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Theoretical Computer Science 310 (2004) 457–467

Theoretical Computer Science

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Competitive facility location: the Voronoi game☆

Hee-Kap Ahn[a], Siu-Wing Cheng[b], Otfried Cheong[c], Mordecai Golin[b], René van Oostrum[c,*]

[a] *Image Media Research Center, Korea Institute of Science & Technology, P.O. Box 131, CheongRyang, Seoul, South Korea*

[b] *Department of Computer Science, HKUST, Clear Water Bay, Kowloon, Hong Kong*

[c] *Institute of Information & Computing Sciences, Utrecht University, Netherlands*

Received 22 August 2003; accepted 10 September 2003
Communicated by G. Ausiello

Abstract

We consider a competitive facility location problem with two players. Players alternate placing points, one at a time, into the playing arena, until each of them has placed n points. The arena is then subdivided according to the nearest-neighbor rule, and the player whose points control the larger area wins. We present a winning strategy for the second player, where the arena is a circle or a line segment. We permit variations where players can play more than one point at a time, and show that the first player can ensure that the second player wins by an arbitrarily small margin.

Keywords: Facility location; Game theory

1. Introduction

The classical facility location problem [6] asks for the optimum location of a new facility (police station, super market, transmitter, etc.) with respect to a given set

☆ The work described in this paper has been supported by the Research Grants Council of Hong Kong, China (Project No. HKUST6074/97E, HKUST8088/99E, HKUST6094/99E, HKUST6162/00E, and HKUST6137/98E). Part of the work was done while the first, third and fifth authors were at the Department of Computer Science, HKUST, Hong Kong.

* Corresponding author. Department of Computer Science, University of Utrecht, P.O. Box 80-089, Utrecht, TB 3508, Netherlands.

E-mail addresses: heekap@kist.re.kr (H.-K. Ahn), scheng@cs.ust.hk (S.-W. Cheng), otfried@cs.uu.nl (O. Cheong), golin@cs.ust.hk (M. Golin), rene@cs.uu.nl (R. van Oostrum).

0304-3975/$ - see front matter
doi:10.1016/j.tcs.2003.09.004

of customers. Typically, the function to be optimized is the maximum distance from customers to the facility—this results in the minimum enclosing disk problem studied by Megiddo [10], Welzl [14] and Aronov et al. [2].

Competitive facility location deals with the placement of sites by competing market players. Geometric arguments are combined with arguments from *game theory* to see how the behavior of these decision makers affect each other. Competitive location models have been studied in many different fields, such as spatial economics and industrial organization [1,11], mathematics [8] and operations research [4,9,13]. Comprehensive overviews of competitive facility locations models are the surveys by Friesz et al. [13], Eiselt and Laporte [4] and Eiselt et al. [5].

We consider a model where the behavior of the customers is deterministic in the sense that a facility can determine the set of customers more attracted to it than to any other facility. This set is called the *market area* of the facility. The collection of market areas forms a tessellation of the underlying space. If customers choose the facility on the basis of distance in some metric, the tessellation is the Voronoi diagram of the set of facilities [12].

We address a competitive facility location problem that we call the *Voronoi game*. It is played by two players, White and Black, who place a specified number, n, of facilities in a region U. They alternate placing their facilities one at a time, with White going first (as in Chess). After all $2n$ facilities have been placed, their decisions are evaluated by considering the *Voronoi diagram* of the $2n$ points. The player whose facilities control the larger area wins.

The most natural Voronoi game is played in a two-dimensional arena U using the Euclidean metric. Unfortunately, nobody seems to know how to win this game, even for very restricted regions U, unless the game is reduced to a single round [3]. In this note we present strategies for winning one-dimensional versions of the game, where the arena is a circle or a line segment, and variations. In other words, we consider competitive facility location on circles and intervals.

Section 3 discusses the simplest game, on the circle. It is obvious that Black can always achieve a tie by playing on the antipode of White's move. One might try to tweak this strategy such that it results in a win for Black. This does not seem to work, and we present instead a quite different winning strategy for Black. Our strategy does not require the players to play one point per round—the game can proceed in batches of points. The rules are made precise in Section 2.

In Section 4 we turn to the line segment arena. It would appear that White has an advantage here, because he can play the midpoint of the segment in his first move. We show that this does not help, and prove that Black still has a winning strategy. The strategy is quite similar to the one for the circle case, but its analysis (because of a loss of symmetry) is more detailed.

In Section 5 we discuss whether Black can win by a higher margin than our strategies permit. It turns out that this is not the case, as White can get as close to a tie as he wishes.

2. The game

We start with a definition of the game as well as a few lemmas that will hold for both the circle and the line segment version.

There are two players, White and Black, each having n points to play, where $n>1$. The players alternate placing points on a smooth curve C, which can be either open or closed. As in Chess, White starts the game, placing the first *batch* of points, Black the second batch of points, White the third batch, etc., until all $2n$ points are played. We assume that points cannot lie upon each other. Let W be the set of white points and B be the set of black ones. After all of the $2n$ points have been played, each player receives a score equal to the total length of the curve that is closer to that player than to the other, that is, White and Black have respective scores

$$\mathscr{W} = \left| \left\{ x \in C\colon \min_{w \in W} d(x,w) < \min_{b \in B} d(x,b) \right\} \right|,$$
$$\mathscr{B} = \left| \left\{ x \in C\colon \min_{b \in B} d(x,b) < \min_{w \in W} d(x,w) \right\} \right|,$$

where distances are measured along the curve C. The player with the highest score (the larger curve length) wins.

Note that since we measure distances along the curve, the actual shape of the curve does not matter. If the curve is closed, we can assume it is a circle, if it is open, we can assume that it is a line segment.

The question that we address here is, *Does either player have a winning strategy and, if yes, what is it*? We will see below that Black always has a winning strategy.

We excluded the degenerate game where $n=1$. The one-point circle game ends in a tie no matter what the players do, while White can win the one-point line segment game by playing on the midpoint of the segment.

We did not specify the exact number of points played by the two players in each round, and indeed our strategies will work for various variations of the game. Suppose that there are $k \leqslant n$ rounds. Let β_i and γ_i be the numbers of points that White and Black play, respectively, in round i. We pose the following restrictions:

- $\forall 1 \leqslant i \leqslant k,\ \beta_i,\ \gamma_i > 0$,
- $\forall 1 \leqslant j \leqslant k,\ \sum_{i=1}^{j} \beta_i \geqslant \sum_{i=1}^{j} \gamma_i$,
- $\sum_{i=1}^{k} \beta_i = \sum_{i=1}^{k} \gamma_i = n$,
- $\beta_1 < n$ (for the circle game), or
- $\beta_1 = 1$ (for the line segment game).

This generalization includes the original Voronoi game, where $\beta_i = \gamma_i = 1$, and a "batched" version, in which each player plays the same number ($\geqslant 1$) of points at each turn. The parameters k, β_i and γ_i need not be fixed in advance. For example, White may decide at every move how many points he will play and then Black plays the same number.

The combinatorially inclined reader may wonder how many ways there are to choose β_i and γ_i according to the restrictions above. One can represent the sequence of moves as a sequence of $2n$ elements that are either $+1$ (a move by White) or -1 (a move by Black). The first three restrictions then translate to the requirement that the sum of the sequence be 0, and that each prefix sum be non-negative. The number of such sequences is C_n, the Catalan number [7]. In the circle game, the requirement $\beta_1 < n$ excludes one possibility, and so the number of sequences is $C_n - 1$. In the line segment game, the requirement $\beta_1 = 1$ implies that the sequence starts with $+1$, -1 and so the number of sequences is C_{n-1}.

Now assume that a set W of white points and a set B of black points has been placed on a *closed* curve. We call an arc between two consecutive white or black points an *interval*. The interior of an interval is free of white or black points. An interval is *monochromatic* if its endpoints have the same color, and *bichromatic* if they have different colors. A *white interval* is a white monochromatic one, a *black interval* a black monochromatic one.

Lemma 1. *Let W be a set of w white points and let B be a set of b black points on a closed curve. Let $n(W)$ be the number of white intervals they form and $n(B)$ the number of black ones. Then $n(W) - n(B) = w - b$.*

Proof. Suppose a point has two faces, facing to each adjacent interval. Let i be the number of bichromatic intervals. Then $2w - i$ white faces are facing white intervals, and $2b - i$ black faces are facing black intervals. It follows that $n(W) = (2w - i)/2$, $n(B) = (2b - i)/2$ and so $n(W) - n(B) = w - b$. □

Our strategies will rely on marking a set of n positions (points) on the curve. We call these positions the *keypoints* of the curve. A white or black point may fall on a keypoint. We call an interval a *key interval* if both of its endpoints are keypoints.

Lemma 2. *Let W be a set of $w \leqslant n$ white points and let B be a set of $b < w$ black points on a closed curve with n keypoints. If $W \cup B$ covers all keypoints, and there is only one white interval, and this is not a key interval, then there exists a bichromatic key interval.*

Proof. We apply the pigeon hole principle: Consider the n curve arcs formed by the n keypoints. We have $|W \cup B| \leqslant 2n - 1$, and n of the points are keypoints. That leaves at most $n - 1$ points that lie inside the n curve arcs. Therefore, one of these arcs must be free of points, forming a key interval. This key interval is not black, as there is no black interval by Lemma 1. It is not white either, as the only white interval is not a key interval, and so it is bichromatic. □

We now describe the basic keypoint strategy, showing how Black can place her first $n - 1$ points to guarantee sole possession of key intervals. Our circle and line segment strategies will be refinements of this keypoint strategy.

Keypoint strategy

Stage I: Black plays onto an empty keypoint.

Stage I ends after the last keypoint is played (by either Black or White).

Stage II: Black plays into a white key interval. We call this *breaking* the white interval.

Stage II ends when the last white key interval is broken. Note that Stage II may be skipped altogether, if no white key interval exists after Stage I.

Stage III: Black breaks a white interval.

Stage III ends before Black plays her last point, which is not included in the basic keypoint strategy.

We make two observations concerning this strategy.

Lemma 3. *After Stage* III *of the keypoint strategy, there is no white key interval.*

Proof. Let k be the number of keypoints played by White during the game. If $k \leqslant 1$, then there certainly is no white key interval, so assume $k > 1$. Because of the condition on β_1, Black has at least one keypoint, and so we have $k < n$, and White can define at most $k-1$ white key intervals.

Black has covered the remaining $n-k$ keypoints by the end of Stage I, and will play $k-1$ points in Stages II and III. Therefore, Stage II will be completed with all white key intervals broken, and no new ones can be created. □

Lemma 4. *After Stage* III *of the keypoint strategy, all black intervals are key intervals.*

Proof. The statement is true at the end of Stage I, as Black has so far only played onto keypoints, and all keypoints are covered. During Stages II and III, Black uses her points to break white intervals, and therefore creates bichromatic intervals only. White cannot create black intervals, and so, at the end of Stage III, all black intervals are indeed key intervals. □

3. The circle game

In this section, we consider the game on a circle C. We parameterize the circle using the interval $[0,1]$, where points 0 and 1 are identified. At any given time during the game the circle is partitioned into intervals. We denote the total length of all white intervals by W_m, and the total length of all the black intervals by B_m. The important thing to notice is that at the end of the game the length of each bichromatic interval is divided equally among the two players, so $\mathscr{B} - \mathscr{W} = B_m - W_m$ and Black wins if and only if $B_m > W_m$. We devise our strategy to force this to happen.

Definition 5. On the circle, the n keypoints are the points $u_i = i/n$, $i = 0, 1, \ldots, n-1$ (Fig. 1).

Since we can parameterize the circle arbitrarily, we can assume without loss of generality that White plays his first point on 0 and thus on a keypoint. We now describe Black's winning strategy. Fig. 2 shows an example.

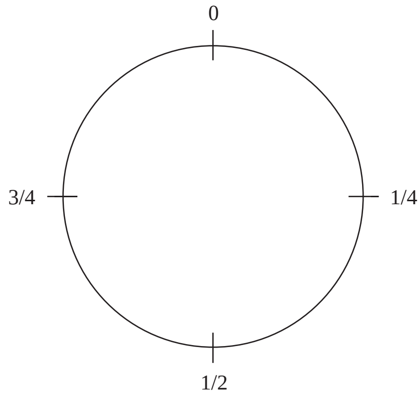

Fig. 1. There are four keypoints when $n = 4$.

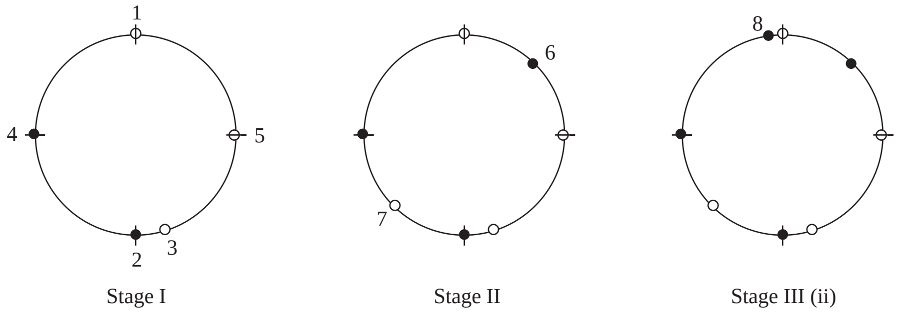

Fig. 2. There are four points to be played for both White and Black. We label the points in chronological order.

Circle strategy
Stage I: Black plays onto an empty keypoint.
Stage I ends after the last keypoint is played (by either Black or White).
Stage II: Black breaks a largest white interval.
Stage II ends before Black's last move.
Stage III: (Black's last move) There are two possibilities:

(i) If more than one white interval exists, then Black breaks a largest one.
(ii) If there is only one white interval, let ℓ be its length. Black places a point in a bichromatic key interval at distance less than $(1/n) - \ell$ from its white endpoint.

Theorem 6. *The circle strategy is a well-defined winning strategy for Black.*

Proof. We start with a simple observation. Since White's first move covers the first keypoint, Black will play onto at most $n - 1$ keypoints. Thus Stage I always ends before Black plays her last point.

Consider Stage II. Before each play by Black we have $b < w$, so by Lemma 1 there is at least one white interval on the circle, and Stage II of the strategy is indeed well defined. During Stage II, any white interval is either a key interval, or has length $< 1/n$. If a white key interval exists, it will always be longer than any non-key interval, and so Stages I and II of the circle strategy are an implementation of the keypoint strategy.

We now show why Stage III is well defined and why Black wins. Suppose that it is time for Black's last move. This implies that White has played all his points and from Lemma 1 we know that $n(W) \geqslant 1$.

If $n(W) > 1$ then the strategy is well defined: Black breaks a largest white interval. This decreases $n(W)$ by 1, so the game ends with $n(W) \geqslant 1$. By Lemma 1 we have $n(B) = n(W) \geqslant 1$, by Lemma 4 all existing black intervals are key intervals, and by Lemma 3 all existing white intervals have length strictly less than $1/n$. Since all black intervals are longer than all white intervals and there are the same number of black ones and white ones we find that $B_m > W_m$ and Black wins.

If $n(W) = 1$, the unique white interval has length $\ell < 1/n$ by Lemma 3. By Lemma 2, there exists a bichromatic key interval, and so the strategy is well defined. After Black places her last point, White still has one white interval of length ℓ while Black has one black interval of length $> \ell$. Thus $B_m > \ell = W_m$ and Black wins. □

Note that we used the condition $\beta_1 < n$ to argue that Black covers at least one keypoint. Without this condition, White can take all the keypoints, and force a tie.

4. The line segment game

We now move on to the version of the game played on a line segment C. We consider it to be horizontal and parameterized as $[0, 1]$. Note that the player with the leftmost point claims everything between 0 and the point, and the player with the rightmost point claims everything between the point and 1.

To re-use the lemmas of Section 2, we extend C into a closed curve C' by connecting the points 1 and 0 using a curve that we will call the *border arc*. The white and black points on C partition C' into intervals. Exactly one of these intervals contains the complete border arc, we call it the *border interval*.

We denote the total length of all of the white intervals by W_m, and the total length of all of the black intervals by B_m. If the border interval is monochromatic, only its part on the segment C is counted, not the part on the border arc.

Unlike other bichromatic intervals, a bichromatic border interval is not shared equally by the two players. In this case, we use W_b to denote the length of the part on C claimed by White, and B_b to denote the length of the part on C claimed by Black. If the border interval is monochromatic, then $W_b = B_b = 0$. We have $\mathscr{B} - \mathscr{W} = (B_m + B_b) - (W_m + W_b)$ and, as in Section 3, we design our strategy so that Black finishes with the right-hand side of the equation > 0.

Definition 7. On the line segment, the n keypoints are the points $u_i = 1/2n + i/n$, $i = 0, 1, \ldots, n - 1$.

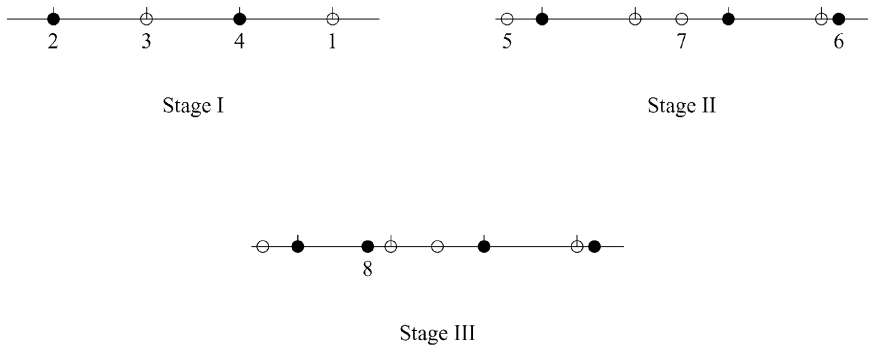

Fig. 3. There are four points to be played for both White and Black, labeled in a chronological order.

We now introduce the *line strategy*, a modified version of the circle strategy. Fig. 3 shows an example.

Line strategy

Stage I: (Black's first move) Black plays onto u_0 or u_{n-1}.

Stage II: Black plays onto an empty keypoint.

Stage II ends after the last keypoint is played by either Black or White. Note that the game may finish in Stage II.

Stage III: We distinguish two cases:

(i) If at least one white non-border interval exists, then Black breaks a largest white non-border interval.

(ii) If the border interval is the only white interval, then there are two possible cases:

   (a) One of the white endpoints of the white border interval is a keypoint: Without loss of generality assume that it is u_0 (the other case is symmetric) and the other endpoint is $1-\ell$. Black now places her new point anywhere in (ℓ, u_0).

   (b) Neither of the endpoints of the border interval are keypoints: Let ℓ be the length of the white border interval. Black places her new point in a bichromatic key interval, at distance less than $1/n-\ell$ from its white endpoint.

Stage III ends before Black's last move.

Stage IV: (Black's last move) We have two mutually exclusive cases:

   (i) If there exists more than one white interval, then Black breaks a largest non-border one.

   (ii) If there exists only one white interval, then let its length be ℓ. Black places her new point in a bichromatic key interval, at distance less than $1/n-\ell$ from its white endpoint.

Theorem 8. *The line strategy is a well-defined winning strategy for Black.*

Proof. We first argue that Stages I–III are well defined. Indeed, Stage I is valid, as White plays a single point in the first round, and cannot cover both u_0 and u_{n-1}. In Stage III, Lemma 1 guarantees the existence of at least one white interval (possibly the border interval). One of the two conditions (i) or (ii) holds, and case (i) is clearly well defined. For case (ii)(a), we observe that u_{n-1} must be black, and so $\ell < 1/2n$. Lemma 2 guarantees the existence of a bichromatic key interval in case (ii)(b).

The line segment strategy differs in at least one major aspect from the circle strategy: since we have lost circular symmetry it cannot be guaranteed that White plays onto at least one keypoint, and so it is possible that the game will end in Stage II, with Black playing all n keypoints. In this case, all black intervals (including, possibly, the border interval) are key intervals and all white intervals have length $< 1/n$. By Lemma 1, White and Black have the same number of monochromatic intervals, so $W_m < B_m$. If the border interval is monochromatic, then $W_b = B_b = 0$ and Black wins. If the border interval is bichromatic, then one of its endpoints must be the black point u_0 or u_{n-1}. This implies that $W_b < B_b = 1/2n$, and Black wins.

In what follows, we may therefore assume that White plays onto at least one keypoint during the game. First, note that under this assumption Stage II always ends with all keypoints covered, and Stage IV is reached. We distinguish two cases, depending on whether case (ii) of Stage III occurred during the game.

We assume first that it did not occur. In this case, Stages I–III of the line segment strategy are an implementation of the keypoint strategy, and so Lemmas 3 and 4 apply. In particular, there is no white key interval before Stage IV, and so Lemmas 1 and 2 prove the validity of Stage IV. Note that the bichromatic key interval in case (ii) could be the border interval.

If Black plays case (ii) of Stage IV, then White ends up with total length ℓ, Black has $> \ell$, and so Black wins. We assume therefore that Black plays case (i) of Stage IV. There are equal numbers of black and white intervals at the end of the game, all white intervals have length $< 1/n$ by Lemma 3, all black intervals are key intervals by Lemma 4, and so we have $W_m < B_m$. If the border interval is monochromatic, then $W_b = B_b = 0$ and Black wins. If the border interval is bichromatic, then its black endpoint must be u_0 or u_{n-1} (since Black plays case (i) of Stage III only), and so $W_b \leqslant B_b = 1/2n$ and Black wins.

We now consider the remaining case, where case (ii) of Stage III does occur. There are no white key intervals when it occurs, and since none can be created later, none exist before Stage IV. Lemmas 1 and 2 again prove the validity of Stage IV. If Black plays case (ii) of Stage IV, then White ends up with total length ℓ, Black has $> \ell$, and so Black wins.

We can therefore assume in the following that Black plays case (i) of Stage IV. Consider the *last* occurrence of case (ii) of Stage III. Before Black's move, the border interval is the only white interval.

If Black plays case (ii)(a), there are no monochromatic intervals at all after Black's move. From this moment until the end of the game, Black only breaks white non-border intervals (case (i) of Stages III and IV). This implies that no new black intervals are created, and therefore, by Lemma 1, there are no monochromatic intervals at the end of the game. The border interval is still bichromatic, and its black endpoint is unchanged. We therefore have $W_m = B_m = 0$ and $W_b < B_b$, which implies $\mathscr{W} < \mathscr{B}$ and Black wins.

If Black plays case (ii)(b), then after Black's move there is a single-black interval of length $\ell' > \ell$ and a single-white (border) interval of length ℓ. As Black only breaks white non-border intervals during the rest of the game, no new black intervals are created, and the white border interval is never broken. By Lemma 1, this implies that the game ends with a single-black interval and the single-white border interval. The black interval is unchanged and has length $\ell' > \ell$, the border interval has length $\leqslant \ell$, and so we have $W_m \leqslant \ell < \ell' = B_m$ and $W_b = B_b = 0$, which implies $\mathscr{W} < \mathscr{B}$, and Black wins. □

While we only required $\beta_1 < n$ in the circle game, we demand $\beta_1 = 1$ in the line segment game. We used this to argue that Black can cover at least one of u_0 or u_{n-1}. Perhaps surprisingly, the restriction is really necessary: it is not hard to give a winning strategy for *White* in the game where $\beta_1 = \gamma_1 \geqslant 2$ and $\beta_i = \gamma_i = 1$ for $i > 1$. On the other hand, it is not true that White always wins if $\beta_1 > 1$. For example, Black has a winning strategy for the game where $\beta_1 = n - 1$, $\gamma_1 = \beta_2 = 1$ and $\beta_2 = n - 1$.

5. White's defense

The strategies given in the previous sections allow Black to win the Voronoi game. The margin by which Black wins is very small, however, and in fact White can make it as small as he wants. Is there a strategy that would allow Black to win by a larger margin? The answer is no.

Theorem 9. *For any* $\varepsilon > 0$, *White can capture at least* $\frac{1}{2} - \varepsilon$ *of the curve, in both the circle and the line segment game.*

Proof. White's strategy is simply: play the n points within distance $\varepsilon/2n$ of the n keypoints. As a result, at the end of the game, any white interval has length at least $1/n - \varepsilon/n$, while any black interval has length at most $1/n + \varepsilon/n$. Since the number of white and black intervals is equal and less than n, we have $B_m - W_m \leqslant 2\varepsilon(n-1)/n$. In the line segment game, we additionally observe that if the border interval is bichromatic, we have $W_b \geqslant 1/2n - \varepsilon/2n$, while $B_b \leqslant 1/2n + \varepsilon/2n$. It follows that $\mathscr{B} - \mathscr{W} \leqslant 2\varepsilon$, and so $\mathscr{W} \geqslant \frac{1}{2} - \varepsilon$. □

6. Conclusions

We have given strategies for one-dimensional competitive facility location, allowing the second player, Black, to win. We have also shown that the first player, White, can keep the winning margin as small as he wishes. For all practical purposes, we can conclude that the one-dimensional Voronoi game ends in a tie.

In fact, our strategies rely on the continuity of the arena—without continuity they do not guarantee a win for Black. Imagine, for instance, a game where both players play 10 points on the line segment, but point locations are restricted to multiples of $\frac{1}{100}$. In the game where $\beta_i = \gamma_i = 1$, White can then achieve a tie by ensuring that all white intervals have length $\frac{1}{10}$ or $\frac{9}{100}$, while all black intervals have length at most $\frac{9}{100}$.

Similarly, if players are allowed to place points infinitesimally close to their opponent (that is, on the same location, but indicating a "side"), then White can enforce a tie by playing the keypoints.

Do our findings have any bearing on the two-dimensional Voronoi game? The concept of *keypoints* turned out to be essential to our strategies. We have seen that a player governing all keypoints cannot possibly lose the game. Surprisingly, the situation in two dimensions is quite different: It can be shown [3] that for any given set of n white points in, say, a unit square, we can find a set of n black points so that the area dominated by Black is at least $\frac{1}{2}+\delta$, for an absolute constant $\delta > 0$ not depending on n (but where n is assumed sufficiently large).

Acknowledgements

The authors would like to thank Dr. Jacob Ecco for initially introducing them to this problem. We also thank Sunil Arya, Mark de Berg and Joachim Gudmundsson for the helpful discussions. We thank an anonymous referee for the simple proof of Lemma 1 and the observation concerning the Catalan numbers.

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