Computational Geometry

Tutorial #3 — Polygon operations & Farthest point pairs

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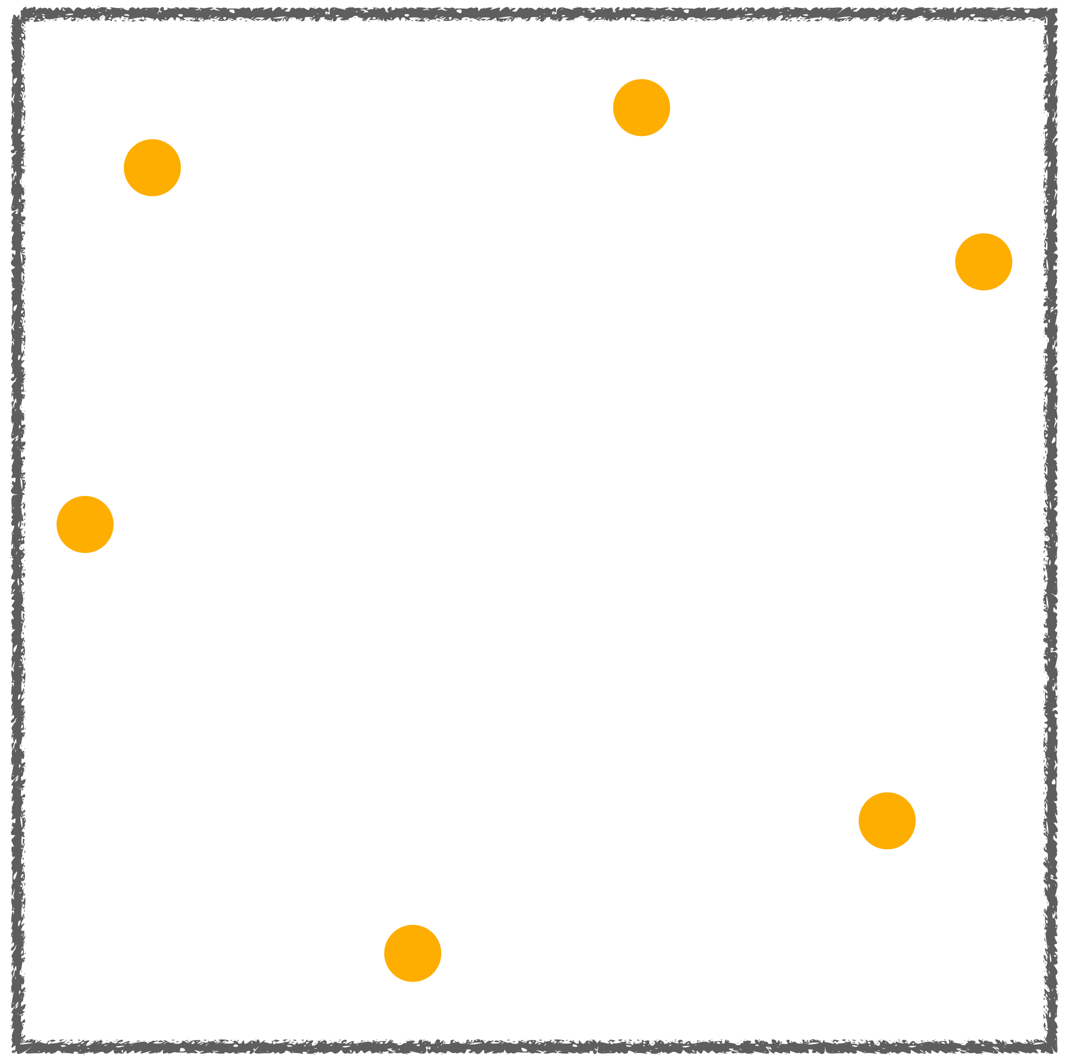
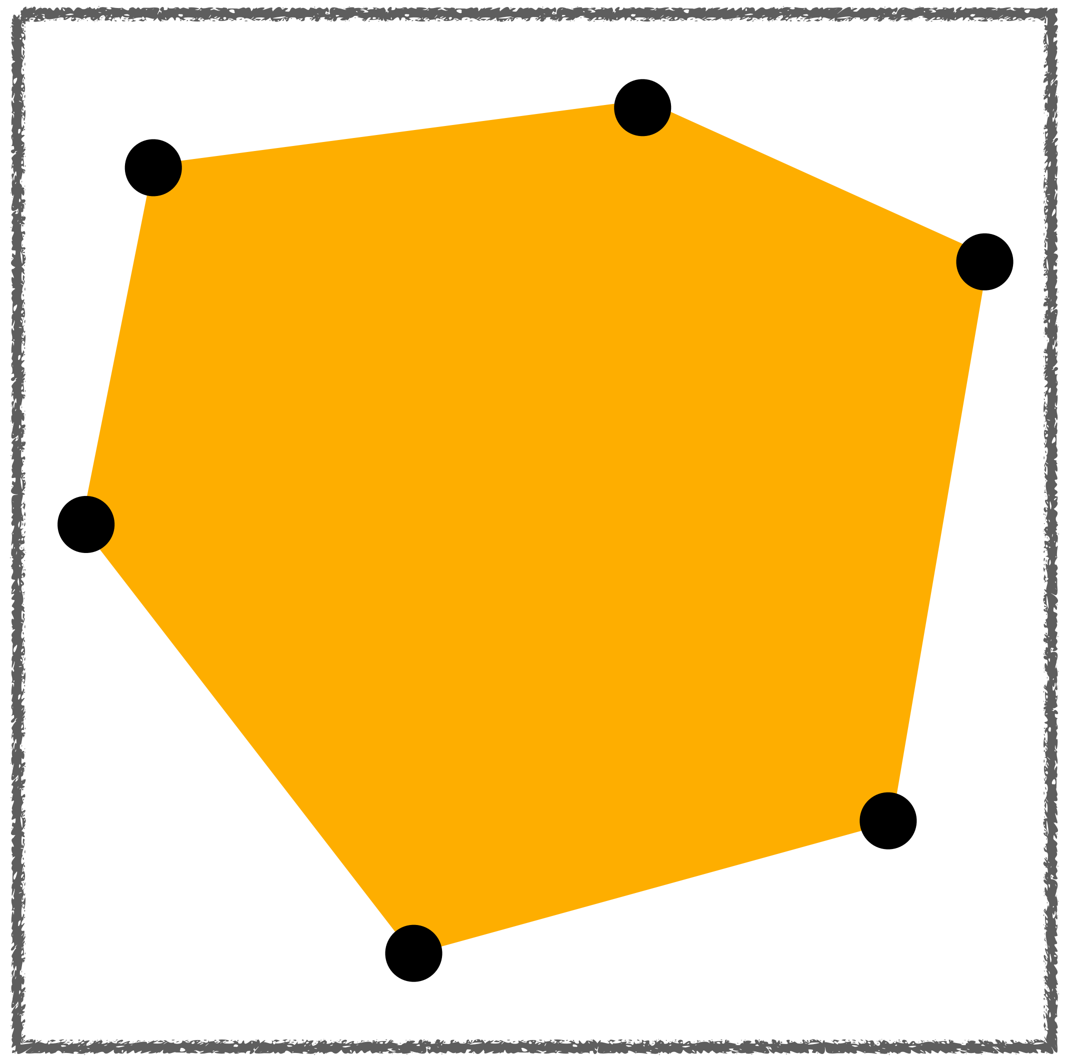
December 7, 2023

Point sets, hulls, and polygons

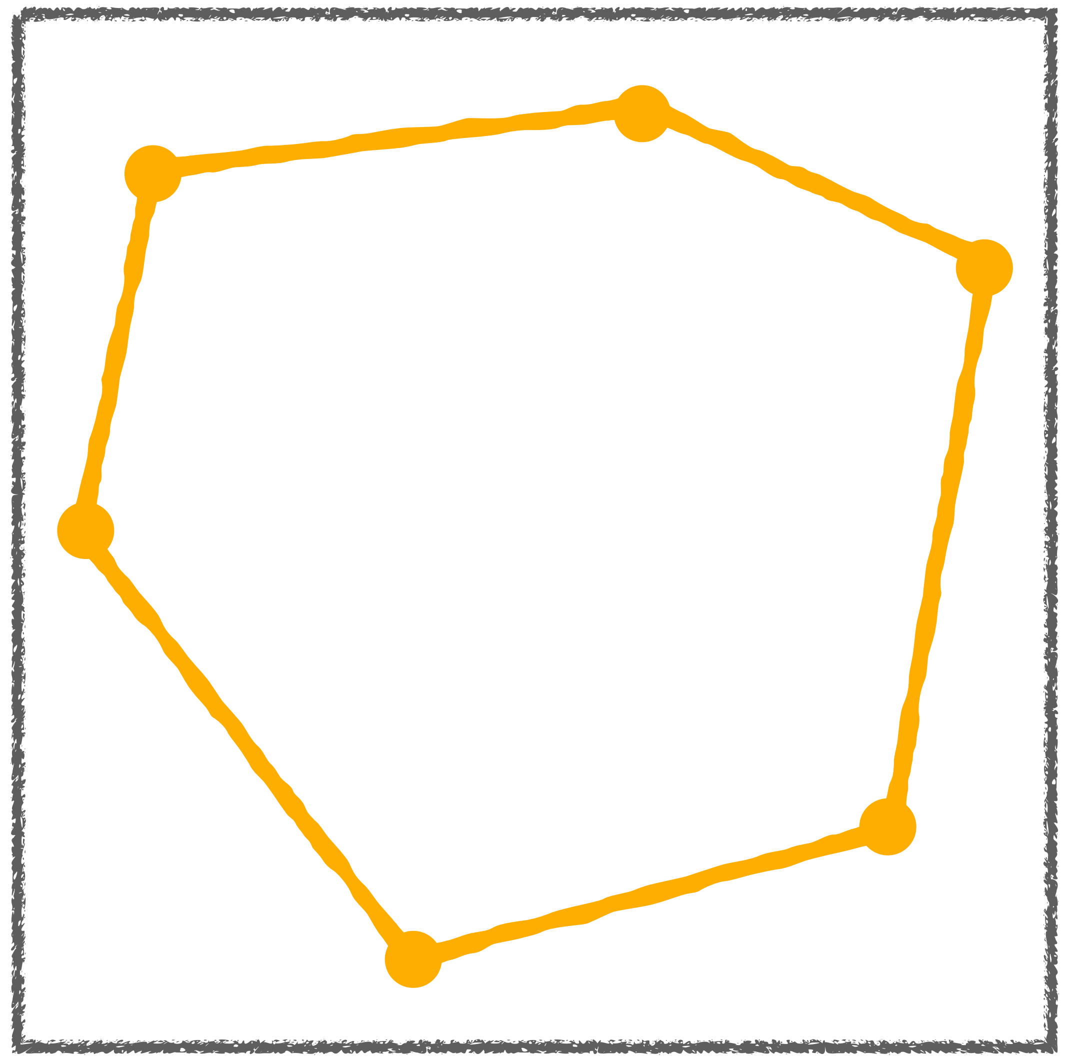
Refresh: What's the difference?



A point set $\mathscr{P}$



$\mathrm{conv}(\mathscr{P})$



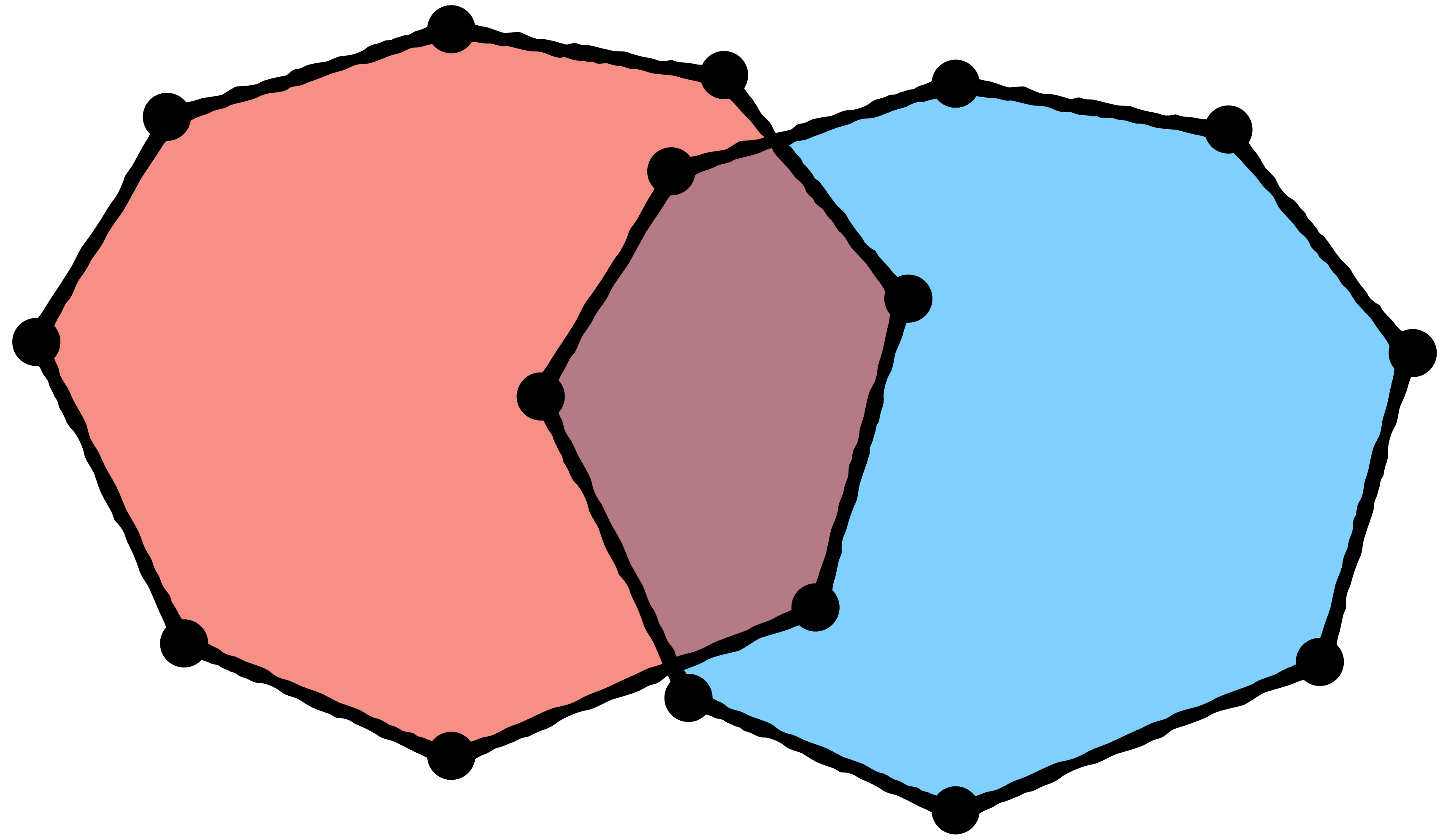
A polygon P on $\mathscr{P}$

Boolean Operations

Boolean Operations on Convex Polygons

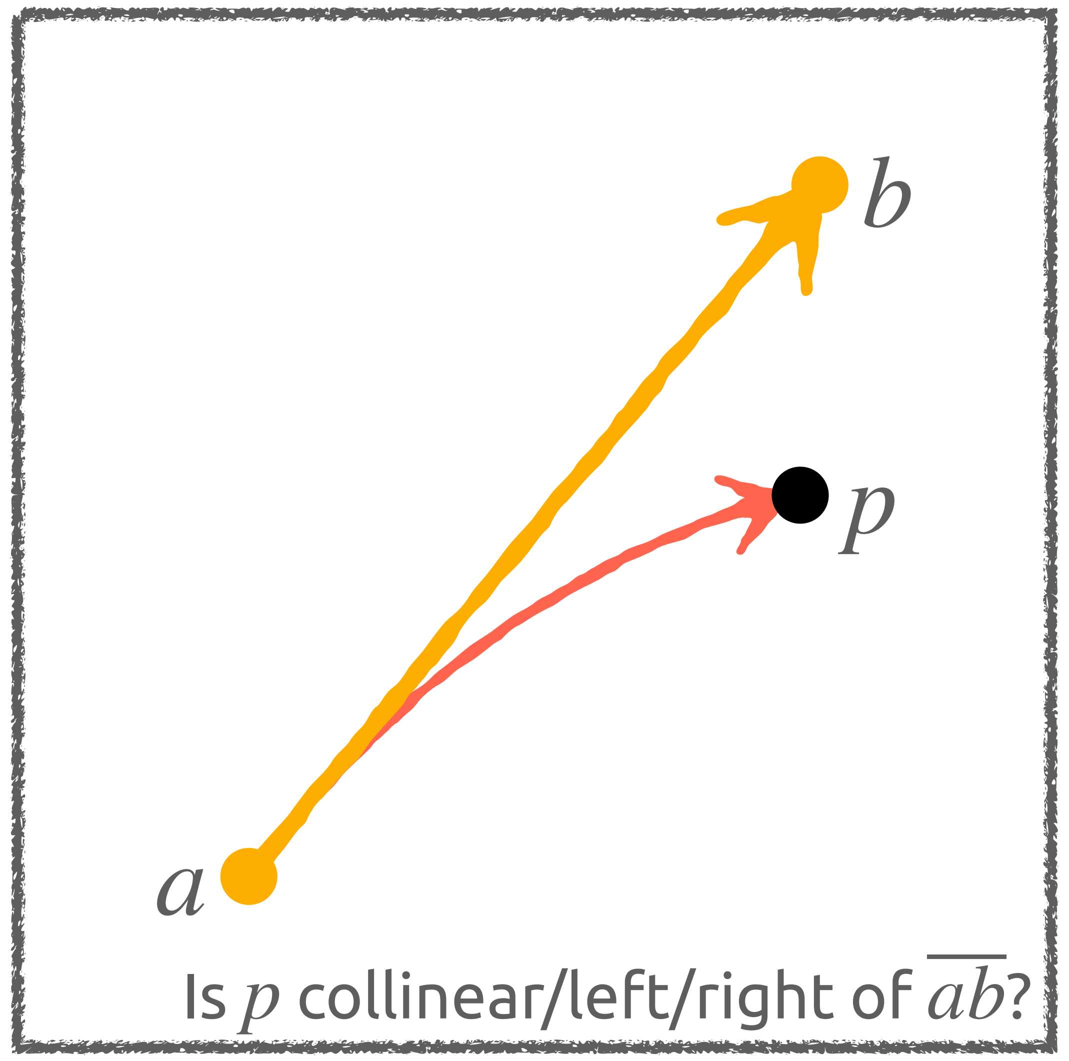
Given two convex Polygons P and Q, we seek to determine:

$$P \cap Q, P \cup Q, P \setminus Q, (Q \setminus P)$$



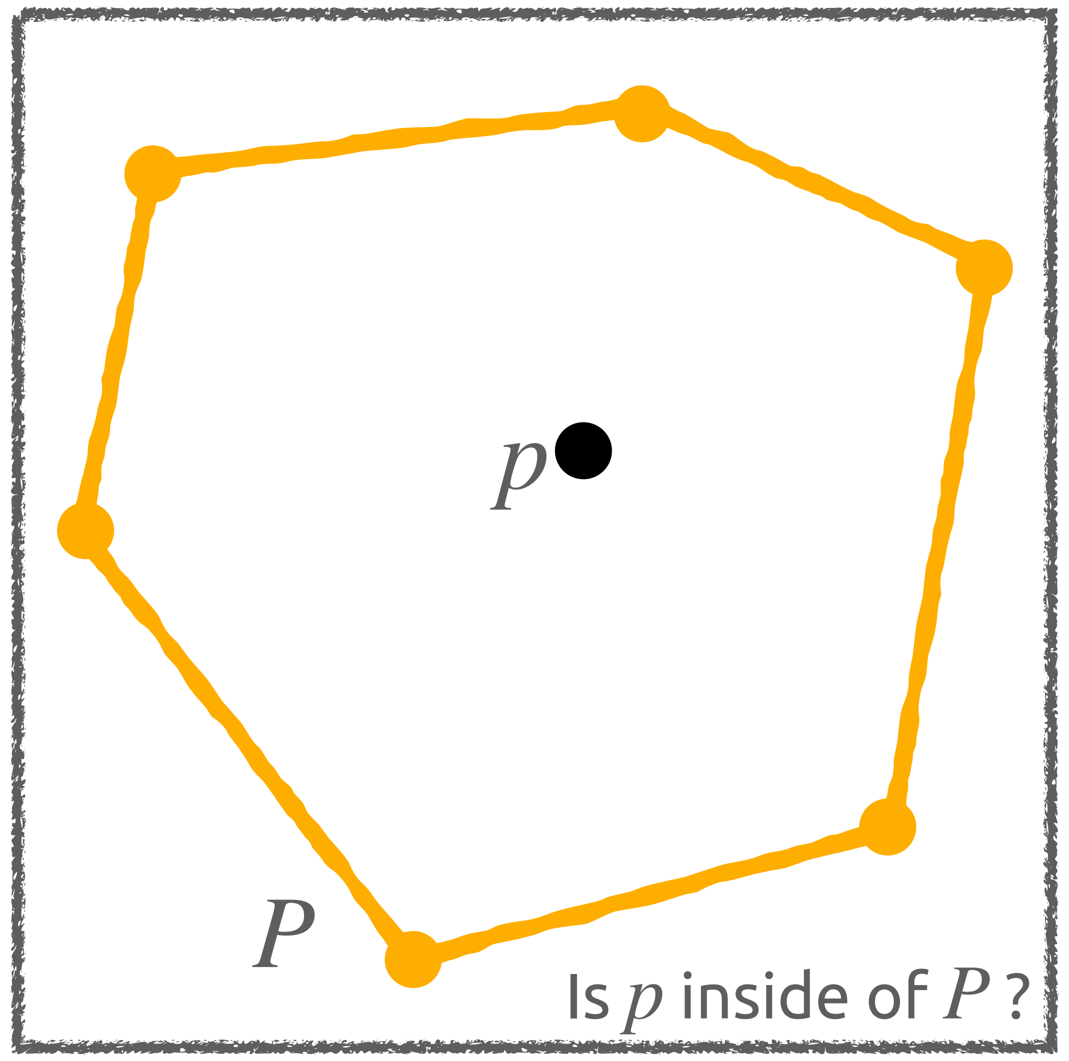
Tools

Point-Line Test



$\mathcal{O}(1)$

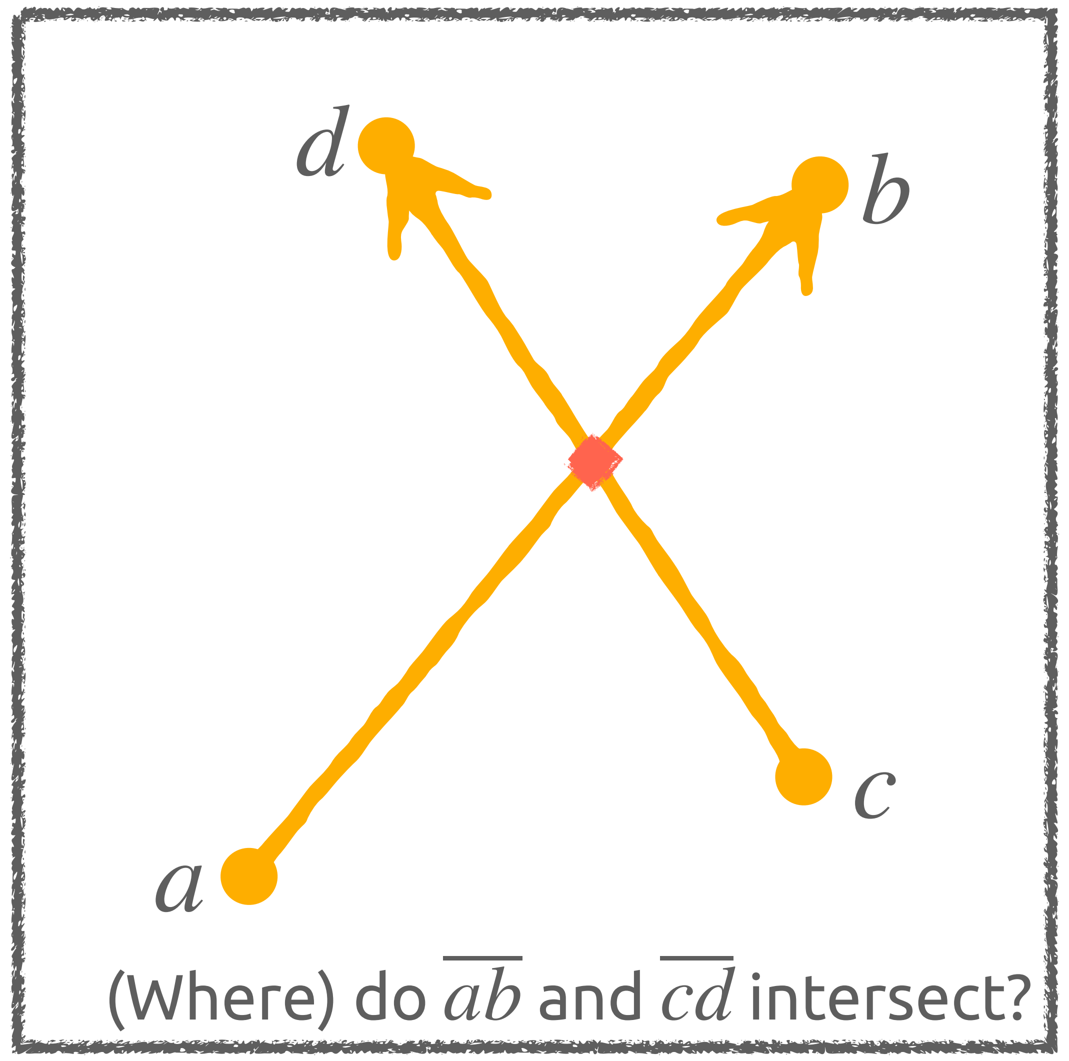
Point Location Problem



$\mathcal{O}(\log n)$

(If P convex)

Intersection Test



$\mathcal{O}(1)$

Naive Algorithm using these tools

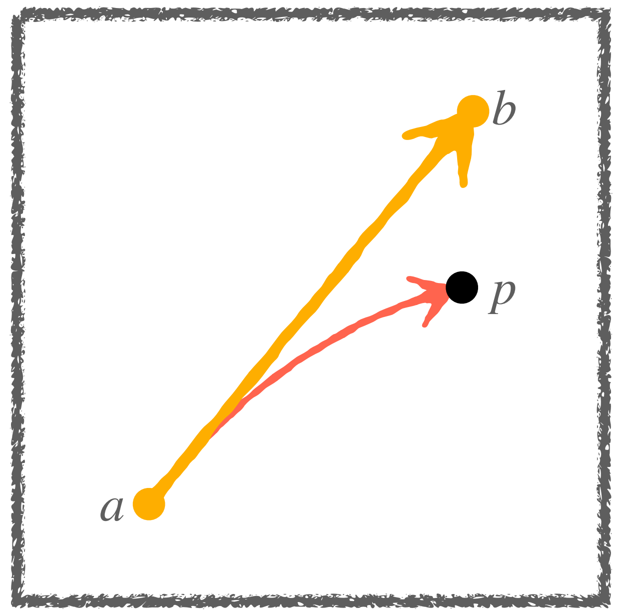
Eliminate/Add points, then recompute convex hull

Given: Convex polygons $P := p_1, \ldots, p_n$ and $Q := q_1, \ldots, q_m$.

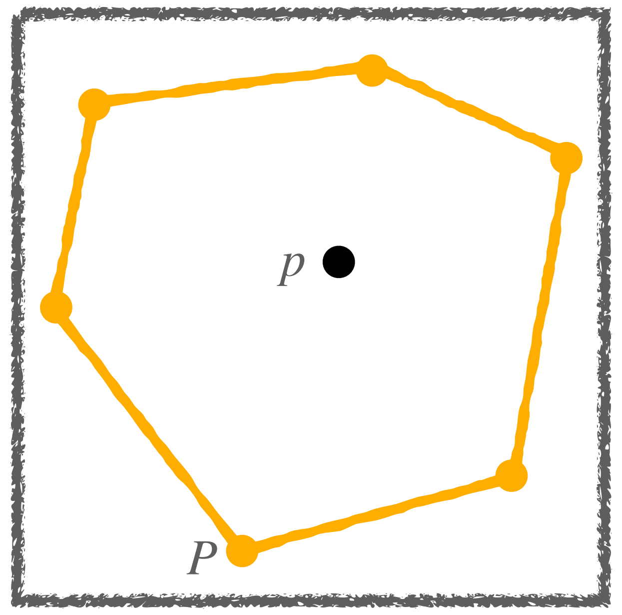
Wanted: The convex polygon $P \cap Q$.

Idea: Determine extreme points of $P \cap Q$ in $\mathcal{O}(n^2)$, then compute the convex hull in $\mathcal{O}(n \log h)$.

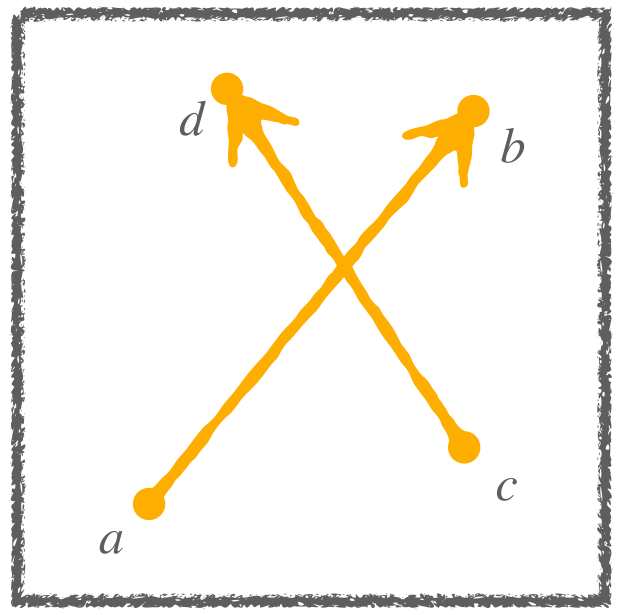
This gives us an $\mathcal{O}(n^2)$-algorithm.



$\mathcal{O}(1)$



$\mathcal{O}(\log n)$

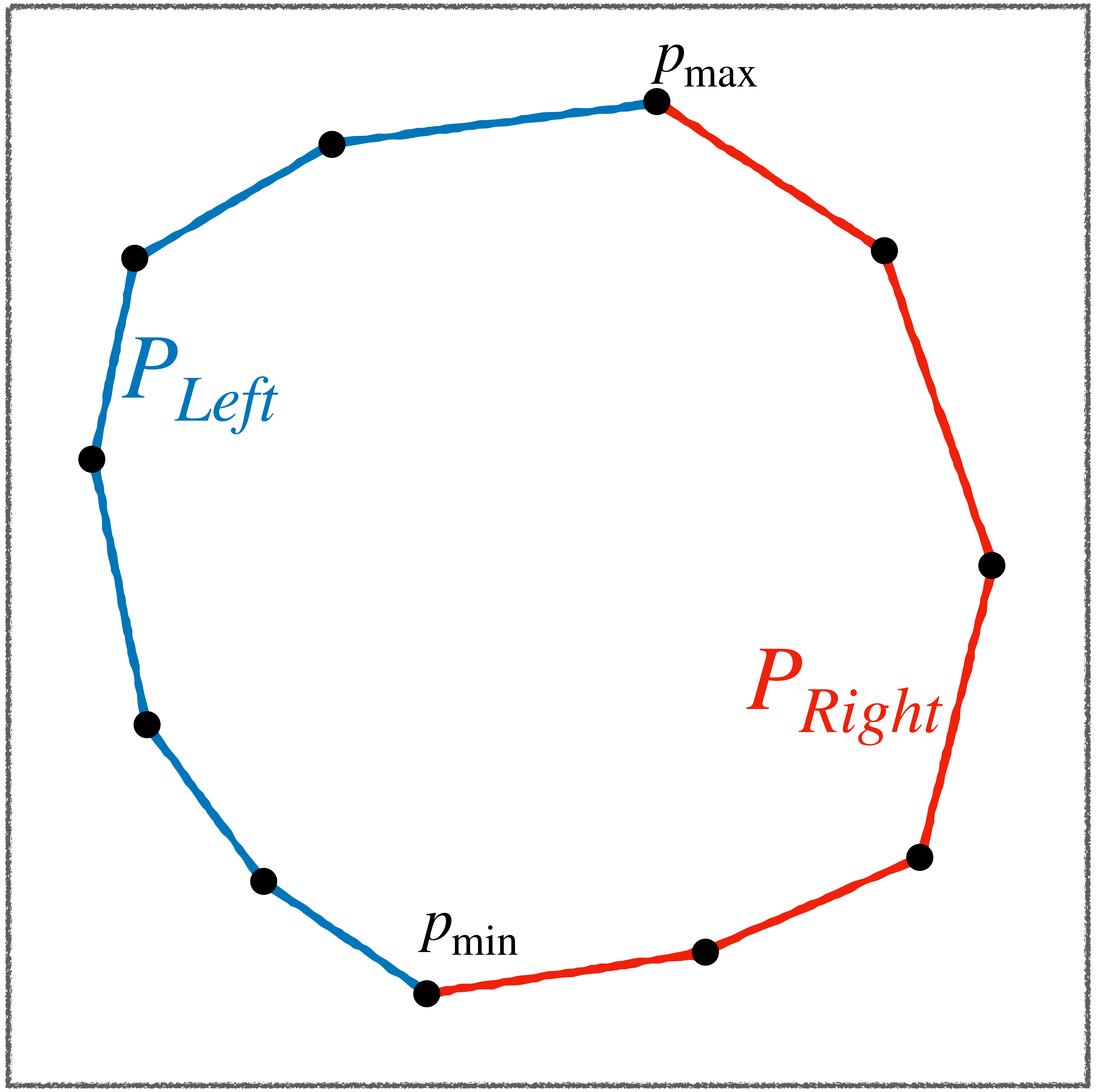


$\mathcal{O}(1)$

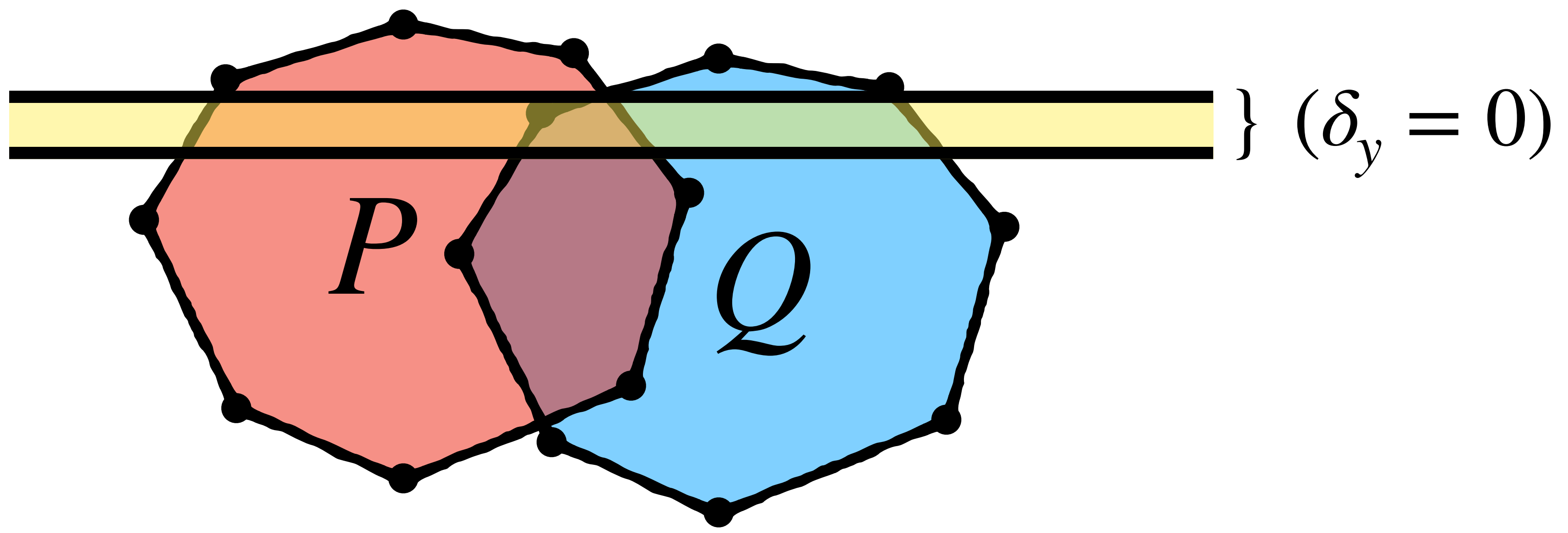
Convex polygons
… can be decomposed!

- Given $P = p_1, \ldots, p_n$
- Compute $p_{\min}$ and $p_{\max}$ in $\mathcal{O}(n)$ along the y-axis
- We obtain P_L and P_R:

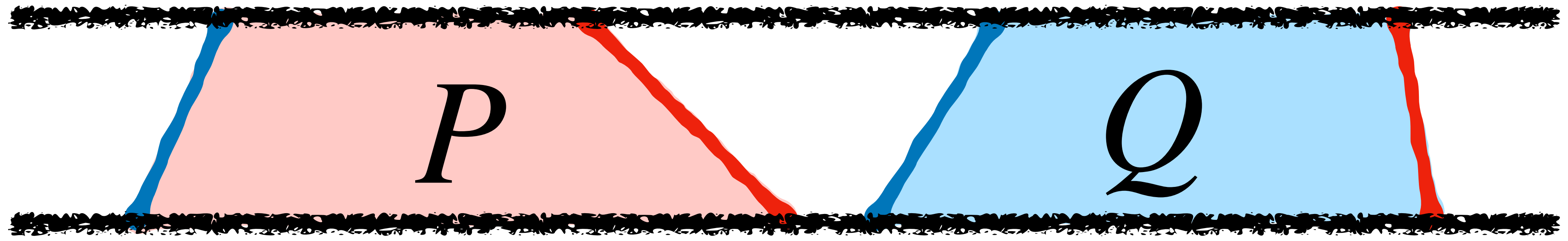
$$P_L = \quad p_{\max}, \ldots, p_{\min}$$
$$P_R = \quad p_{\min}, \ldots, p_{\max}$$

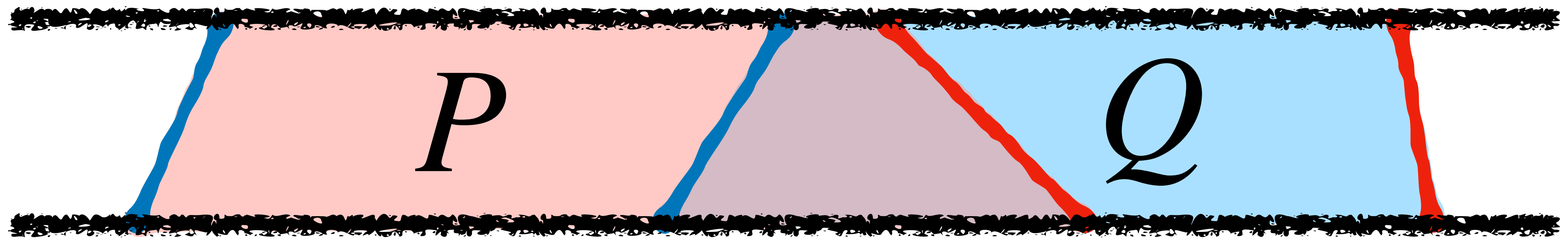


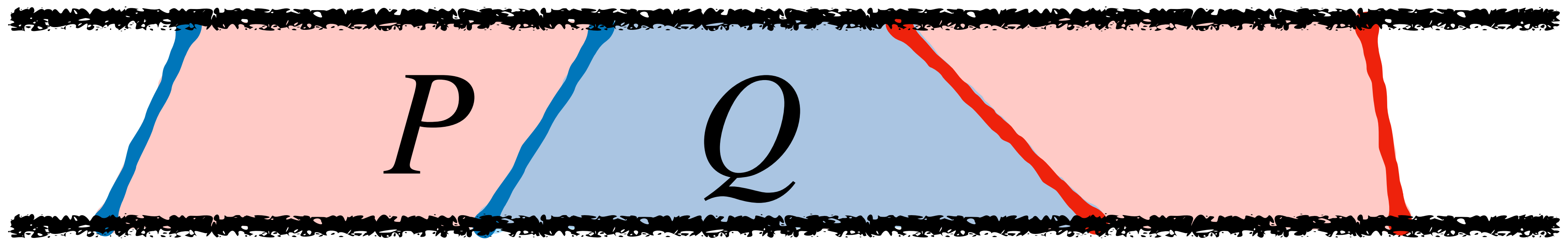
Towards an $\mathcal{O}(n)$-Algorithm
Using left- and right decomposition



- If we slice through P and Q horizontally, at some y (see right, exaggerated):
  (a) P and Q do not intersect,
  (b) P and Q overlap partially, or
  (c) one contains the other.

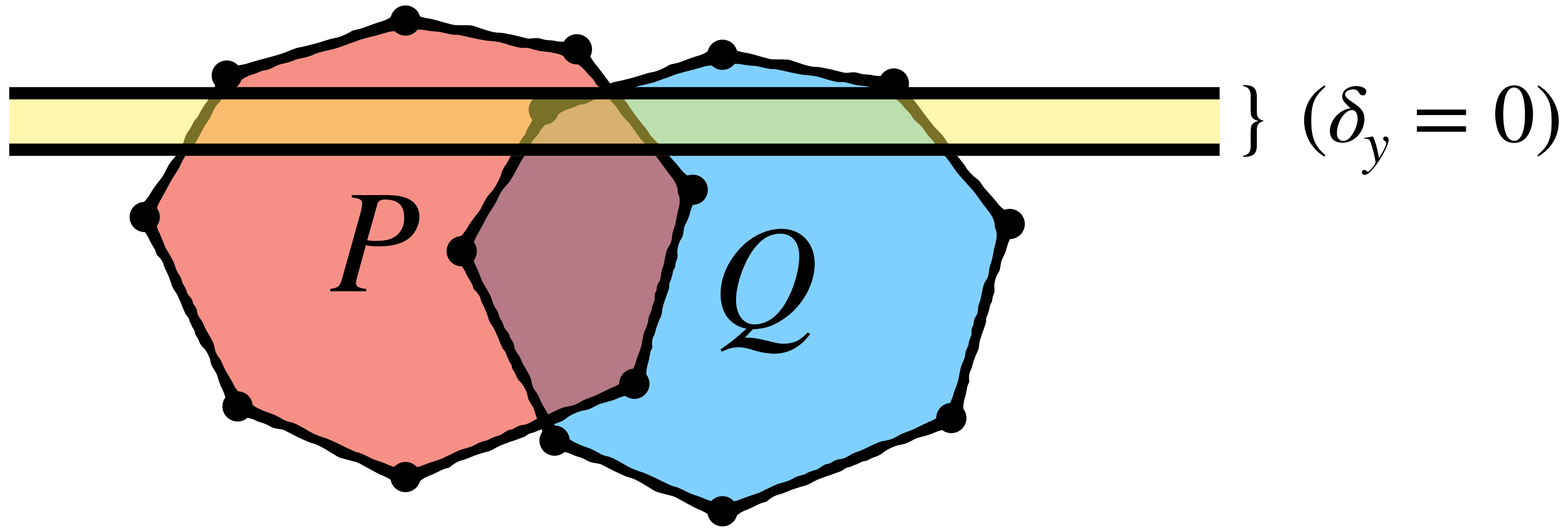




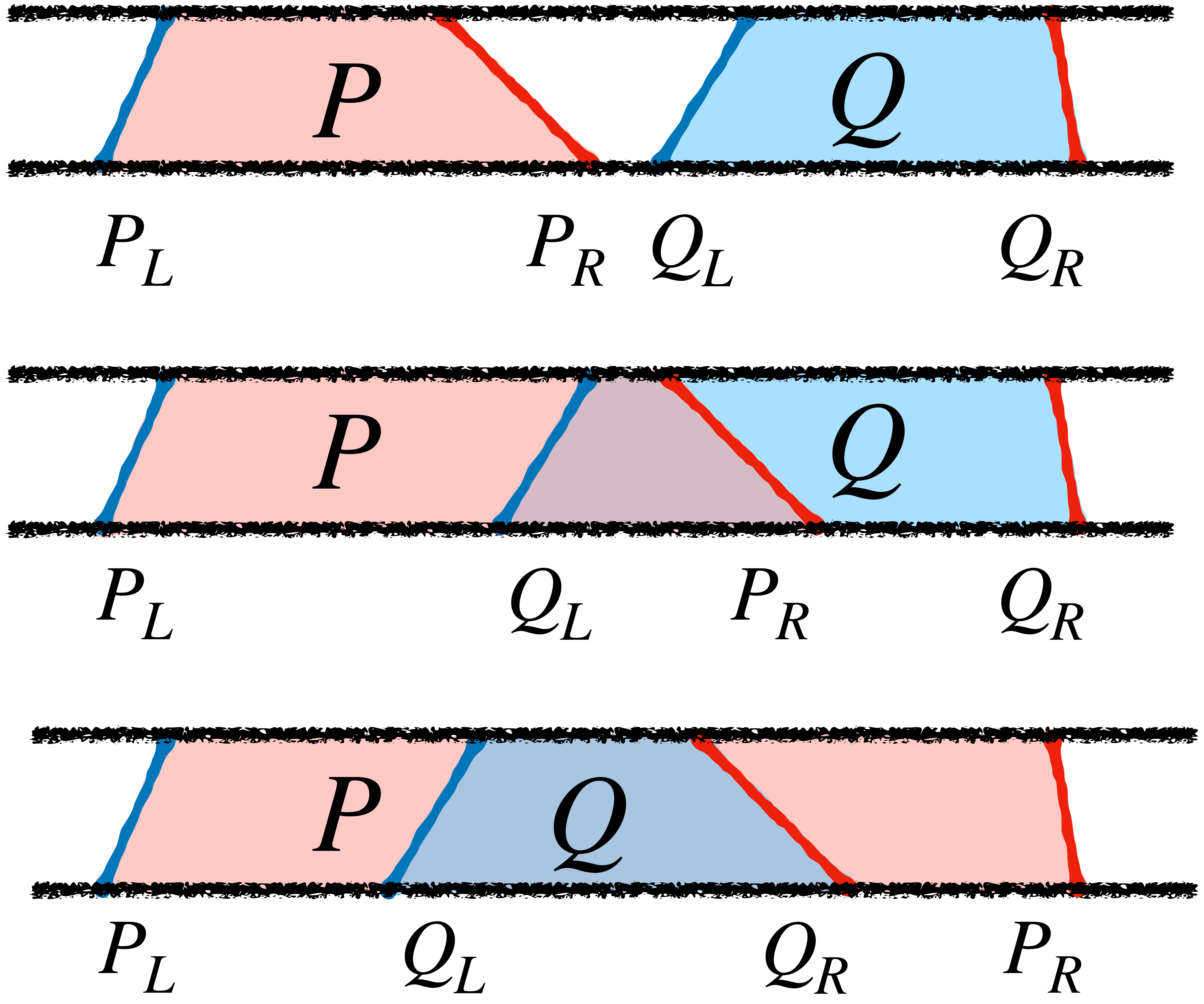


Towards an $\mathcal{O}(n)$-Algorithm

Using left- and right decomposition



- If we slice through P and Q horizontally, at some y (see right, exaggerated):
  (a) P and Q do not intersect,
  (b) P and Q overlap partially, or
  (c) one contains the other.
- Each case corresponds to an x-order of the chains at that y-coordinate.

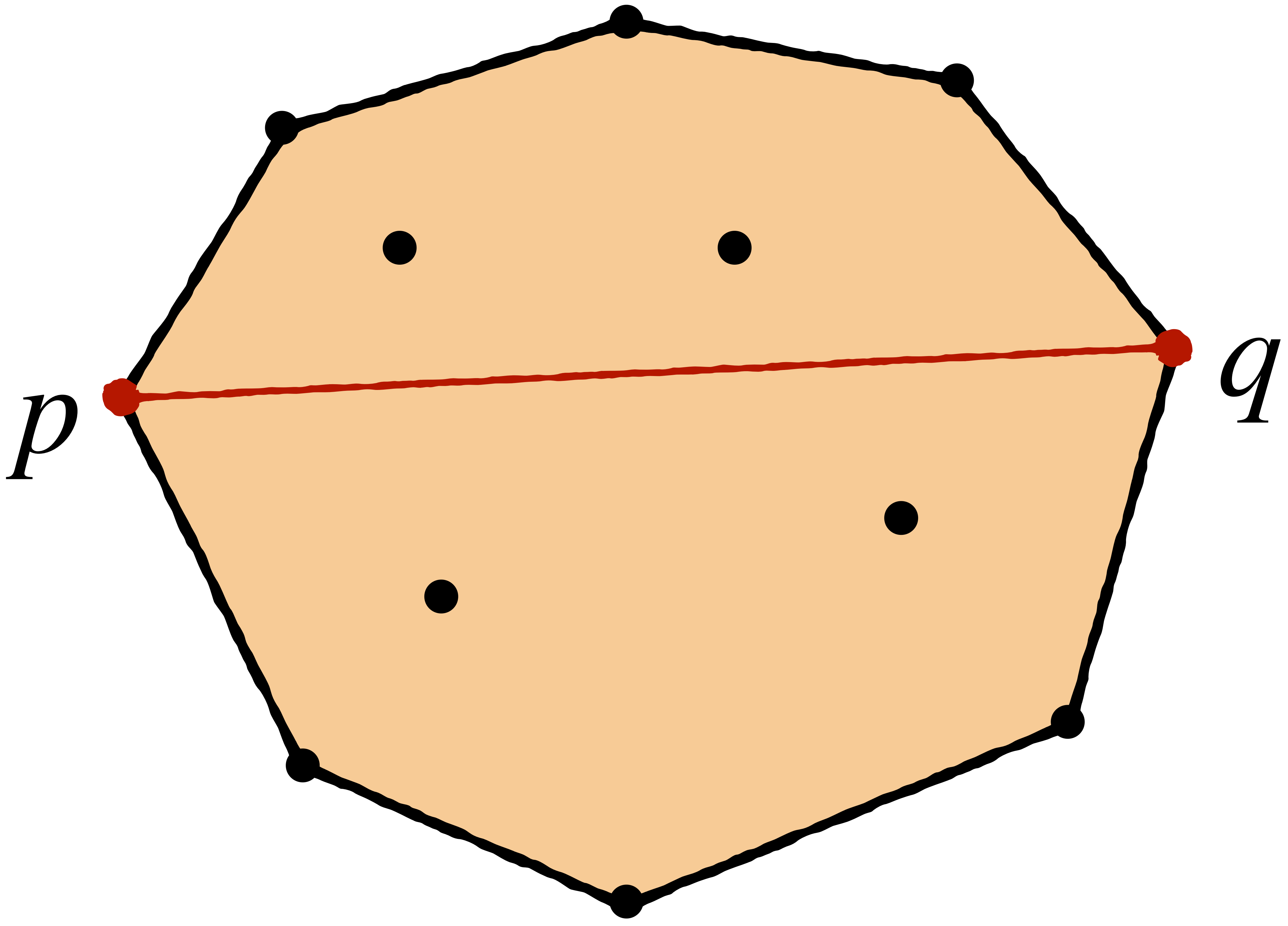


Farthest point pairs

Farthest Point Pairs

Let $\mathscr{P}$ be a finite point set in general position.

The **farthest point pair of $\mathscr{P}$** consists of two vertices of the convex hull.



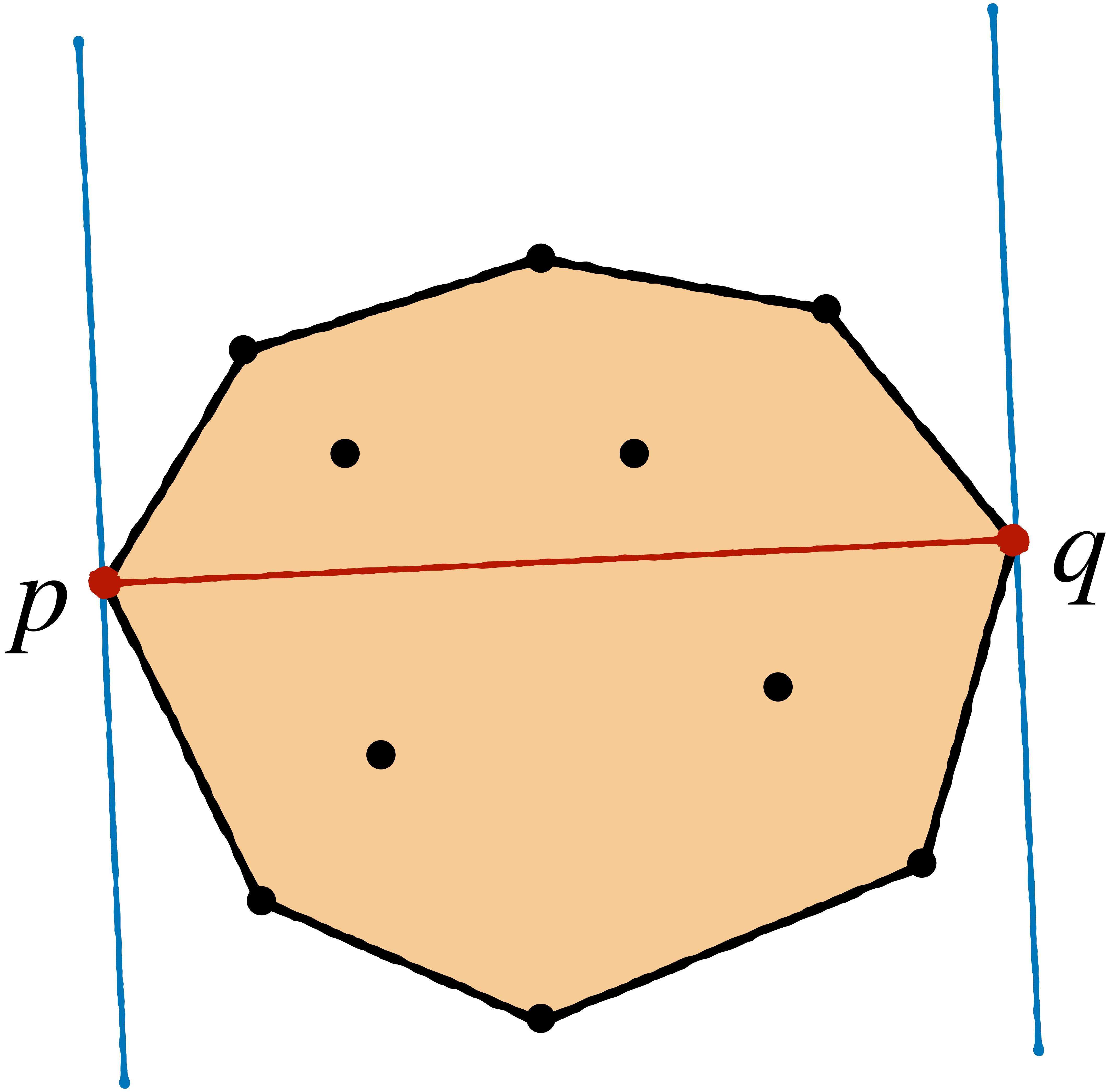
Farthest Point Pairs

Let $\mathscr{P}$ be a finite point set in general position.

The **farthest point pair of** $\mathscr{P}$ consists of two vertices of the convex hull.

Two points of $\mathscr{P}$ **are antipodal** if there exist parallel lines through them which do not cut the hull.

Argue that the farthest pair is antipodal.



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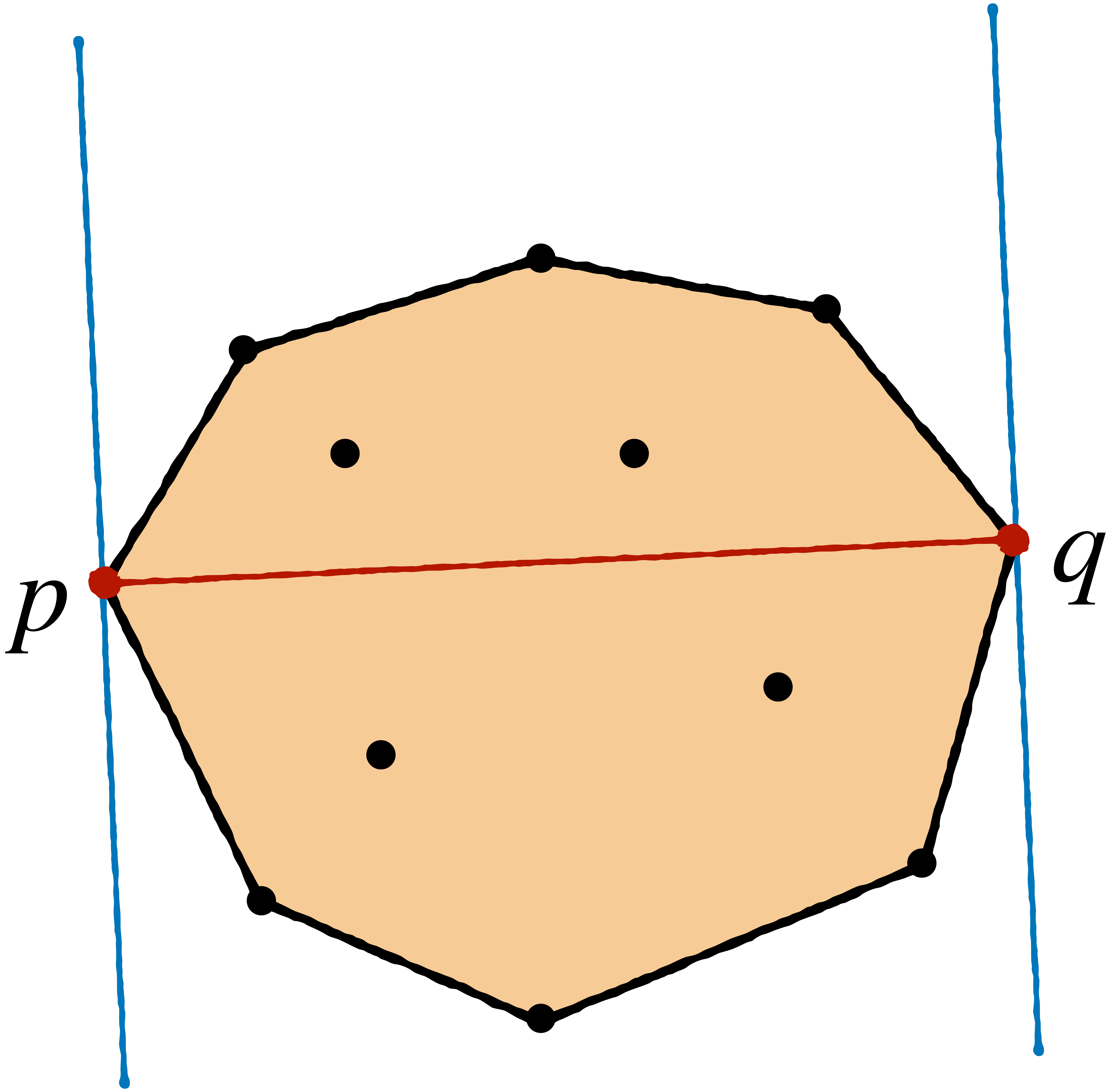
How can we use this?

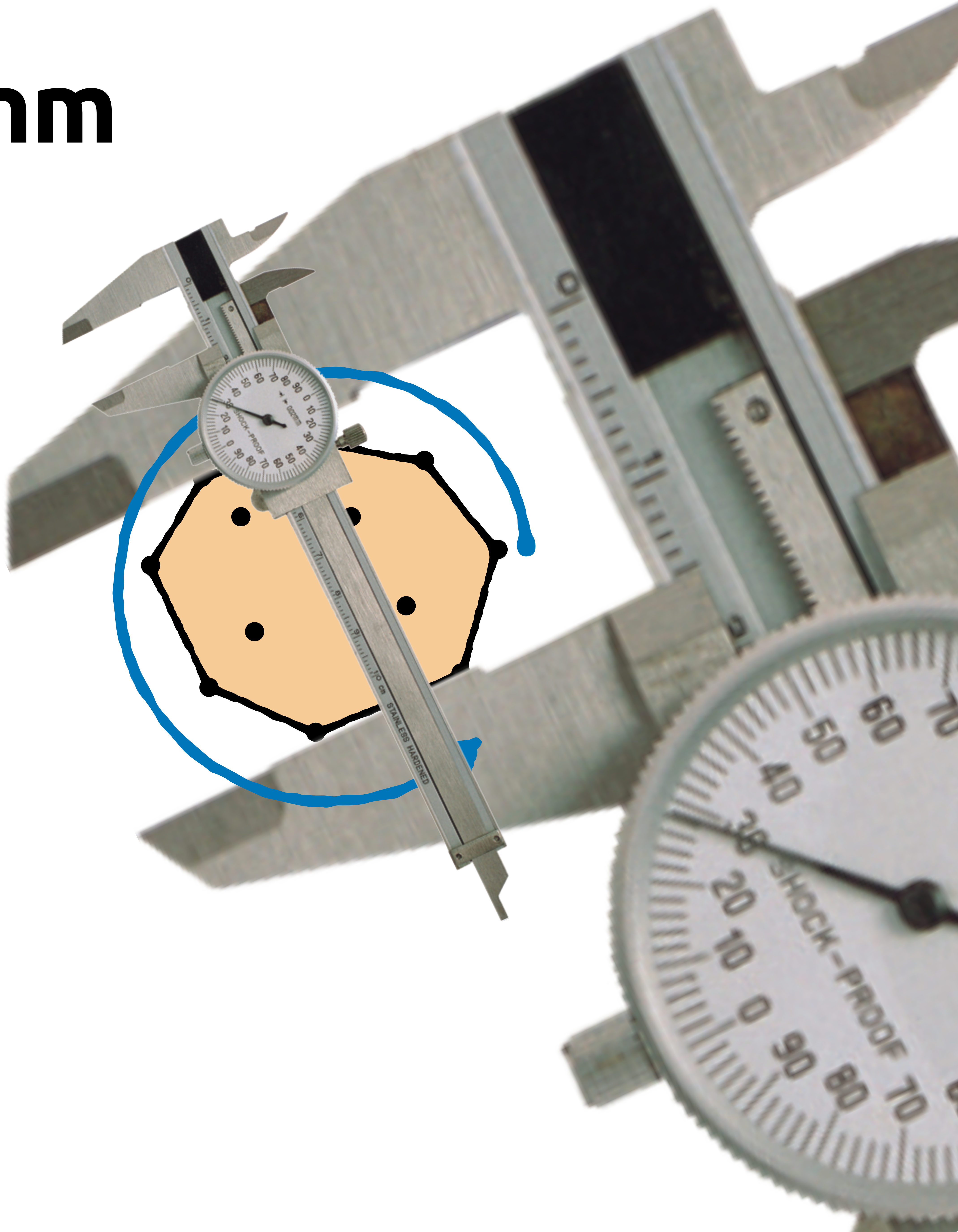


Rotating Callipers Algorithm

Michael Shamos, 1978

Idea: Compute convex hull, then enumerate antipodal pairs and track the farthest one.

To achieve this, “rotate” parallel lines around the point set.



Rotating Callipers Algorithm

Michael Shamos, 1978

Distances [edit]

- Diameter (maximum width) of a convex polygon[6][7]
- Maximum distance between two convex polygons[9][10]
- Minimum distance between two convex polygons[11][12]

machine

Bounding boxes [edit]

- Minimum area oriented bounding box
- Minimum perimeter oriented bounding box

Triangulations [edit]

- Onion triangulations
- Spiral triangulations
- Quadrangulation
- Nice triangulation
- Art gallery problem
- Wedge placement optimization problem[15]

Multi-polygon operations [edit]

- Union of two convex polygons
- Common tangents to two convex polygons
- Intersection of two convex polygons[16]
- Critical support lines of two convex polygons
- Vector sums (or Minkowski sum) of two convex polygons[17]
- Convex hull of two convex polygons

Traversals [edit]

- Shortest transversals[18][19]
- Thinnest-strip transversals[20]

Others [edit]

- Non parametric decision rules for machine learned classification[21]
- Aperture angle optimizations for visibility problems in computer vision[22]
- Finding longest cells in millions of biological cells[23]
- Comparing precision of two people at firing range
- Classify sections of brain from scan images