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AN EFFICIENT ALGORITH FOR DETERMINING THE CONVEX HULL OF A FINITE PLANAR SET

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convex hull algorithm

Given a finite set $S = \{s_1, \ldots s_n\}$ in the plane, it is frequently of interest to find the convex hull CH(S) of S. In this note we describe an algorithm which determines CH(S) in no more than $(n \log n)/(\log 2) + cn$ "operations" where c is a small positive constant which depends upon what is meant by an "operation".

The algorithm we give determines which points of S are the extreme points of CH(S). These, of course, define CH(S). The algorithm proceeds in five steps.

Step 1: Find a point P in the plane which is in the interior of CH(S). At worst, this can be done in $c_1 n$ steps by testing 3 element subsets of S for collinearity, discarding middle points of collinear sets and stopping when the first noncollinear set (if there is one), say x, y and z, is found. P can be chosen to be the centroid of the triangle formed by x, y and z.

Step 2: Express each $s_i \in S$ in polar coordinates with origin P and $\theta = 0$ in the direction of an arbitrary fixed half-line L from P. This conversion can be done in $c_2 n$ operations for some fixed constant c_2.

Step 3: Order the elements $\rho_k \exp(\mathrm{i}\theta_k)$ of S in terms of increasing θ_k. This is well known to be possible in essentially $(n \log n)/\log 2$ comparisons (cf. [1]). We now have S in the form $S = \{r_1 \exp(\mathrm{i}\varphi_1), \ldots, r_n \exp(\mathrm{i}\varphi_n)\}$ with $0 \leqslant \varphi_1 \leqslant \ldots \leqslant \varphi_n < 2\pi$ and $r_i \geqslant 0$ (cf. fig. 1). Note that by the choice of P, $\varphi_{k-1} - \varphi_k < \pi$ where the index addition is modulo n.

Step 4: If $\varphi_i = \varphi_{i+1}$ then we may delete the point with the smaller amplitude since it clearly cannot be an extreme point of CH(S). Also any point with $r_i = 0$ can be deleted. We can eliminate all these points in less than n comparisons, and by relabelling the remaining points, we can set $S' = \{r_1 \exp(\mathrm{i}\varphi_1), \ldots, r_{n'}, \exp(\mathrm{i}\varphi_{n'})\}$ where $n' \leqslant n$.

Step 5: Start with three consecutive points in S', say, $r_k \exp(\mathrm{i}\varphi_k)$, $r_{k+1} \exp(\mathrm{i}\varphi_{k+1})$, $r_{k+2} \exp(\mathrm{i}\varphi_{k+2})$ with $\varphi_k < \varphi_{k+1} < \varphi_{k+2}$ (cf. fig. 2). There are two possibilities:

(i) $\alpha + \beta \geqslant \pi$. Then we delete the point $r_{k+1} \exp(\mathrm{i}\varphi_{k+1})$ from S' since it cannot be an extreme point of CH(S), and return to the beginning of step 5 with the points $r_k \exp(\mathrm{i}\varphi_k)$, $r_{k+1} \exp(\mathrm{i}\varphi_{k+1})$, $r_{k+2} \exp(\mathrm{i}\varphi_{k+2})$ replaced by $r_{k-1} \exp(\mathrm{i}\varphi_{k-1})$, $r_k \exp(\mathrm{i}\varphi_k)$, $r_{k+2} \exp(\mathrm{i}\varphi_{k+2})$ (where indices are reduced modulo n').

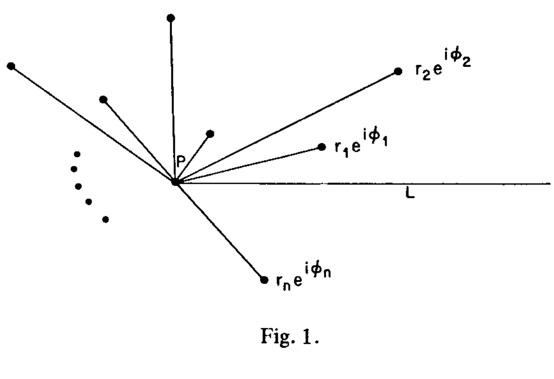

Fig. 1.

(ii) $\alpha + \beta < \pi$. Return to the beginning of step 5 with the points $r_k \exp(i\varphi_k)$, $r_{k+1} \exp(i\varphi_{k+1})$, $r_{k+2} \exp(i\varphi_{k+2})$ replaced by $r_{k+1} \exp(i\varphi_{k+1})$, $r_{k+2} \exp(i\varphi_{k+2})$, $r_{k+3} \exp(i\varphi_{k+3})$.

By noting that each application of step 5 *either* reduces the number of possible points of CH(S) by one *or* increases the current total number of points of S' considered by one, an easy induction argument shows that with less than $2n'$ iterations of step 5, we must be left with exactly the subset of S of all extreme points of CH(S). This completes the algorithm.

The reader may find it instructive to consider a small example of ten points or so. Computer implementation of this algorithm makes it quite feasible to consider examples with $n = 50\,000$.

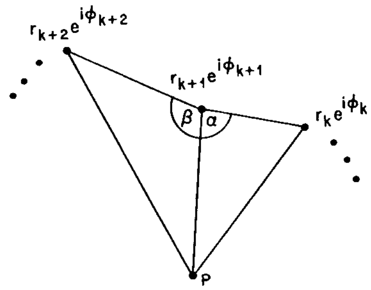

Fig. 2.

Reference

[1] L.R. Ford and S.M. Johnson, A tournament problem, Amer. Math. Monthly 66, 5 (1959) 387.