, choose `true` or `false` by picking all even or all odd black positions.
- For each clause, a satisfying truth assignment picks additional area.



Locating a Single Site [Cheong, Efrat and Har-Peled 2007]

Technische
Universität
Braunschweig

Locating a Single Site [Cheong, Efrat and Har-Peled 2007]

Discrete Comput Geom 37:545–563 (2007)
DOI: 10.1007/s00454-007-1328-5

Discrete & Computational Geometry

Finding a Guard that Sees Most and a Shop that Sells Most*

Otfried Cheong,[1] Alon Efrat,[2] and Sariel Har-Peled[3]

[1]Division of Computer Science, KAIST,
Gwahangno 335, Yuseong-gu, Daejeon 305-701, South Korea
otfried@tclab.kaist.ac.kr

[2]Department of Computer Science, University of Arizona,
Tucson, AZ 85721, USA
alon@cs.arizona.edu

[3]Department of Computer Science, University of Illinois,
201 N. Goodwin Avenue, Urbana, IL 61801, USA
sariel@uiuc.edu

Abstract. We present a near-quadratic time algorithm that computes a point inside a simple polygon P in the plane having approximately the largest visibility polygon inside P, and a near-linear time algorithm for finding the point that will have approximately the largest Voronoi region when added to an n-point set in the plane. We apply the same technique to find the translation that approximately maximizes the area of intersection of two polygonal regions in near-quadratic time, and the rigid motion doing so in near-cubic time.

1. Introduction

We consider two problems where our goal is to find a point x such that the area of the region $V(x)$ "controlled" by x is as large as possible. In the first problem we are given a simple polygon P, and $V(x)$ is the *visibility polygon* of x, that is, the region of points y inside P such that the segment xy does not intersect the boundary of P. In the second problem we are given a set of points T, and $V(x)$ is the *Voronoi cell* of x in the Voronoi diagram of the set $T \cup \{x\}$, that is, the set of points that are closer to x than to any point in T.

* Work on this paper by Otfried Cheong was supported by the Brain Korea 21 Project, The School of Information Technology, KAIST, and that by Sariel Har-Peled was partially supported by NSF CAREER award CCR-0132901.

Discrete Comput Geom 37:545–563 (2007)
DOI: 10.1007/s00454-007-1328-5

Discrete & Computational Geometry
© 2007 Springer Science+Business Media, Inc.

Finding a Guard that Sees Most and a Shop that Sells Most*

Otfried Cheong,[1] Alon Efrat,[2] and Sariel Har-Peled[3]

[1]Division of Computer Science, KAIST,
Gwahangno 335, Yuseong-gu, Daejeon 305-701, South Korea
otfried@tclab.kaist.ac.kr

[2]Department of Computer Science, University of Arizona,
Tucson, AZ 85721, USA
alon@cs.arizona.edu

[3]Department of Computer Science, University of Illinois,
201 N. Goodwin Avenue, Urbana, IL 61801, USA
sariel@uiuc.edu

Abstract. We present a near-quadratic time algorithm that computes a point inside a simple polygon P in the plane having approximately the largest visibility polygon inside P, and a near-linear time algorithm for finding the point that will have approximately the largest Voronoi region when added to an n-point set in the plane. We apply the same technique to find the translation that approximately maximizes the area of intersection of two polygonal regions in near-quadratic time, and the rigid motion doing so in near-cubic time.

1. Introduction

We consider two problems where our goal is to find a point x such that the area of the region $V(x)$ "controlled" by x is as large as possible. In the first problem we are given a simple polygon P, and $V(x)$ is the *visibility polygon of* x, that is, the region of points y inside P such that the segment xy does not intersect the boundary of P. In the second problem we are given a set of points T, and $V(x)$ is the *Voronoi cell* of x in the Voronoi diagram of the set $T \cup \{x\}$, that is, the set of points that are closer to x than to any point in T.

* Work on this paper by Otfried Cheong was supported by the Brain Korea 21 Project, The School of Information Technology, KAIST, and that by Sariel Har-Peled was partially supported by NSF CAREER award CCR-0132901.

Abstract. We present a near-quadratic time algorithm that computes a point inside a simple polygon P in the plane having approximately the largest visibility polygon inside P, and a near-linear time algorithm for finding the point that will have approximately the largest Voronoi region when added to an n-point set in the plane. We apply the same technique to find the translation that approximately maximizes the area of intersection of two polygonal regions in near-quadratic time, and the rigid motion doing so in near-cubic time.

Locating a Single Site [Cheong, Efrat and Har-Peled 2007]

Discrete Comput Geom 37:545–563 (2007)
DOI: 10.1007/s00454-007-1328-5

Discrete & Computational Geometry

Finding a Guard that Sees Most and a Shop that Sells Most*

Otfried Cheong,[1] Alon Efrat,[2] and Sariel Har-Peled[3]

[1] Division of Computer Science, KAIST,
Gwahangno 335, Yuseong-gu, Daejeon 305-701, South Korea
otfried@tclab.kaist.ac.kr

[2] Department of Computer Science, University of Arizona,
Tucson, AZ 85721, USA
alon@cs.arizona.edu

[3] Department of Computer Science, University of Illinois,
201 N. Goodwin Avenue, Urbana, IL 61801, USA
sariel@uiuc.edu

Abstract. We present a near-quadratic time algorithm that computes a point inside a simple polygon P in the plane having approximately the largest visibility polygon inside P, and a near-linear time algorithm for finding the point that will have approximately the largest Voronoi region when added to an n-point set in the plane. We apply the same technique to find the translation that approximately maximizes the area of intersection of two polygonal regions in near-quadratic time, and the rigid motion doing so in near-cubic time.

1. Introduction

We consider two problems where our goal is to find a point x such that the area [cut off] region $V(x)$ "controlled" by x is as large as possible. In the first problem we are gi[cut off] simple polygon P, and $V(x)$ is the *visibility polygon* of x, that is, the region of poi[cut off] inside P such that the segment xy does not intersect the boundary of P. In the se[cut off] problem we are given a set of points T, and $V(x)$ is the *Voronoi cell* of x in the Voronoi diagram of the set $T \cup \{x\}$, that is, the set of points that are closer to x than to any point in T.

* Work on this paper by Otfried Cheong was supported by the Brain Korea 21 Project, The School of Information Technology, KAIST, and that by Sariel Har-Peled was partially supported by NSF CAREER award CCR-0132901.

Abstract. We present a near-quadratic time algorithm that computes a point inside a simple polygon P in the plane having approximately the largest visibility polygon inside P, and a near-linear time algorithm for finding the point that will have approximately the largest Voronoi region when added to an n-point set in the plane. We apply the same technique to find the translation that approximately maximizes the area of intersection of two polygonal regions in near-quadratic time, and the rigid motion doing so in near-cubic time.

Theorem 3.3. *Given a set T of n points in the plane and a parameter $\delta > 0$, one can deterministically compute, in time $O(n/\delta^4 + n \log n)$, a point x_{app} such that $\mu(x_{app}) \geq (1-\delta)\mu_{opt}$.*

Technische Universität Braunschweig

The One-Round Manhattan Game [Byrne, Fekete, Kalcsics and Kleist 2021]

Competitive Location Problems: Balanced Facility Location and the One-Round Manhattan Voronoi Game *

Thomas Byrne[1][0000-0003-3548-4086], Sándor P. Fekete[2][0000-0002-9062-4241], Jörg Kalcsics[1][0000-0002-5013-3448], and Linda Kleist[2][0000-0002-3790-916X]

[1] School of Mathematics, University of Edinburgh, United Kingdom
{tbyrne,joerg.kalcsics}@ed.ac.uk
[2] Department of Computer Science, TU Braunschweig, Germany
{s.fekete,l.kleist}@tu-bs.de

Abstract. We study competitive location problems in a continuous setting, in which facilities have to be placed in a rectangular domain R of normalized dimensions of 1 and $\rho \geq 1$, and distances are measured according to the Manhattan metric. We show that the family of *balanced* configurations (in which the Voronoi cells of individual facilities are equalized with respect to geometric properties) is richer in this metric than for Euclidean distances. Our main result considers the *One-Round Voronoi Game* with Manhattan distances, in which first player White and then player Black each place n points in R; each player scores the area for which one of its facilities is closer than the facilities of the opponent. We give a tight characterization: White has a winning strategy if and only if $\rho \geq n$; for all other cases, we present a winning strategy for Black.

Keywords: Facility location · competitive location · Manhattan distances · Voronoi game · geometric optimization.

1 Introduction

Problems of optimal location are arguably among the most important in a wide range of areas, such as economics, engineering, and biology, as well as in mathematics and computer science. In recent years, they have gained importance through clustering problems in artificial intelligence. In all scenarios, the task is to choose a set of positions from a given domain, such that some optimality criteria for the resulting distances to a set of demand points are satisfied; in a geometric setting, Euclidean or Manhattan distances are natural choices. Another challenge is that facility location problems often happen in a *competitive* setting, in which two or more players contend for the best locations. This change to competitive, multi-player versions can have a serious impact on the algorithmic difficulty of optimization problems: e.g., the classic Travelling Salesman Problem is NP-hard, while the competitive two-player variant is even PSPACE-complete [10].

* A full version can be found at arXiv: 2011.13275 [6].

Technische Universität Braunschweig

Competitive Location Problems: Balanced Facility Location and the One-Round Manhattan Voronoi Game *

Thomas Byrne[1[0000-0003-3548-4086]], Sándor P. Fekete[2[0000-0002-9062-4241]], Jörg Kalcsics[1[0000-0002-5013-3448]], and Linda Kleist[2[0000-0002-3786-916X]]

[1] School of Mathematics, University of Edinburgh, United Kingdom
{tbyrne,joerg.kalcsics}@ed.ac.uk
[2] Department of Computer Science, TU Braunschweig, Germany
{s.fekete,l.kleist}@tu-bs.de

Abstract. We study competitive location problems in a continuous setting, in which facilities have to be placed in a rectangular domain R of normalized dimensions of 1 and $\rho \geq 1$, and distances are measured according to the Manhattan metric. We show that the family of *balanced* configurations (in which the Voronoi cells of individual facilities are equalized with respect to geometric properties) is richer in this metric than for Euclidean distances. Our main result considers the *One-Round Voronoi Game* with Manhattan distances, in which first player White and then player Black each place n points in R; each player scores the area for which one of its facilities is closer than the facilities of the opponent. We give a tight characterization: White has a winning strategy if and only if $\rho \geq n$; for all other cases, we present a winning strategy for Black.

Keywords: Facility location · competitive location · Manhattan distances · Voronoi game · geometric optimization.

1 Introduction

Problems of optimal location are arguably among the most important in a wide range of areas, such as economics, engineering, and biology, as well as in mathematics and computer science. In recent years, they have gained importance through clustering problems in artificial intelligence. In all scenarios, the task is to choose a set of positions from a given domain, such that some optimality criteria for the resulting distances to a set of demand points are satisfied; in a geometric setting, Euclidean or Manhattan distances are natural choices. Another challenge is that facility location problems often happen in a *competitive* setting, in which two or more players contend for the best locations. This change to competitive, multi-player versions can have a serious impact on the algorithmic difficulty of optimization problems: e.g., the classic Travelling Salesman Problem is NP-hard, while the competitive two-player variant is even PSPACE-complete [10].

* A full version can be found at arXiv: 2011.13275 [6].

- **Manhattan instead of Euclidean distances**

The One-Round Manhattan Game [Byrne, Fekete, Kalcsics and Kleist 2021]

Competitive Location Problems: Balanced Facility Location and the One-Round Manhattan Voronoi Game *

Thomas Byrne[1][0000−0003−3548−4086], Sándor P. Fekete[2][0000−0002−9062−4241], Jörg Kalcsics[1][0000−0002−5013−3448], and Linda Kleist[2][0000−0002−3786−916X]

[1] School of Mathematics, University of Edinburgh, United Kingdom
{tbyrne,joerg.kalcsics}@ed.ac.uk
[2] Department of Computer Science, TU Braunschweig, Germany
{s.fekete,l.kleist}@tu-bs.de

Abstract. We study competitive location problems in a continuous setting, in which facilities have to be placed in a rectangular domain R of normalized dimensions of 1 and $\rho \geq 1$, and distances are measured according to the Manhattan metric. We show that the family of *balanced* configurations (in which the Voronoi cells of individual facilities are equalized with respect to geometric properties) is richer in this metric than for Euclidean distances. Our main result considers the *One-Round Voronoi Game* with Manhattan distances, in which first player White and then player Black each place n points in R; each player scores the area for which one of its facilities is closer than the facilities of the opponent. We give a tight characterization: White has a winning strategy if and only if $\rho \geq n$; for all other cases, we present a winning strategy for Black.

Keywords: Facility location · competitive location · Manhattan distances · Voronoi game · geometric optimization.

1 Introduction

Problems of optimal location are arguably among the most important in a wide range of areas, such as economics, engineering, and biology, as well as in mathematics and computer science. In recent years, they have gained importance through clustering problems in artificial intelligence. In all scenarios, the task is to choose a set of positions from a given domain, such that some optimality criteria for the resulting distances to a set of demand points are satisfied; in a geometric setting, Euclidean or Manhattan distances are natural choices. Another challenge is that facility location problems often happen in a *competitive* setting, in which two or more players contend for the best locations. This change to competitive, multi-player versions can have a serious impact on the algorithmic difficulty of optimization problems: e.g., the classic Travelling Salesman Problem is NP-hard, while the competitive two-player variant is even PSPACE-complete [10].

* A full version can be found at arXiv: 2011.13275 [6].

- **Manhattan instead of Euclidean distances**
- **Neutral zones cause additional twists.**

Technische Universität Braunschweig

Competitive Location Problems: Balanced Facility Location and the One-Round Manhattan Voronoi Game *

Thomas Byrne[1][0000−0003−3548−4086], Sándor P. Fekete[2][0000−0002−9062−4241], Jörg Kalcsics[1][0000−0002−5013−0448], and Linda Kleist[2][0000−0002−3786−916X]

[1] School of Mathematics, University of Edinburgh, United Kingdom
{tbyrne,joerg.kalcsics}@ed.ac.uk
[2] Department of Computer Science, TU Braunschweig, Germany
{s.fekete,l.kleist}@tu-bs.de

Abstract. We study competitive location problems in a continuous setting, in which facilities have to be placed in a rectangular domain R of normalized dimensions of 1 and $\rho \geq 1$, and distances are measured according to the Manhattan metric. We show that the family of *balanced* configurations (in which the Voronoi cells of individual facilities are equalized with respect to geometric properties) is richer in this metric than for Euclidean distances. Our main result considers the *One-Round Voronoi Game* with Manhattan distances, in which first player White and then player Black each place n points in R; each player scores the area for which one of its facilities is closer than the facilities of the opponent. We give a tight characterization: White has a winning strategy if and only if $\rho \geq n$; for all other cases, we present a winning strategy for Black.

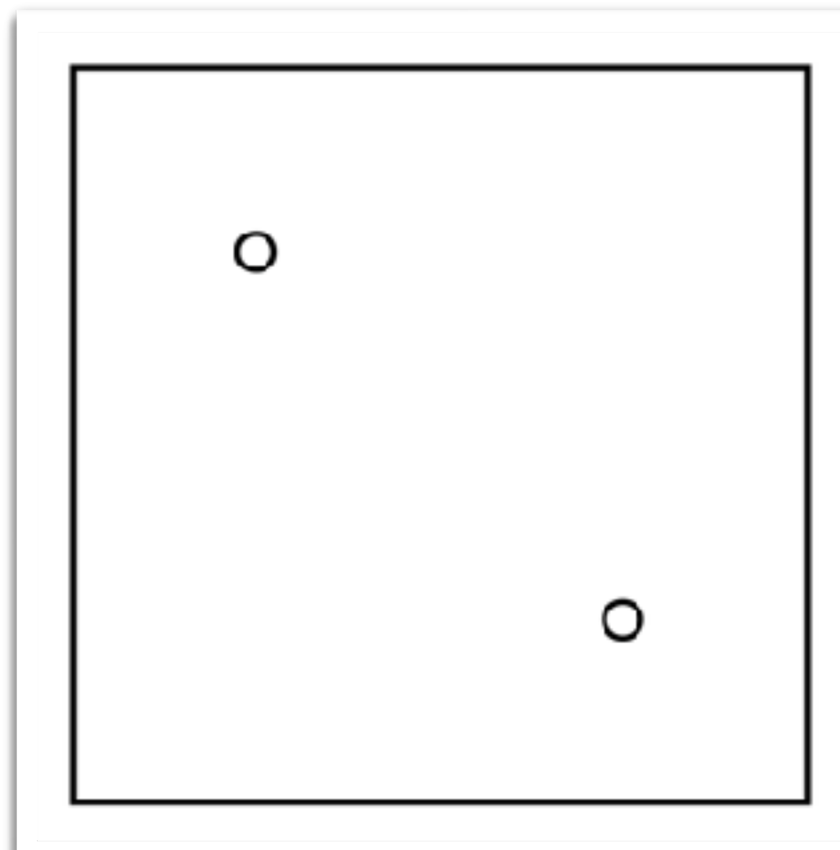
Keywords: Facility location · competitive location · Manhattan distances · Voronoi game · geometric optimization.

1 Introduction

Problems of optimal location are arguably among the most important in a wide range of areas, such as economics, engineering, and biology, as well as in mathematics and computer science. In recent years, they have gained importance through clustering problems in artificial intelligence. In all scenarios, the task is to choose a set of positions from a given domain, such that some optimality criteria for the resulting distances to a set of demand points are satisfied; in a geometric setting, Euclidean or Manhattan distances are natural choices. Another challenge is that facility location problems often happen in a *competitive* setting, in which two or more players contend for the best locations. This change to competitive, multi-player versions can have a serious impact on the algorithmic difficulty of optimization problems: e.g., the classic Travelling Salesman Problem is NP-hard, while the competitive two-player variant is even PSPACE-complete [10].

* A full version can be found at arXiv: 2011.13275 [6].

- **Manhattan instead of Euclidean distances**
- **Neutral zones cause additional twists.**



The One-Round Manhattan Game [Byrne, Fekete, Kalcsics and Kleist 2021]

Competitive Location Problems:
Balanced Facility Location and the
One-Round Manhattan Voronoi Game *

Thomas Byrne[1], Sándor P. Fekete[2], Jörg Kalcsics[1], and Linda Kleist[2]

[1] School of Mathematics, University of Edinburgh, United Kingdom
{tbyrne,joerg.kalcsics}@ed.ac.uk
[2] Department of Computer Science, TU Braunschweig, Germany
{s.fekete,l.kleist}@tu-bs.de

Abstract. We study competitive location problems in a continuous setting, in which facilities have to be placed in a rectangular domain R of normalized dimensions of 1 and $\rho \geq 1$, and distances are measured according to the Manhattan metric. We show that the family of *balanced* configurations (in which the Voronoi cells of individual facilities are equalized with respect to geometric properties) is richer in this metric than for Euclidean distances. Our main result considers the *One-Round Voronoi Game* with Manhattan distances, in which first player White and then player Black each place n points in R; each player scores the area for which one of its facilities is closer than the facilities of the opponent. We give a tight characterization: White has a winning strategy if and only if $\rho \geq n$; for all other cases, we present a winning strategy for Black.

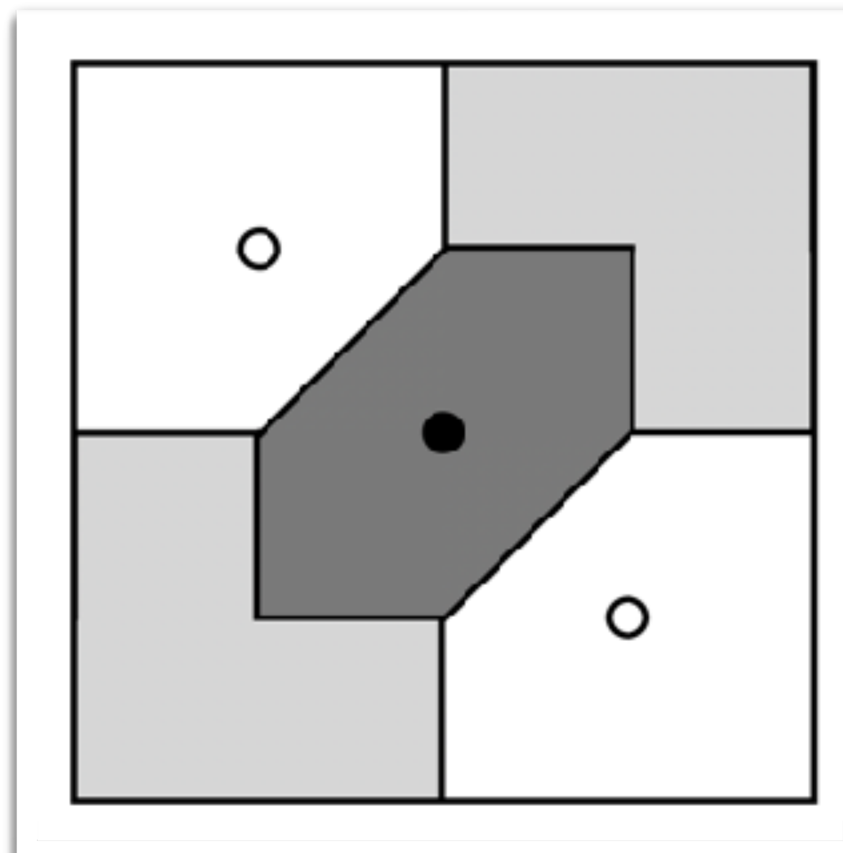
Keywords: Facility location · competitive location · Manhattan distances · Voronoi game · geometric optimization.

1 Introduction

Problems of optimal location are arguably among the most important in a wide range of areas, such as economics, engineering, and biology, as well as in mathematics and computer science. In recent years, they have gained importance through clustering problems in artificial intelligence. In all scenarios, the task is to choose a set of positions from a given domain, such that some optimality criteria for the resulting distances to a set of demand points are satisfied; in a geometric setting, Euclidean or Manhattan distances are natural choices. Another challenge is that facility location problems often happen in a *competitive* setting, in which two or more players contend for the best locations. This change to competitive, multi-player versions can have a serious impact on the algorithmic difficulty of optimization problems: e.g., the classic Travelling Salesman Problem is NP-hard, while the competitive two-player variant is even PSPACE-complete [10].

* A full version can be found at arXiv: 2011.13275 [6].

- **Manhattan instead of Euclidean distances**
- **Neutral zones cause additional twists.**



Technische Universität Braunschweig

The One-Round Manhattan Game [Byrne, Fekete, Kalcsics and Kleist 2021]

Competitive Location Problems: Balanced Facility Location and the One-Round Manhattan Voronoi Game *

Thomas Byrne[1[0000-0003-3548-4086]], Sándor P. Fekete[2[0000-0002-9062-4241]], Jörg Kalcsics[1[0000-0002-5013-3448]], and Linda Kleist[2[0000-0002-3786-916X]]

[1] School of Mathematics, University of Edinburgh, United Kingdom
{tbyrne,joerg.kalcsics}@ed.ac.uk
[2] Department of Computer Science, TU Braunschweig, Germany
{s.fekete,l.kleist}@tu-bs.de

Abstract. We study competitive location problems in a continuous setting, in which facilities have to be placed in a rectangular domain R of normalized dimensions of 1 and $\rho \geq 1$, and distances are measured according to the Manhattan metric. We show that the family of *balanced* configurations (in which the Voronoi cells of individual facilities are equalized with respect to geometric properties) is richer in this metric than for Euclidean distances. Our main result considers the *One-Round Voronoi Game* with Manhattan distances, in which first player White and then player Black each place n points in R; each player scores the area for which one of its facilities is closer than the facilities of the opponent. We give a tight characterization: White has a winning strategy if and only if $\rho \geq n$; for all other cases, we present a winning strategy for Black.

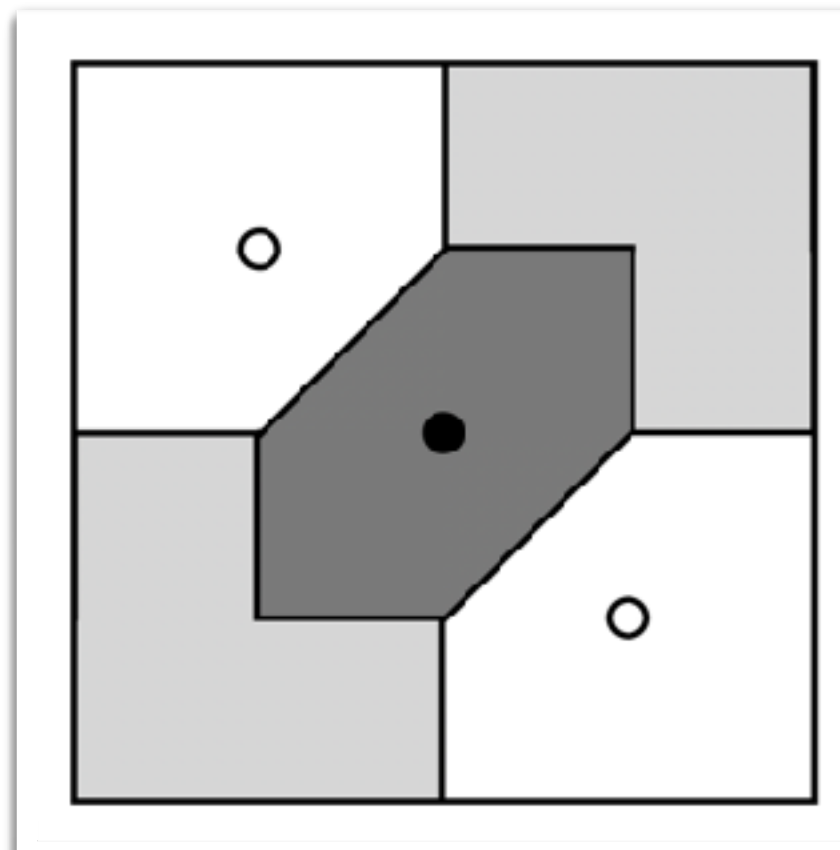
Keywords: Facility location · competitive location · Manhattan distances · Voronoi game · geometric optimization.

1 Introduction

Problems of optimal location are arguably among the most important in a wide range of areas, such as economics, engineering, and biology, as well as in mathematics and computer science. In recent years, they have gained importance through clustering problems in artificial intelligence. In all scenarios, the task is to choose a set of positions from a given domain, such that some optimality criteria for the resulting distances to a set of demand points are satisfied; in a geometric setting, Euclidean or Manhattan distances are natural choices. Another challenge is that facility location problems often happen in a *competitive* setting, in which two or more players contend for the best locations. This change to competitive, multi-player versions can have a serious impact on the algorithmic difficulty of optimization problems: e.g., the classic Travelling Salesman Problem is NP-hard, while the competitive two-player variant is even PSPACE-complete [10].

* A full version can be found at arXiv: 2011.13275 [6].

- **Manhattan instead of Euclidean distances**
- **Neutral zones cause additional twists.**
- **Other „balanced“ configurations.**



The One-Round Manhattan Game [Byrne, Fekete, Kalcsics and Kleist 2021]

Competitive Location Problems: Balanced Facility Location and the One-Round Manhattan Voronoi Game *

Thomas Byrne[1], Sándor P. Fekete[2], Jörg Kalcsics[1], and Linda Kleist[2]

[1] School of Mathematics, University of Edinburgh, United Kingdom
{tbyrne,joerg.kalcsics}@ed.ac.uk
[2] Department of Computer Science, TU Braunschweig, Germany
{s.fekete,l.kleist}@tu-bs.de

Abstract. We study competitive location problems in a continuous setting, in which facilities have to be placed in a rectangular domain R of normalized dimensions of 1 and $\rho \geq 1$, and distances are measured according to the Manhattan metric. We show that the family of *balanced* configurations (in which the Voronoi cells of individual facilities are equalized with respect to geometric properties) is richer in this metric than for Euclidean distances. Our main result considers the *One-Round Voronoi Game* with Manhattan distances, in which first player White and then player Black each place n points in R; each player scores the area for which one of its facilities is closer than the facilities of the opponent. We give a tight characterization: White has a winning strategy if and only if $\rho \geq n$; for all other cases, we present a winning strategy for Black.

Keywords: Facility location · competitive location · Manhattan distances · Voronoi game · geometric optimization.

1 Introduction

Problems of optimal location are arguably among the most importan[t] wide range of areas, such as economics, engineering, and biology, as well [as] mathematics and computer science. In recent years, they have gained impo[rtance] through clustering problems in artificial intelligence. In all scenarios, the tas[k is to] choose a set of positions from a given domain, such that some optimality c[riteria] for the resulting distances to a set of demand points are satisfied; in a geo[metric] setting, Euclidean or Manhattan distances are natural choices. Another cha[llenge] is that facility location problems often happen in a *competitive* setting, in [which] two or more players contend for the best locations. This change to compe[titive] multi-player versions can have a serious impact on the algorithmic difficu[lty of] optimization problems: e.g., the classic Travelling Salesman Problem is NP[-hard,] while the competitive two-player variant is even PSPACE-complete [10].

* A full version can be found at arXiv: 2011.13275 [6].

- **Manhattan instead of Euclidean distances**
- **Neutral zones cause additional twists.**
- **Other „balanced“ configurations.**

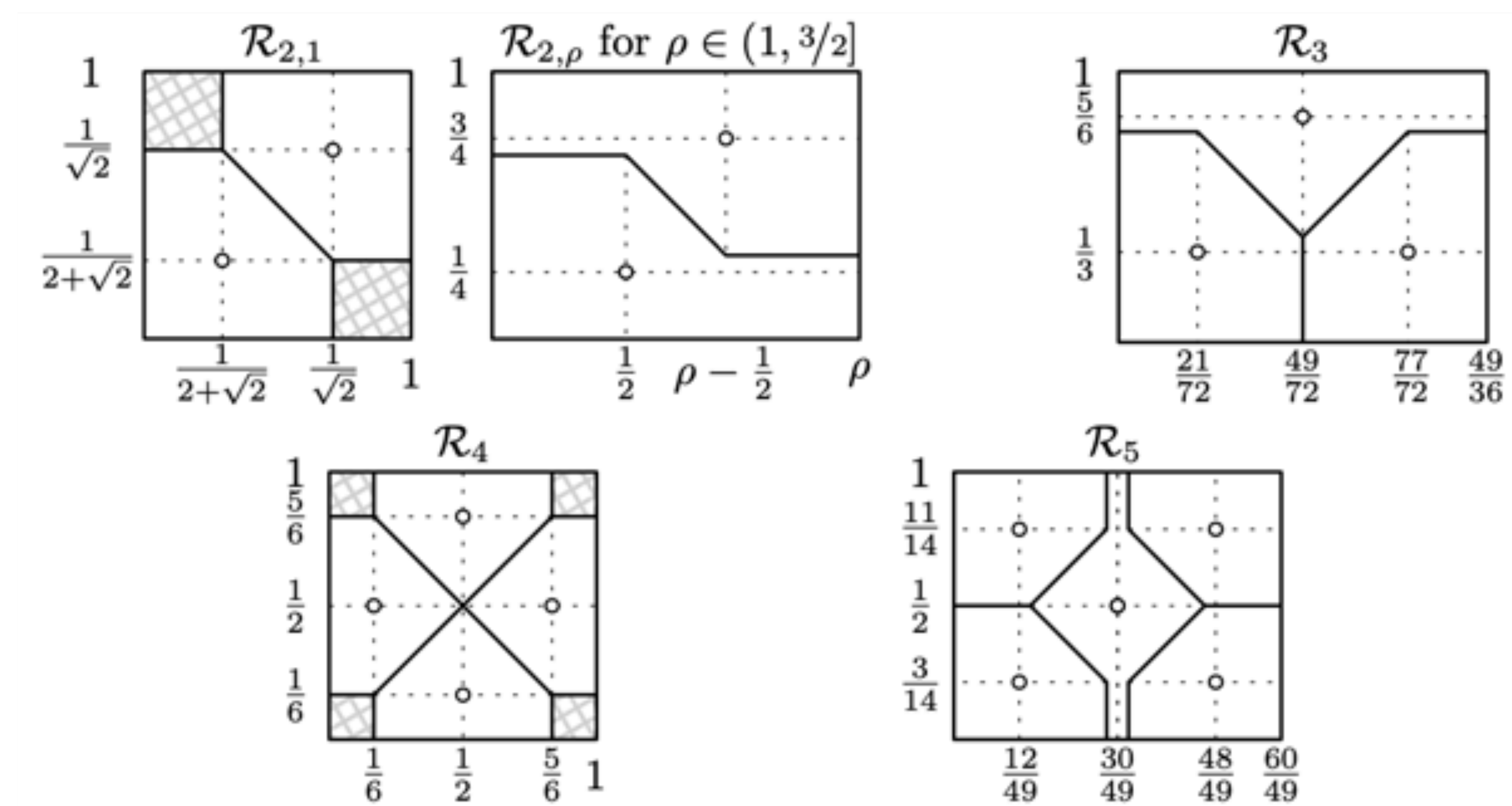


Fig. 4. Non-grid examples of balanced point sets of cardinality 2, 3, 4, and 5.

The One-Round Manhattan Game [Byrne, Fekete, Kalcsics and Kleist 2021]

Theorem 15. *White has a winning strategy for placing n points in a $(1 \times \rho)$ rectangle with $\rho \geq 1$ if and only if $\rho \geq n$; otherwise Black has a winning strategy. Moreover, if $\rho \geq n$, the unique winning strategy for White is to place a $1 \times n$ grid.*

Theorem 15. *White has a winning strategy for placing n points in a $(1 \times \rho)$ rectangle with $\rho \geq 1$ if and only if $\rho \geq n$; otherwise Black has a winning strategy. Moreover, if $\rho \geq n$, the unique winning strategy for White is to place a $1 \times n$ grid.*

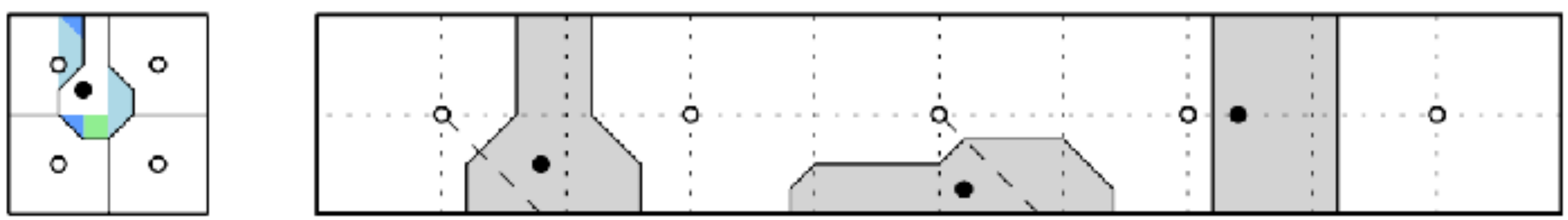


Fig. 9. Illustration of the proof of Theorem 15. (Left) A black winning point in a 2×2 grid. (Right) Every black cell has an area $\leq 1/2n \cdot \mathcal{A}(R)$. Moreover, only $n-1$ locations result in cells of that size.

Technische Universität Braunschweig

Voronoi Game in Graphs [Teramoto, Demaine and Uehara 2011]

Technische Universität Braunschweig

Voronoi Game in Graphs [Teramoto, Demaine and Uehara 2011]

ELSEVIER

Available at
www.ElsevierComputerScience.com
POWERED BY SCIENCE DIRECT•
Theoretical Computer Science 313 (2004) 377–392

Theoretical Computer Science

www.elsevier.com/locate/tcs

Traveling salesmen in the presence of competition

Sándor P. Fekete[a,*], Rudolf Fleischer[b], Aviezri Fraenkel[c], Matthias Schmitt[d]

[a] *Department of Mathematical Optimization, Braunschweig University of Technology, Pockelsstrasse 14, D-38106 Braunschweig, Germany*
[b] *Department of Computer Science, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong*
[c] *Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, 76100 Rehovot, Israel*
[d] *Center for Parallel Computing, Universität zu Köln, 50931 Köln, Germany*

Received 20 May 2002; received in revised form 6 November 2002; accepted 2 December 2002

Abstract

We propose the "competing salesmen problem" (CSP), a two-player competitive version of the classical traveling salesman problem. This problem arises when considering two competing salesmen instead of just one. The concern for a shortest tour is replaced by the necessity to reach any of the customers before the opponent does.

Voronoi Game in Graphs [Teramoto, Demaine and Uehara 2011]

Available at
www.ElsevierComputerScience.com
POWERED BY SCIENCE DIRECT·
Theoretical Computer Science 313 (2004) 377–392

Theoretical Computer Science

www.elsevier.com/locate/tcs

ELSEVIER

Traveling salesmen in the presence of competition

Sándor P. Fekete[a,*], Rudolf Fleischer[b], Aviezri Fraenkel[c], Matthias Schmitt[d]

[a] *Department of Mathematical Optimization, Braunschweig University of Technology, Pockelsstrasse 14, D-38106 Braunschweig, Germany*

[b] *Department of Computer Science, Hong Kong University of Science and Technology, Clear Wat*

[c] *Department of Computer Science an*

7

[d] *Center for Parallel Comput*

Received 20 May 2002; received in re

Theorem 1. *The decision problem whether player* I *can win in* CSP(1,1) *is* PSPACE-*complete, even for the special case of bipartite graphs, with both players starting at distance* 2 *from each other.*

Abstract

We propose the "competing salesmen problem" (CSP), a two-player competitive version of the classical traveling salesman problem. This problem arises when considering two competing salesmen instead of just one. The concern for a shortest tour is replaced by the necessity to reach any of the customers before the opponent does.

Voronoi Game in Graphs [Teramoto, Demaine and Uehara 2011]

Technische Universität Braunschweig

Voronoi Game in Graphs [Teramoto, Demaine and Uehara 2011]

Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 15, no. 4, pp. 485–501 (2011)

The Voronoi game on graphs and its complexity

Sachio Teramoto [1] *Erik D. Demaine* [2] *Ryuhei Uehara* [3]

[1] Service Platforms Research Laboratories, NEC Corporation, Tokyo, Japan.
[2] Computer Science and Artificial Intelligence Lab, the Massachusetts Institute of Technology, Cambridge, Massachusetts, USA.
[3] School of Information Science, Japan Advanced Institute of Science and Technology, Ishikawa, Japan.

Abstract

The Voronoi game is a two-person game which is a model for a competitive facility location. The game is played on a continuous domain, and only two special cases (one-dimensional case and one-round case) are well investigated. We introduce the *discrete* Voronoi game in which the game arena is given as a graph. We first analyze the game when the arena is a large complete k-ary tree, and give an optimal strategy. When both players play optimally, the first player wins when k is odd, and the game ends in a tie for even k. Next we show that the discrete Voronoi game is intractable in general. Even for the one-round case in which the strategy adopted by the first player consist of a fixed single node, deciding whether the second player can win is NP-complete. We also show that deciding whether the second player can win is PSPACE-complete in general.

Submitted:	Reviewed:	Revised:	Accepted:
May 2009	October 2009	July 2011	August 2011
	Final:	Published:	
	August 2011	August 2011	

Article type: Regular Paper
Communicated by: H. Meijer

A preliminary version was presented at the 2nd IEEE Symposium on Computational Intelligence and Games (CIG 2006). A part of this work was done while the third author was visiting MIT, USA.

E-mail addresses: s-teramoto@bx.jp.nec.com (Sachio Teramoto) edemaine@mit.edu (Erik D.

Voronoi Game in Graphs [Teramoto, Demaine and Uehara 2011]

Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 15, no. 4, pp. 485–501 (2011)

The Voronoi game on graphs and its complexity

Sachio Teramoto[1] *Erik D. Demaine*[2] *Ryuhei Uehara*[3]

[1]Service Platforms Research Laboratories, NEC Corporation, Tokyo, Japan.
[2]Computer Science and Artificial Intelligence Lab, the Massachusetts Institute of Technology, Cambridge, Massachusetts, USA.
[3]School of Information Science, Japan Advanced Institute of Science and Technology, Ishikawa, Japan.

Abstract

The Voronoi game is a two-person game which is a model for a competitive facility location. The game is played on a continuous domain, and only two special cases (one-dimensional case and one-round case) are well investigated. We introduce the *discrete* Voronoi game in which the game arena is given as a graph. We first analyze the game when the arena is a large complete k-ary tree, and give an optimal strategy. When both players play optimally, the first player wins when k is odd, and the game ends in a tie for even k. Next we show that the discrete Voronoi game is intractable in general. Even for the one-round case in which the strategy adopted by the first player consist of a fixed single node, deciding whether the second player can win is NP-complete. We also show that deciding whether the second player can win is PSPACE-complete in general.

Submitted:	Reviewed:	Revised:	Accepted:
May 2009	October 2009	July 2011	August 2011
	Final:	Published:	
	August 2011	August 2011	

Article type:	Communicated by:
Regular Paper	H. Meijer

A preliminary version was presented at the 2nd IEEE Symposium on Computational Intelligence and Games (CIG 2006). A part of this work was done while the third author was visiting MIT, USA.

E-mail addresses: s-teramoto@bx.jp.nec.com (Sachio Teramoto) edemaine@mit.edu (Erik D.

G_{pos}(Pos DNF):

Input: A positive DNF formula A (that is, a DNF formula containing no negative literal).

Rule: Two players alternately choose some variable of A which has not been chosen yet. The game ends after all variables of A have been chosen. The first player wins if and only if A is true when all variables chosen by the first player are set to 1 and all variables chosen by the second player are set to 0. (In other words, the first player wins if and only if he takes every variable of some disjunct.)

Output: Determine whether the first player has the winning strategy for A.

Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 15, no. 4, pp. 485–501 (2011)

The Voronoi game on graphs and its complexity

Sachio Teramoto[1] *Erik D. Demaine*[2] *Ryuhei Uehara*[3]

[1]Service Platforms Research Laboratories, NEC Corporation, Tokyo, Japan.
[2]Computer Science and Artificial Intelligence Lab, the Massachusetts Institute of Technology, Cambridge, Massachusetts, USA.
[3]School of Information Science, Japan Advanced Institute of Science and Technology, Ishikawa, Japan.

Abstract

The Voronoi game is a two-person game which is a model for a competitive facility location. The game is played on a continuous domain, and only two special cases (one-dimensional case and one-round case) are well investigated. We introduce the *discrete* Voronoi game in which the game arena is given as a graph. We first analyze the game when the arena is a large complete k-ary tree, and give an optimal strategy. When both players play optimally, the first player wins when k is odd, and the game ends in a tie for even k. Next we show that the discrete Voronoi game is intractable in general. Even for the one-round case in which the strategy adopted by the first player consist of a fixed single node, deciding whether the second player can win is NP-complete. We also show that deciding whether the second player can win is PSPACE-complete in general.

Submitted:	Reviewed:	Revised:	Accepted:
May 2009	October 2009	July 2011	August 2011
	Final:	Published:	
	August 2011	August 2011	

Article type: Regular Paper
Communicated by: H. Meijer

A preliminary version was presented at the 2nd IEEE Symposium on Computational Intelligence and Games (CIG 2006). A part of this work was done while the third author was visiting MIT, USA.

E-mail addresses: s-teramoto@bx.jp.nec.com (Sachio Teramoto) edemaine@mit.edu (Erik D.

G_{pos}(Pos Dnf):

Input: A positive DNF formula A (that is, a DNF formula containing no negative literal).

Rule: Two players alternately choose some variable of A which has not been chosen yet. The game ends after all variables of A have been chosen. The first player wins if and only if A is true when all variables chosen by the first player are set to 1 and all variables chosen by the second player are set to 0. (In other words, the first player wins if and only if he takes every variable of some disjunct.)

Output: Determine whether the first player has the winning strategy for A.

Theorem 4 *The discrete Voronoi game is* PSPACE*-complete in general.*

Technische Universität Braunschweig

Thank you for today!