Computational Geometry

Tutorial #4 — Voronoi diagrams, cutting and glueing

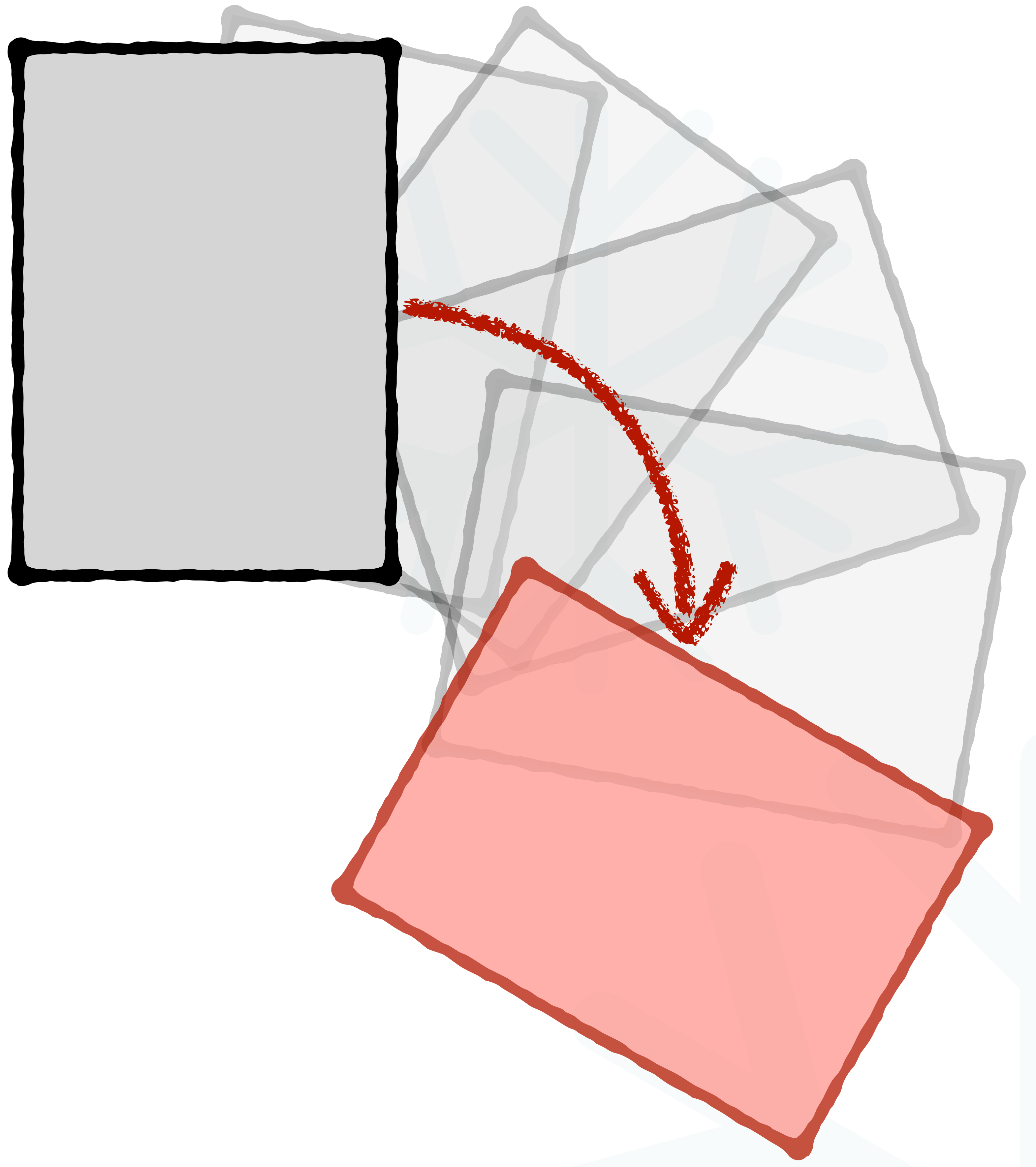
Peter Kramer

December 21, 2023

Scissors congruence

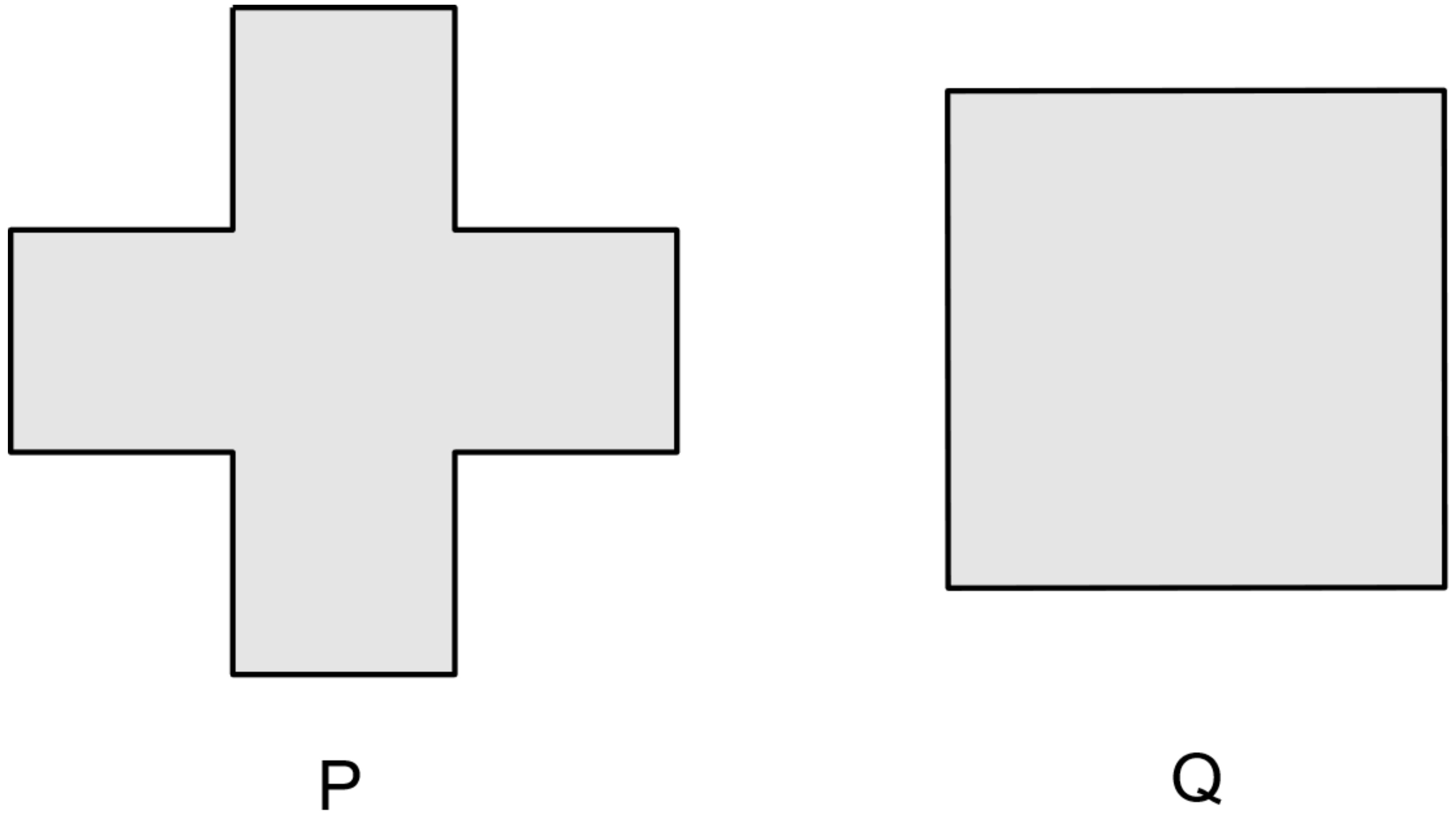
Congruence

Two polygons are **congruent** if there exists a transformation consisting of **only translation and rotation** for one into the other.



Scissors congruence

Two simple polygons P and Q are **scissors congruent** if we can subdivide their area into polygons $P_1, \ldots, P_k$ and $Q_1, \ldots, Q_k$ such that for any $i \in [1,k]$, the polygon P_i is congruent to Q_i. *(This corresponds to cutting and glueing…)*



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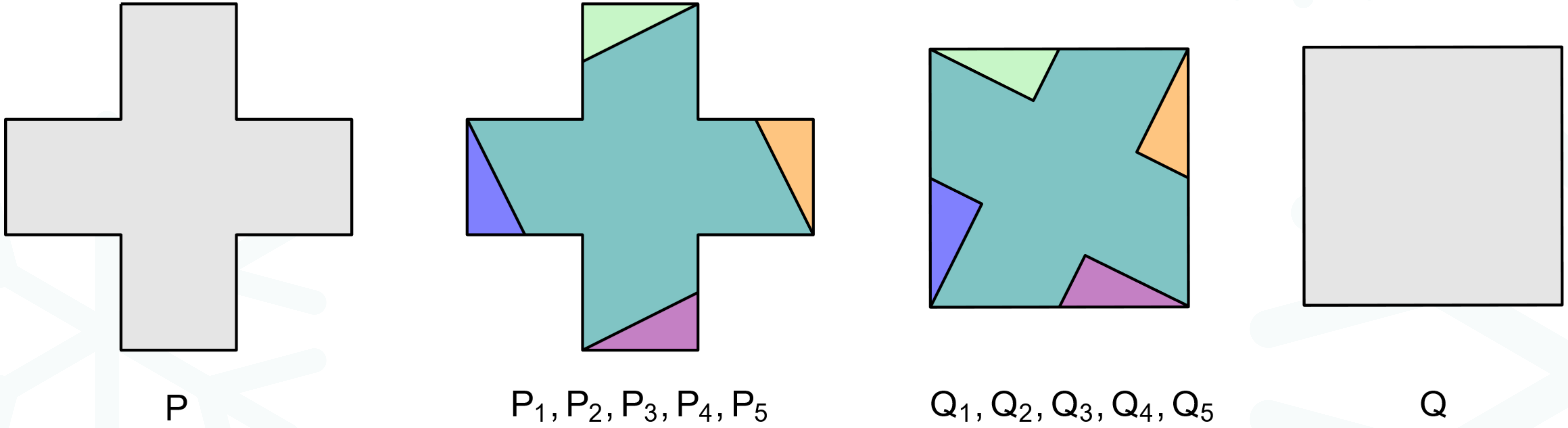

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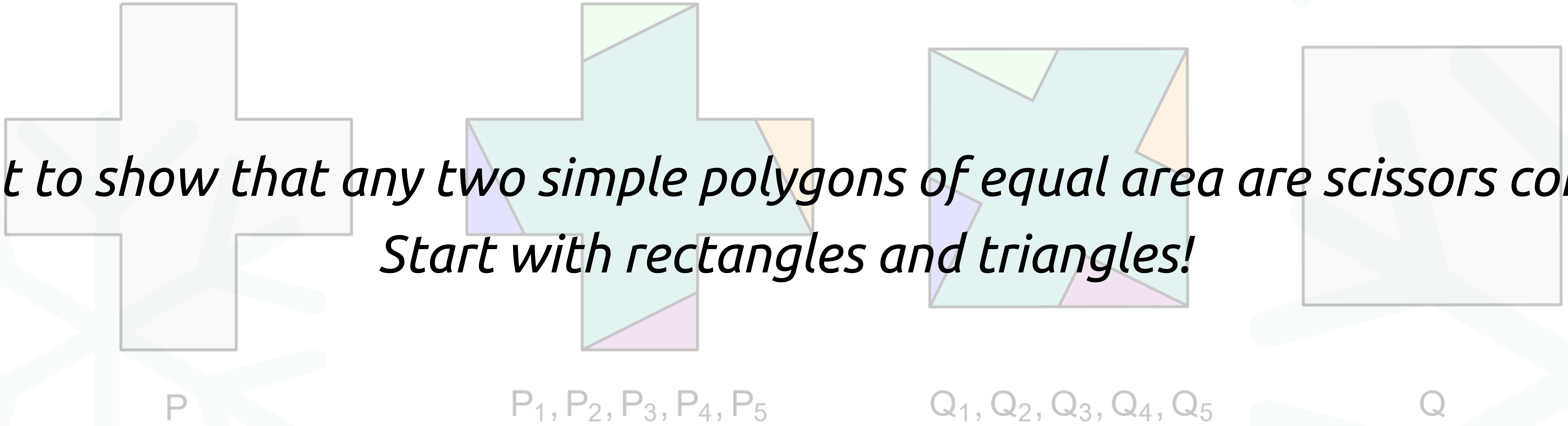
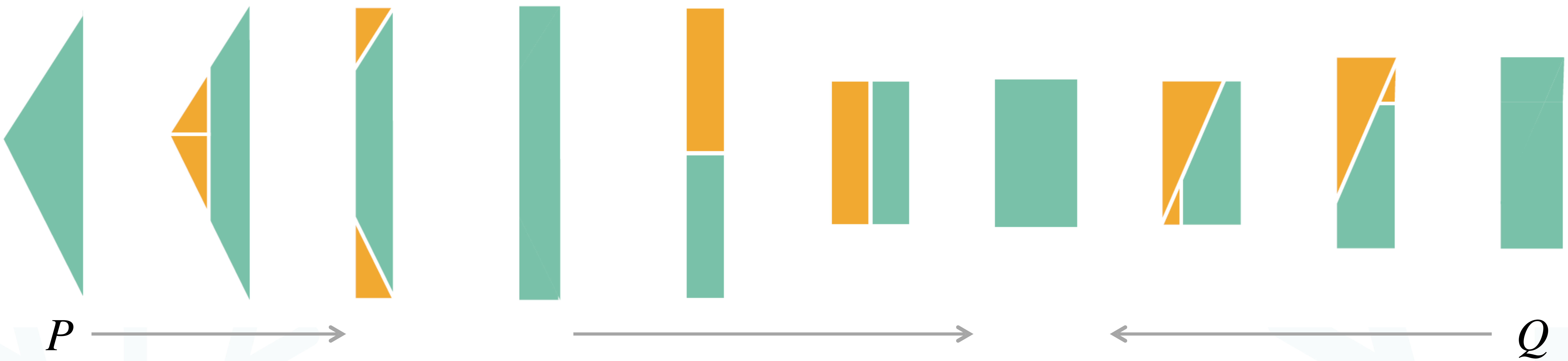


We want to show that any two simple polygons of equal area are scissors congruent.

Start with rectangles and triangles!

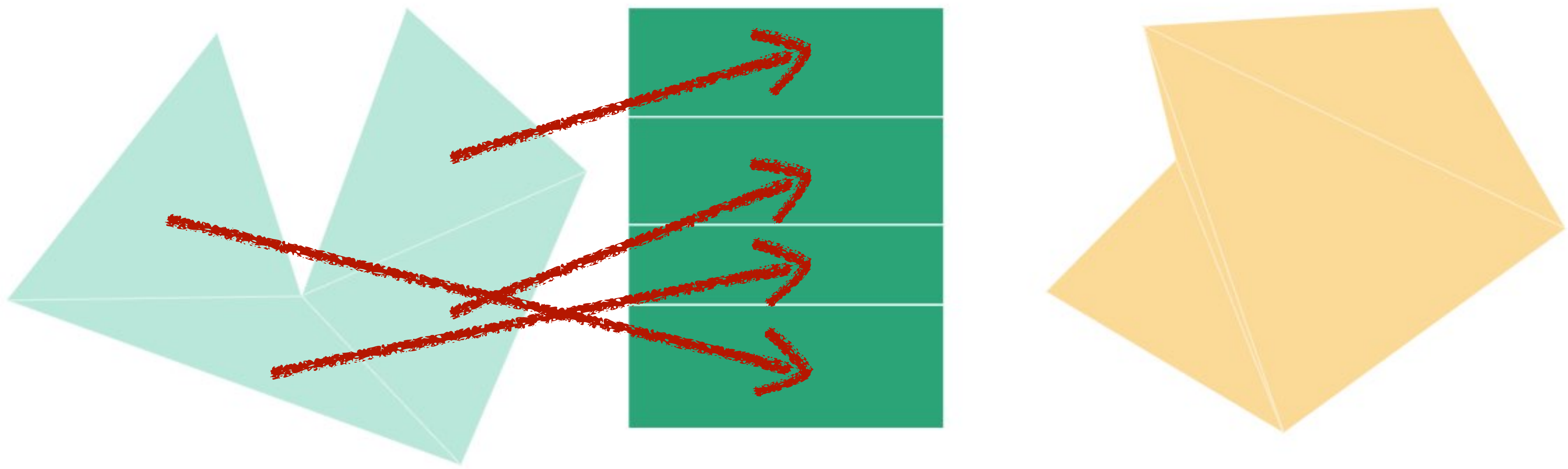
Scissors congruence

Triangles to rectangles… to other rectangles

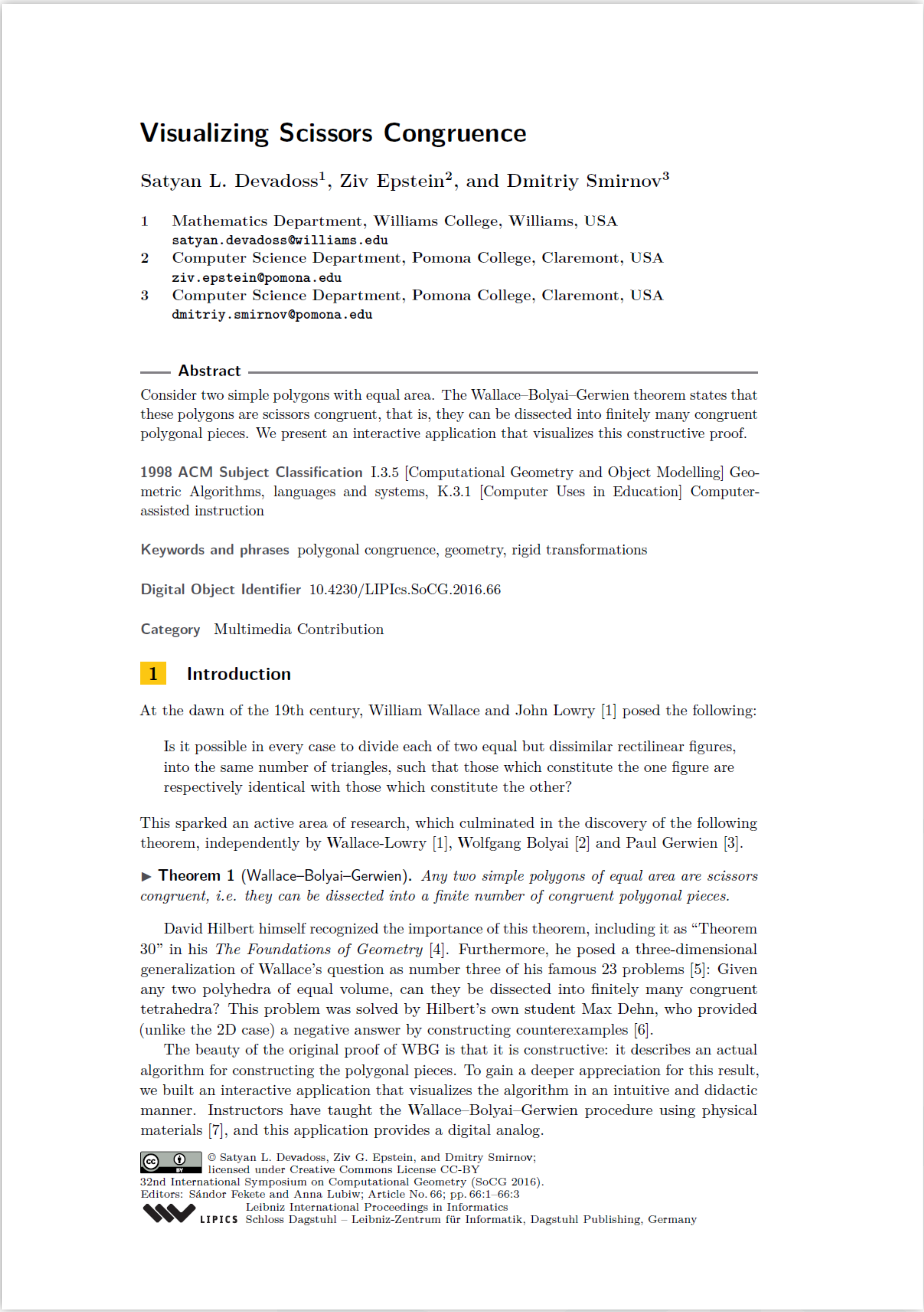


Visualizing scissors congruence

The Wallace–Bolyai–Gerwien theorem



http://dmsm.github.io/scissors-congruence/



Visualizing Scissors Congruence

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Abstract

Consider two simple polygons with equal area. The Wallace–Bolyai–Gerwien theorem states that these polygons are scissors congruent, that is, they can be dissected into finitely many congruent polygonal pieces. We present an interactive application that visualizes this constructive proof.

1998 ACM Subject Classification I.3.5 [Computational Geometry and Object Modelling] Geometric Algorithms, languages and systems, K.3.1 [Computer Uses in Education] Computer-assisted instruction

Keywords and phrases polygonal congruence, geometry, rigid transformations

Digital Object Identifier 10.4230/LIPIcs.SoCG.2016.66

Category Multimedia Contribution

1 Introduction

At the dawn of the 19th century, William Wallace and John Lowry [1] posed the following:

> Is it possible in every case to divide each of two equal but dissimilar rectilinear figures, into the same number of triangles, such that those which constitute the one figure are respectively identical with those which constitute the other?

This sparked an active area of research, which culminated in the discovery of the following theorem, independently by Wallace-Lowry [1], Wolfgang Bolyai [2] and Paul Gerwien [3].

▶ **Theorem 1** (Wallace–Bolyai–Gerwien). *Any two simple polygons of equal area are scissors congruent, i.e. they can be dissected into a finite number of congruent polygonal pieces.*

David Hilbert himself recognized the importance of this theorem, including it as "Theorem 30" in his *The Foundations of Geometry* [4]. Furthermore, he posed a three-dimensional generalization of Wallace's question as number three of his famous 23 problems [5]: Given any two polyhedra of equal volume, can they be dissected into finitely many congruent tetrahedra? This problem was solved by Hilbert's own student Max Dehn, who provided (unlike the 2D case) a negative answer by constructing counterexamples [6].

The beauty of the original proof of WBG is that it is constructive: it describes an actual algorithm for constructing the polygonal pieces. To gain a deeper appreciation for this result, we built an interactive application that visualizes the algorithm in an intuitive and didactic manner. Instructors have taught the Wallace–Bolyai–Gerwien procedure using physical materials [7], and this application provides a digital analog.

32nd International Symposium on Computational Geometry (SoCG 2016).
Editors: Sándor Fekete and Anna Lubiw; Article No. 66; pp. 66:1–66:3
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Scissors congruence – Notes and open problems

MATHEMATICAL NOTES

EDITED BY DAVID DRASIN

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ON DIVIDING A SQUARE INTO TRIANGLES

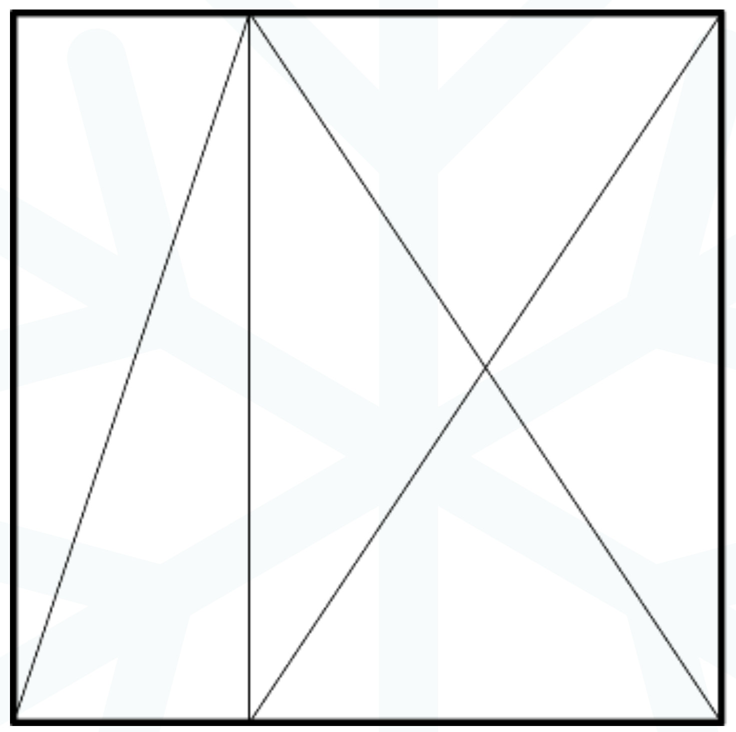
PAUL MONSKY, Brandeis University and Kyoto University

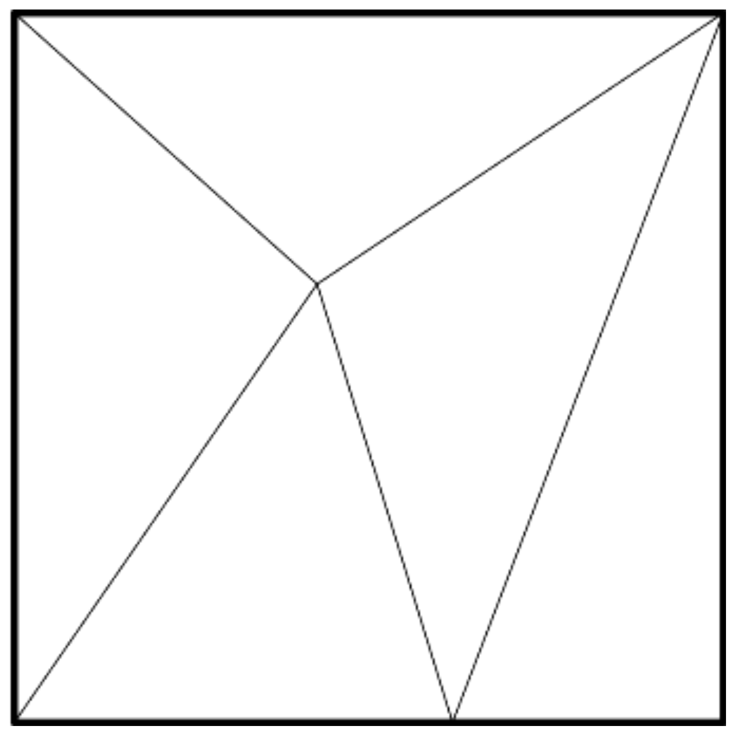
Sometime ago in this MONTHLY, Fred Richman and John Thomas [1] asked the following puzzling question:

Can a square S be divided into an odd number of nonoverlapping triangles T_i, all of the same area?

Monsky's Theorem (1970)

A square can never be divided into an odd number of non-overlapping triangles of equal area.





https://en.wikipedia.org/wiki/Monsky%27s_theorem

EXPLORATIONS ON THE WALLACE-BOLYAI-GERWIEN THEOREM

RYAN KAVANAGH

ABSTRACT. In this survey paper, we present a proof of the Wallace-Bolyai-Gerwien theorem, namely, that any two plane polygons of the same area may be decomposed into the same number of pairwise congruent triangles. Several generalisations and closely related theorems will be considered, and an original example will be explored.

1. INTRODUCTION

In 1814, Wallace [WL14] posed:

Is it possible in every case to divide each of two equal but dissimilar rectilinear figures, into the same number of triangles, such, that those which constitute the one figure are respectively identical with those which constitute the other?

https://rak.ac/files/papers/wallace-bolyai-gerwien.pdf

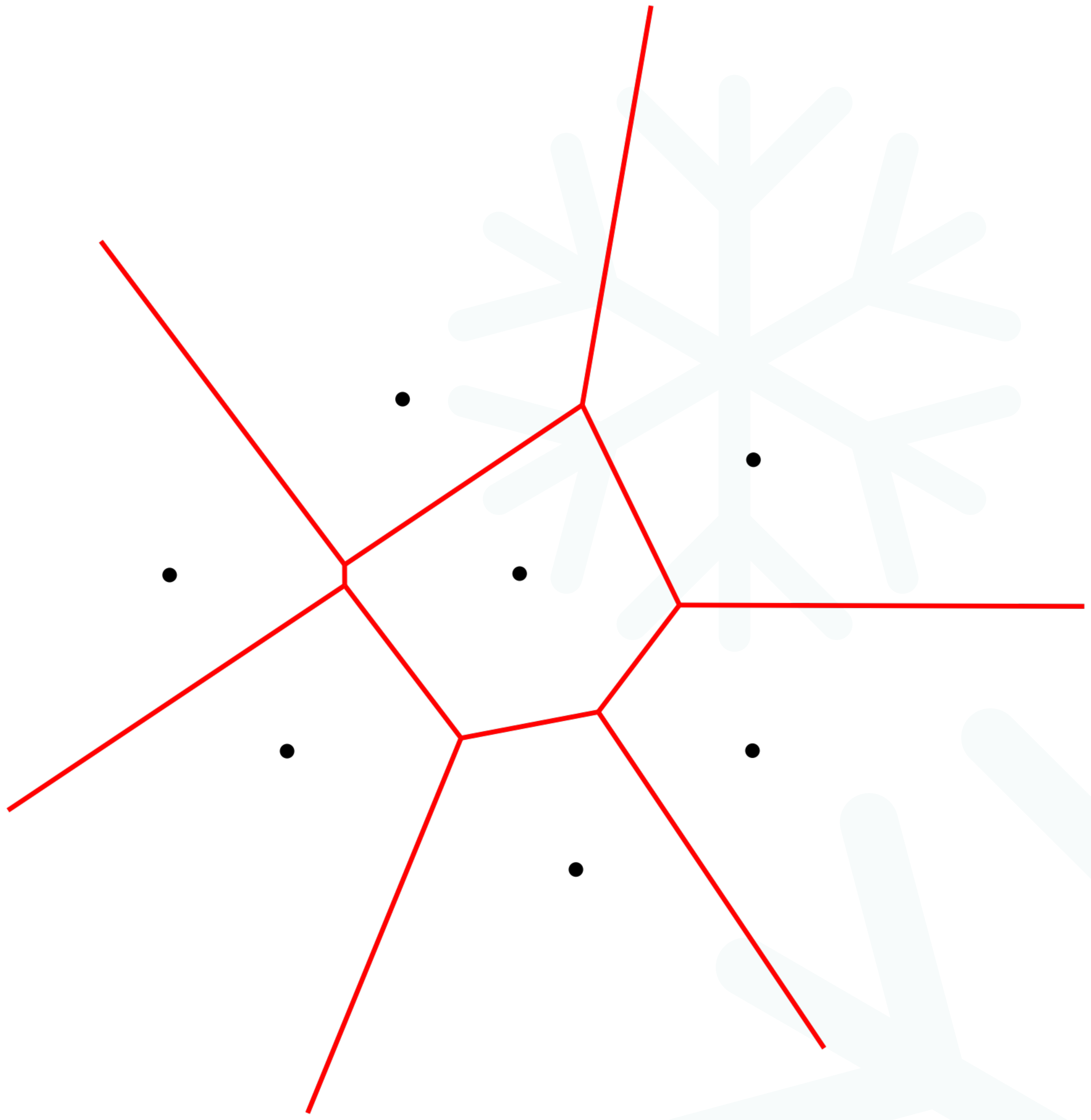
Open Question #1

Can Monsky's Theorem be generalized for cubes of higher dimension?

Open Question #2

Is it possible to bound from below the number of cuts required to show that two polygons have the same area?

Voronoi diagrams

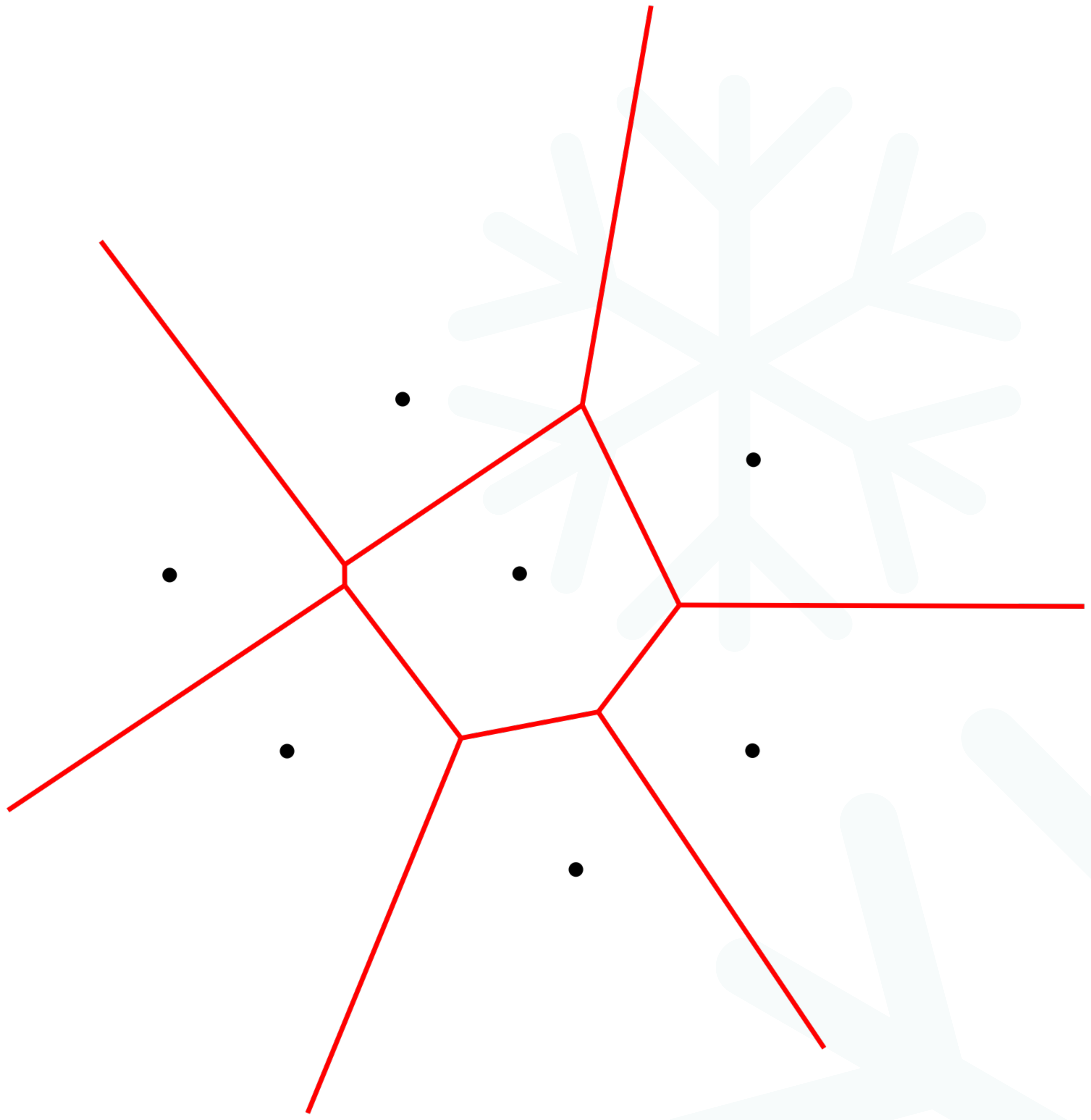


Voronoi diagrams
Properties

A Voronoi diagram $\mathrm{Vor}(P)$ divides the hyperplane based on which element of a discrete point set P is closest by some metric.

How do the unbounded faces relate to the convex hull $\mathrm{conv}(P)$*?*

What if we wanted to divide based on which ***two*** *points are closest?*



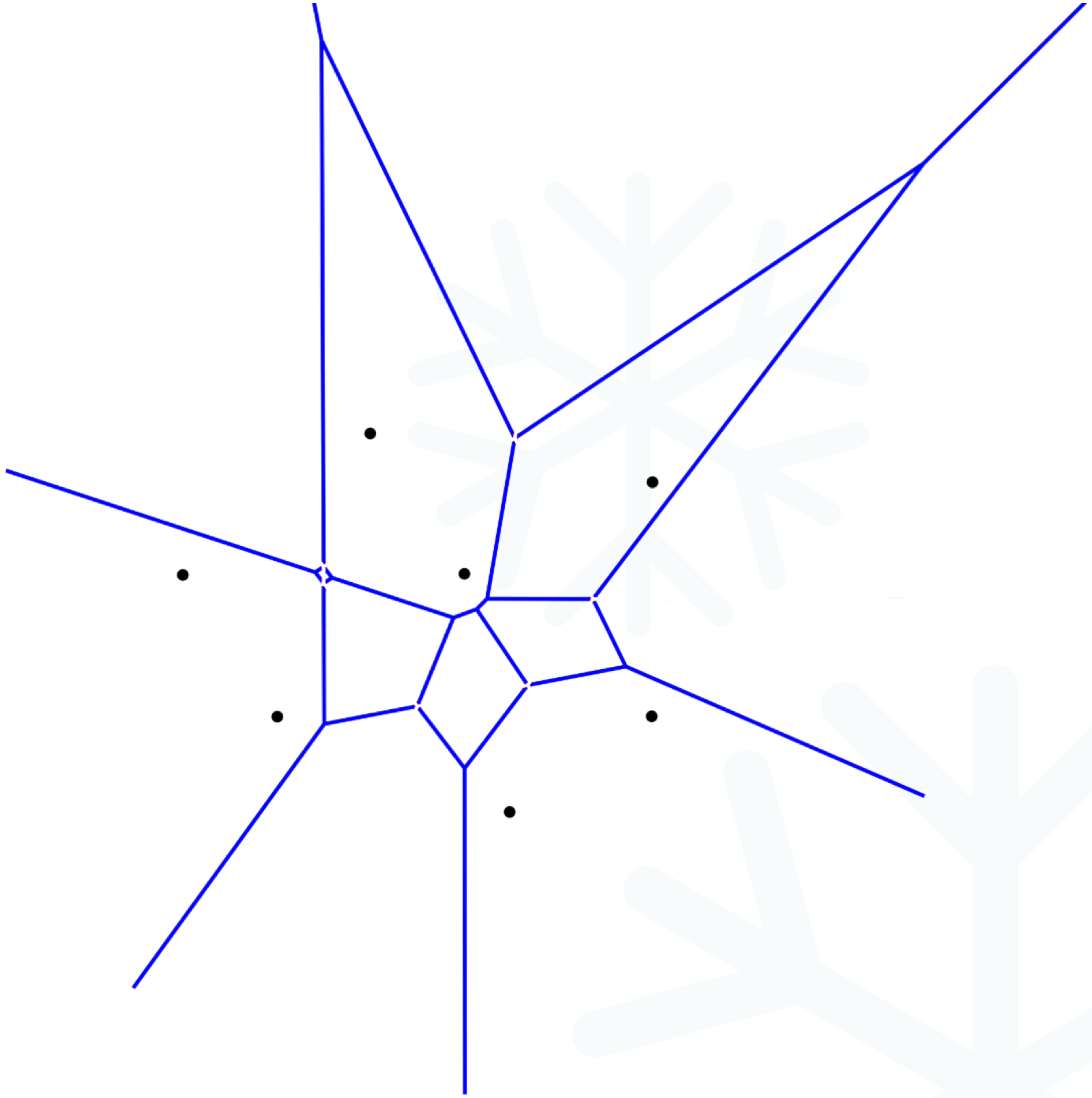
Voronoi diagrams

Higher orders

An ith order Voronoi diagram of P divides the hyperplane based on **which i points** of P are closest.

Here: 2nd order Voronoi diagram.

How can we compute this?



Voronoi diagrams

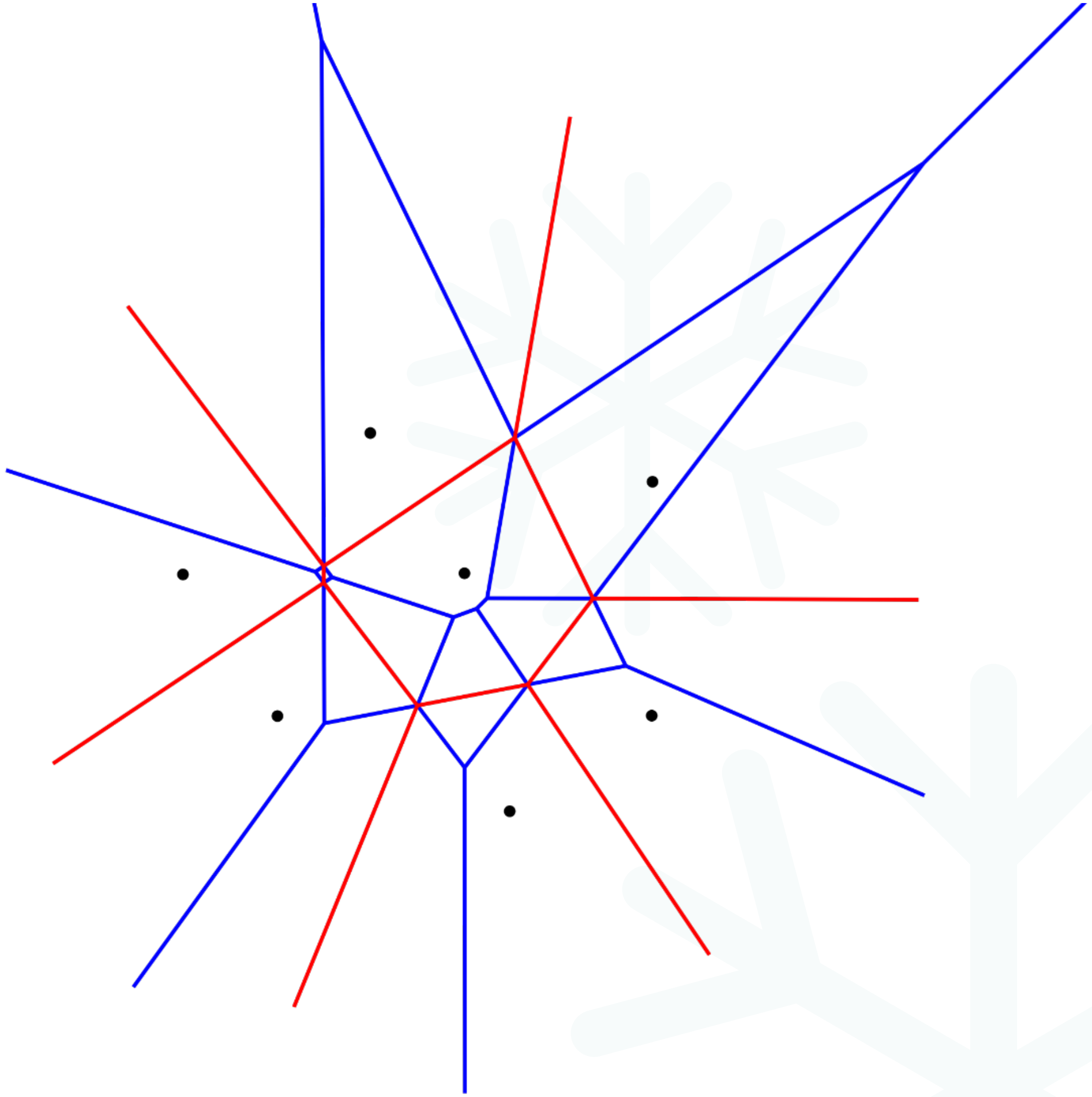
Higher orders

An ith order Voronoi diagram of P divides the hyperplane based on **which i points** of P are closest.

Here: 2nd order Voronoi diagram.

How can we compute this?

… *using* $\mathrm{Vor}(P)$.



Voronoi diagrams

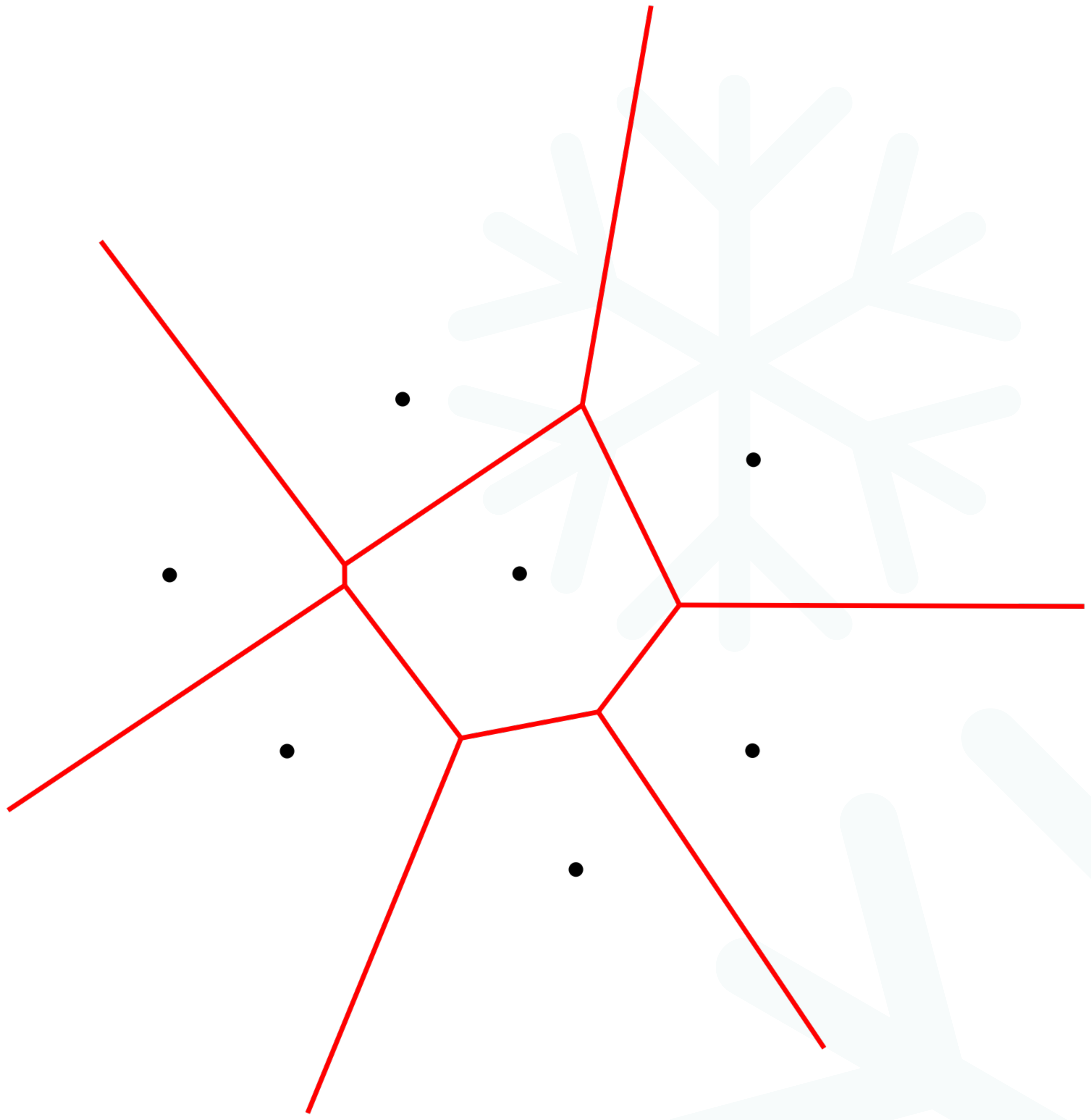
Higher orders

An ith order Voronoi diagram of P divides the hyperplane based on **which i points** of P are closest.

Here: 1st order Voronoi diagram.

How can we compute this?

… *using* $\mathrm{Vor}(P)$.

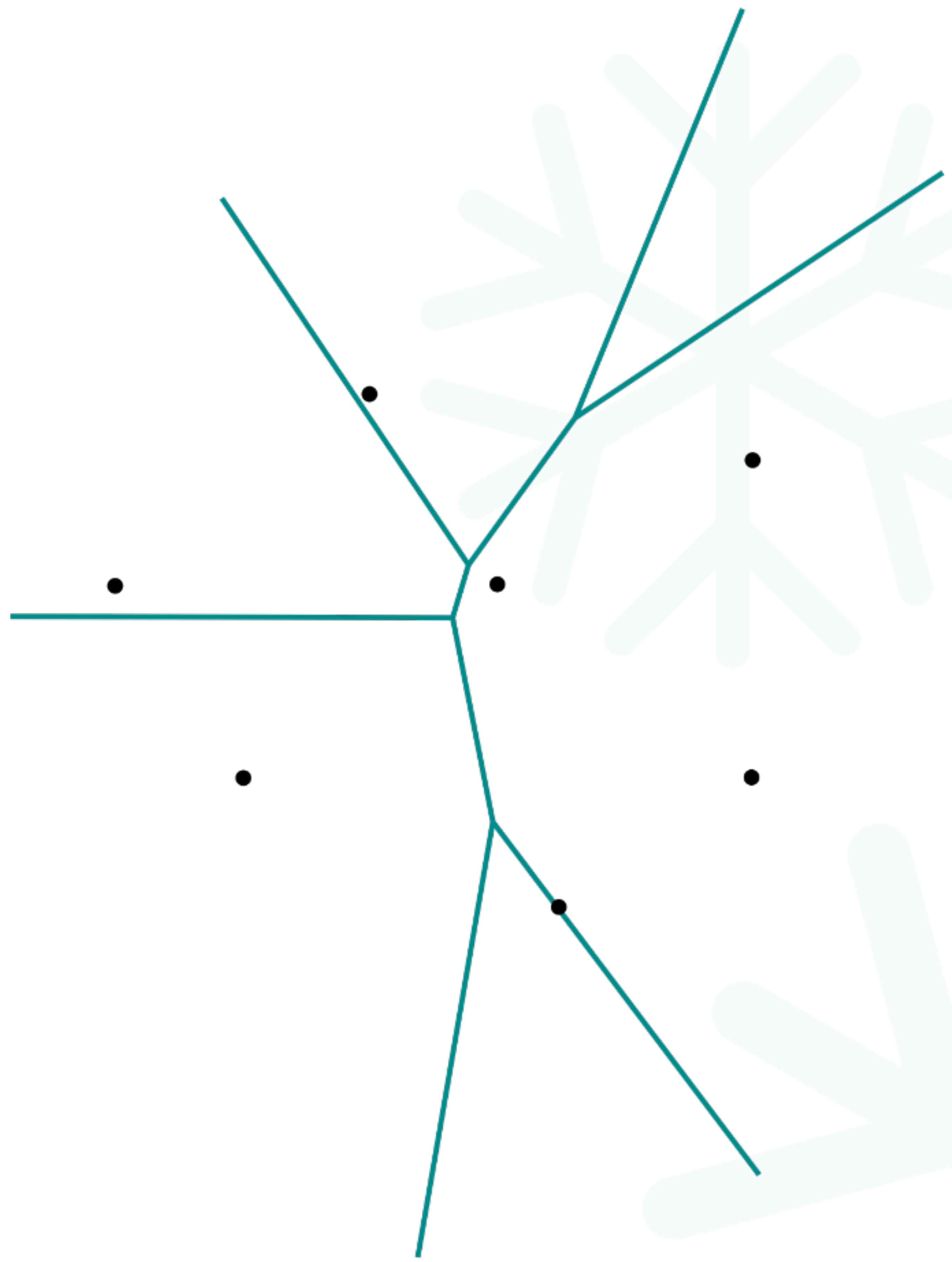


Voronoi diagrams

Farthest-point, $(n-1)$th order

An $(n-1)$th order Voronoi diagram divides the hyperplane based on which element of a discrete point set P is **farthest** by some metric.

Can you think of some relation to the convex hull $\mathrm{conv}(P)$*?*

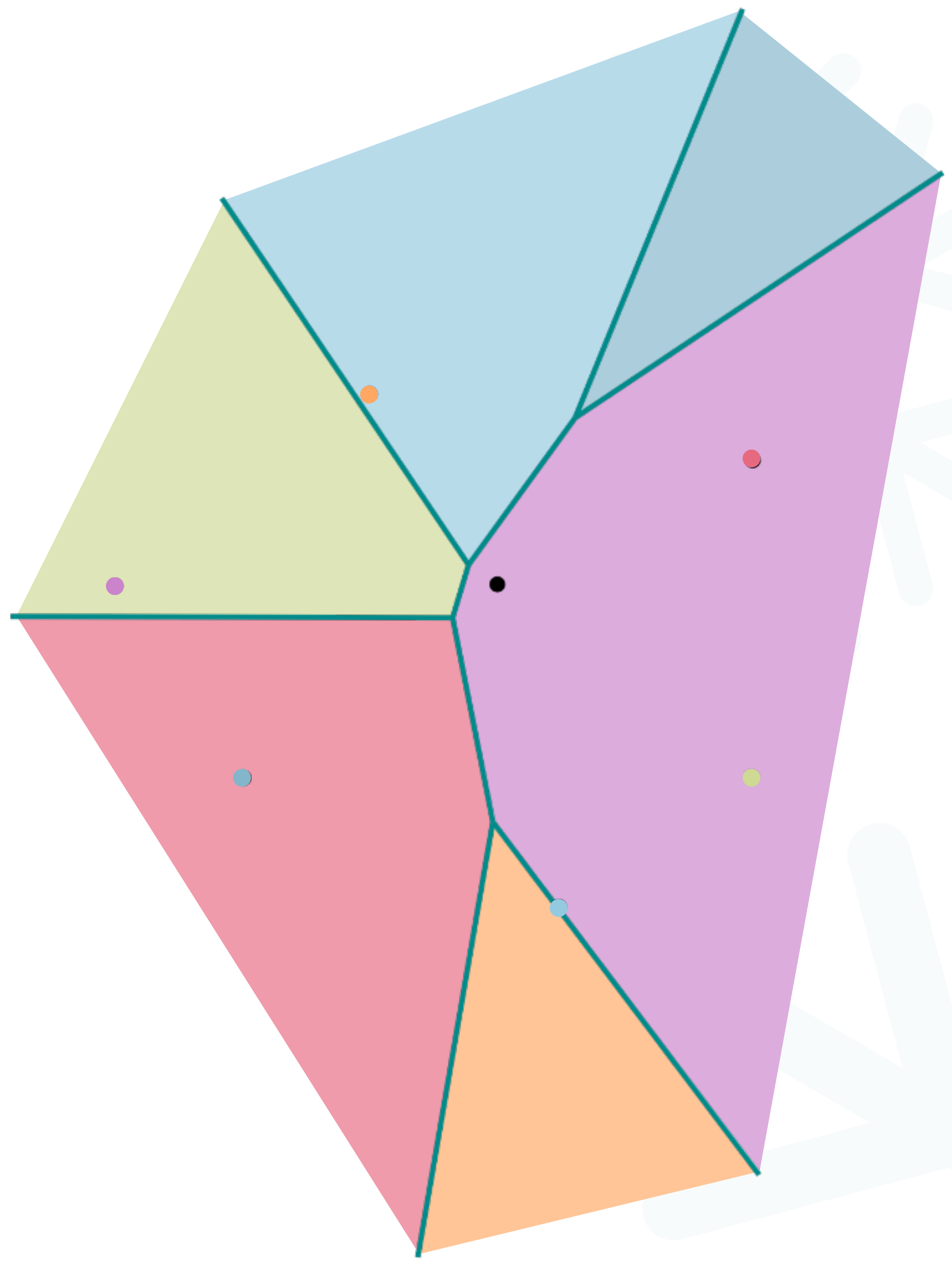


Voronoi diagrams

Farthest-point, $(n-1)$th order

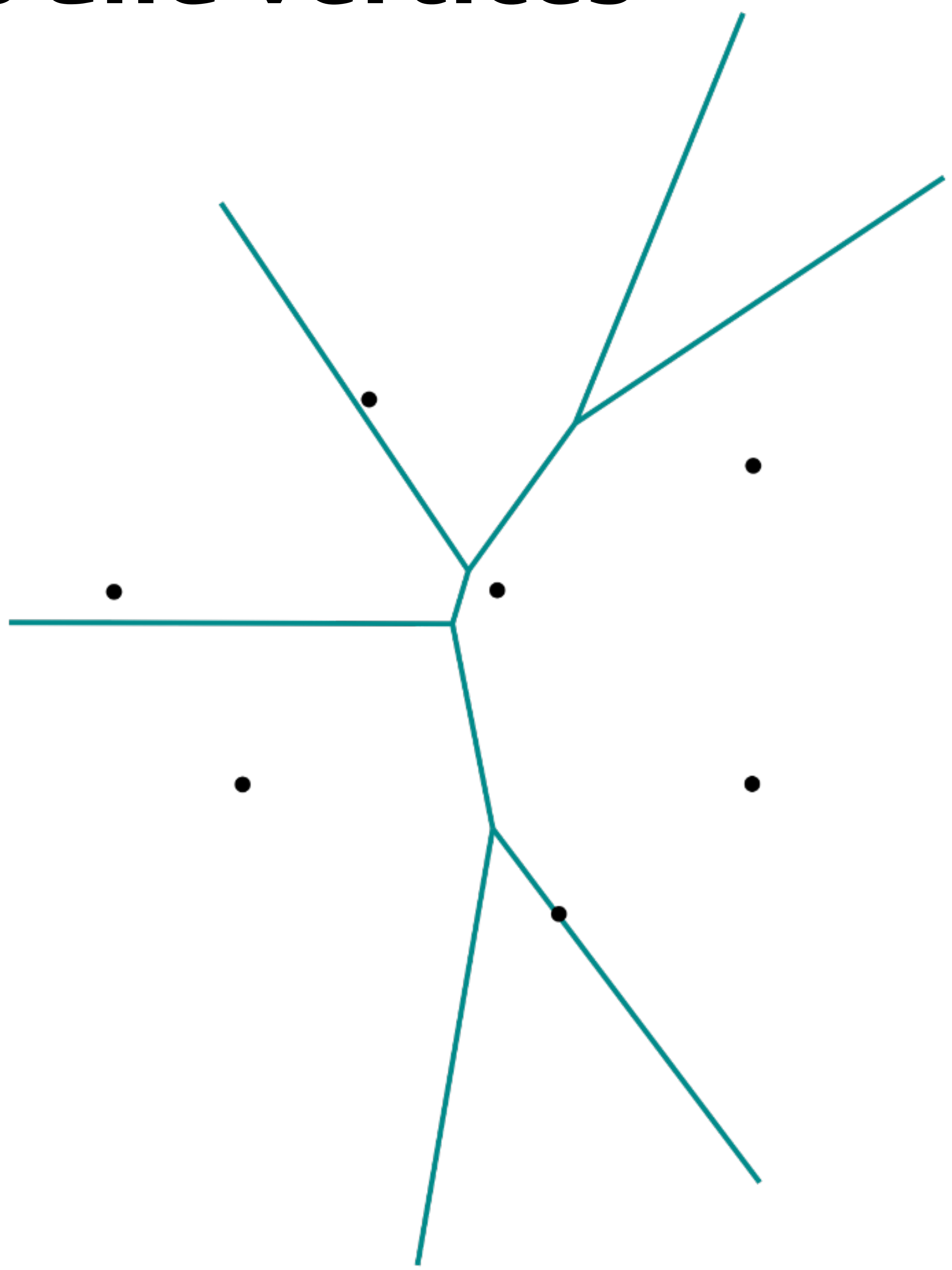
An $(n-1)$th order Voronoi diagram divides the hyperplane based on which element of a discrete point set P is **farthest** by some metric.

All cells are unbounded, i.e., the dual graph is a tree. A point $p \in P$ has a non-empty Voronoi region exactly if it lies on the boundary of the convex hull $\mathrm{conv}(P)$.



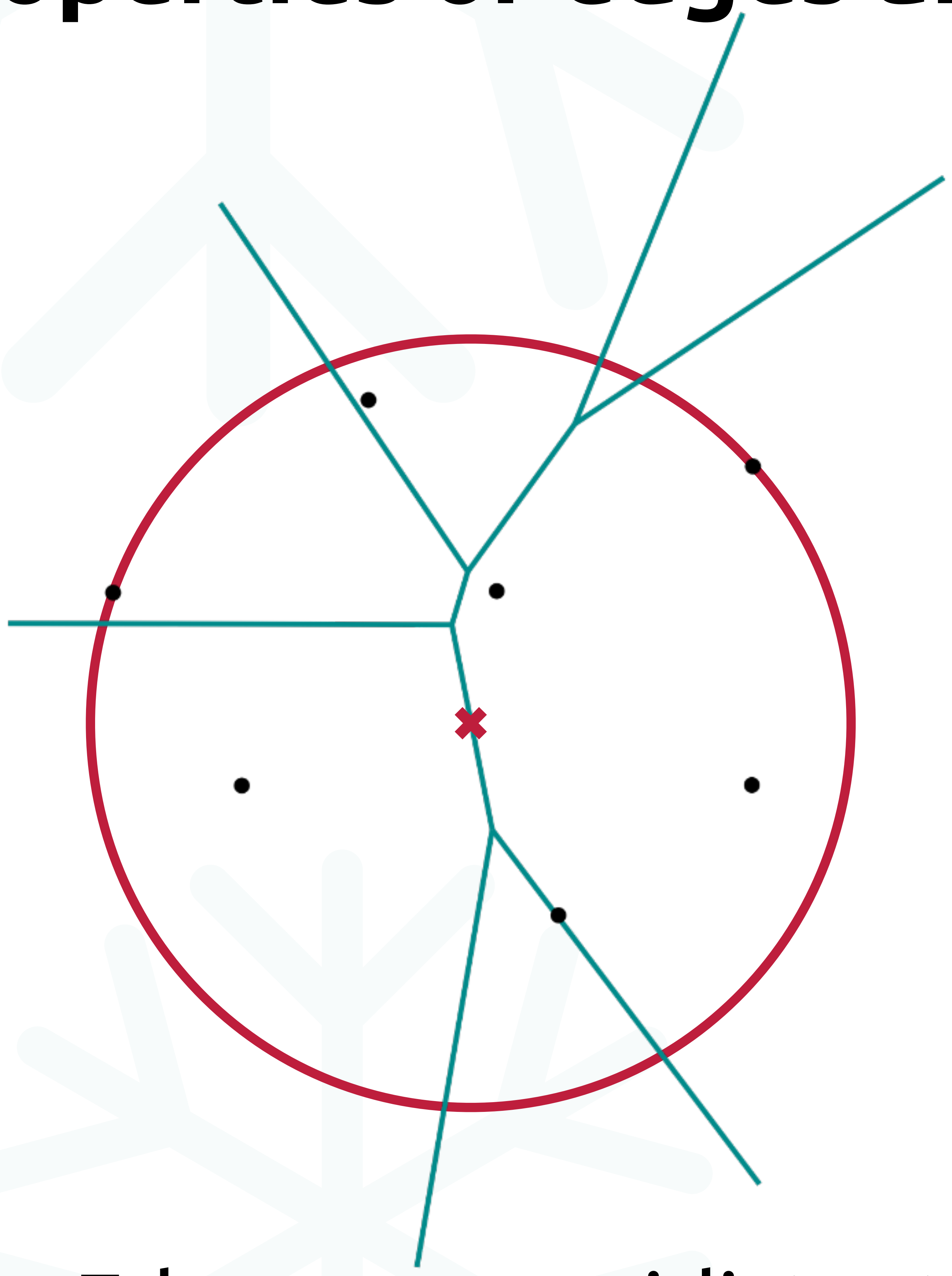
Farthest-point Voronoi diagrams

Properties of edges and vertices

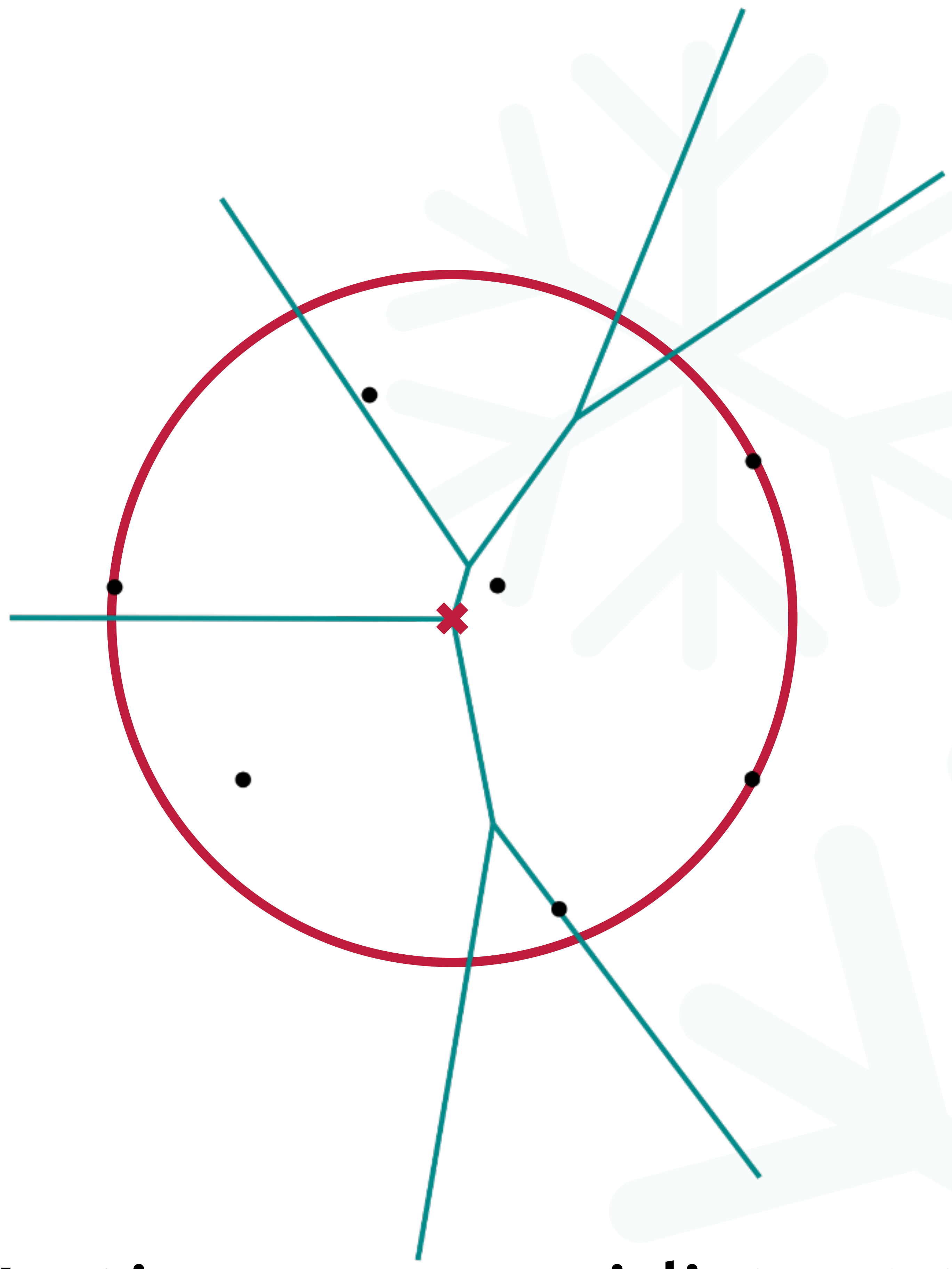


Farthest-point Voronoi diagrams

Properties of edges and vertices



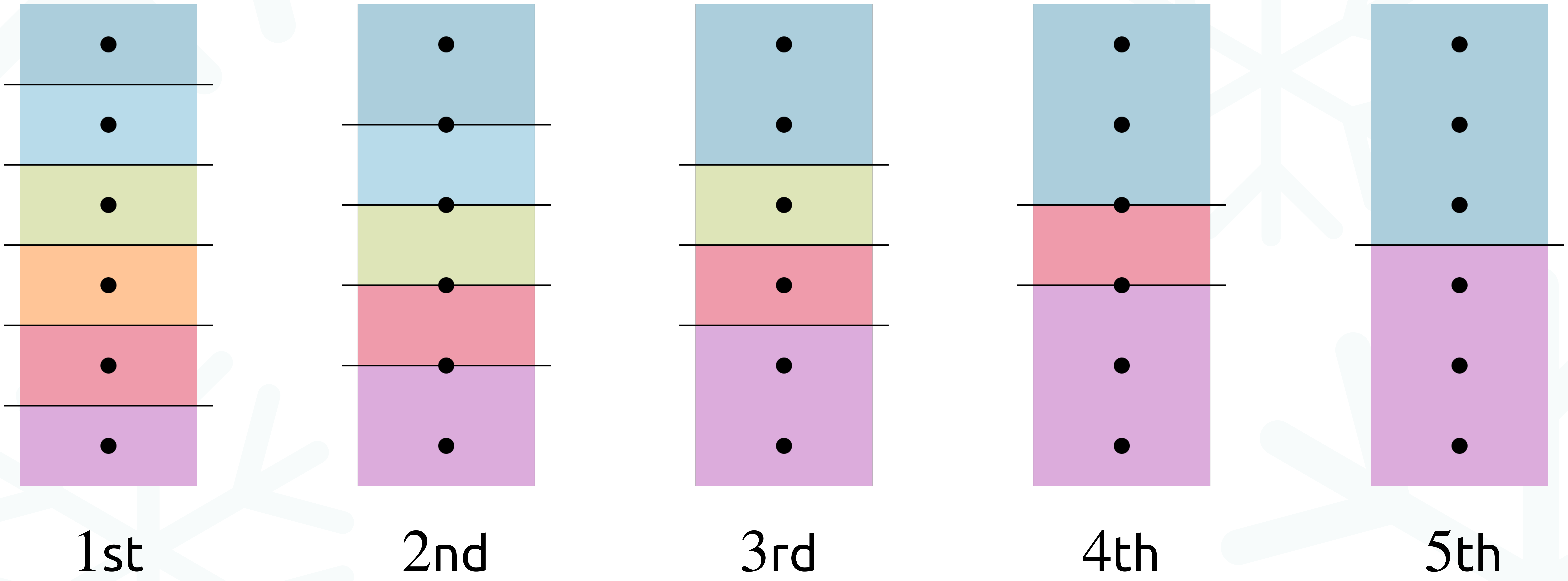
Edges are equidistant to two sites, closer to all others.



Vertices are equidistant to at least three sites, closer to all others.

Degenerate cases: Collinearity

What if all points lie on a line?



Enclosing disks

Smallest enclosing disk

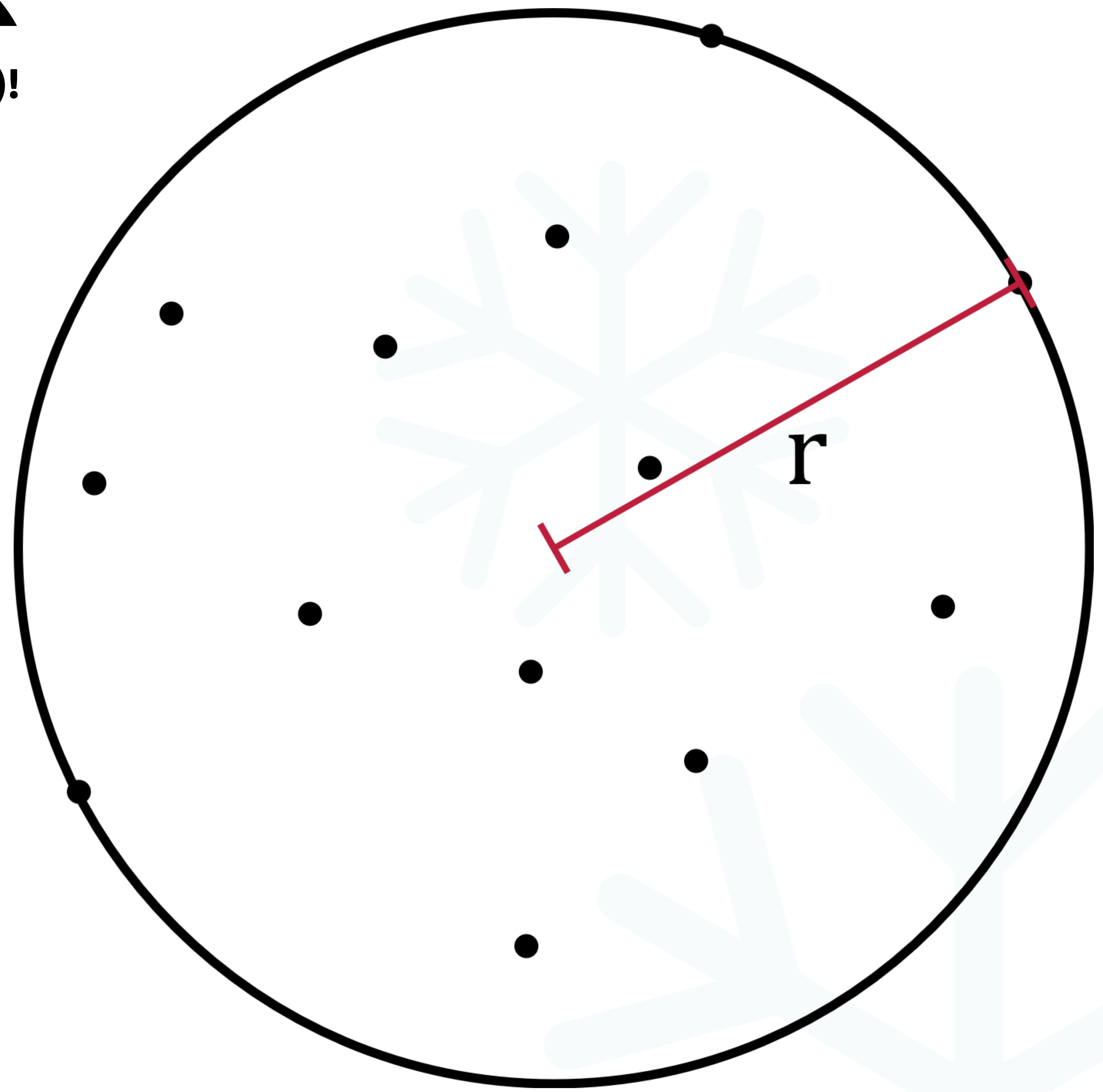
In general position (no four points on a common circle this time)!

Given: Points $P := p_1, \ldots, p_n$ in the plane, in general position.

Wanted: An enclosing disk $\mathrm{md}(P)$ of minimal radius r.

Can you characterise $\mathrm{md}(P)$ *based on* P*?*

Can you think of a fast approximation method? Which factor can you achieve?



Smallest enclosing disk

A $\sqrt{2}$-approximation

Given: Points $P := p_1, \ldots, p_n$ in the plane, in general position.

Idea: Compute in $\mathcal{O}(n)$ an axis-aligned bounding box via $\min$ and $\max$ coordinates, use the smallest enclosing disk.

The diameter of this disk is larger than $\max\{\delta_x, \delta_y\}$, which bounds the diameter of any enclosing disk from below, by a factor of no more than $\sqrt{2}$.