Computational Geometry

Tutorial #2 — Convex Hulls

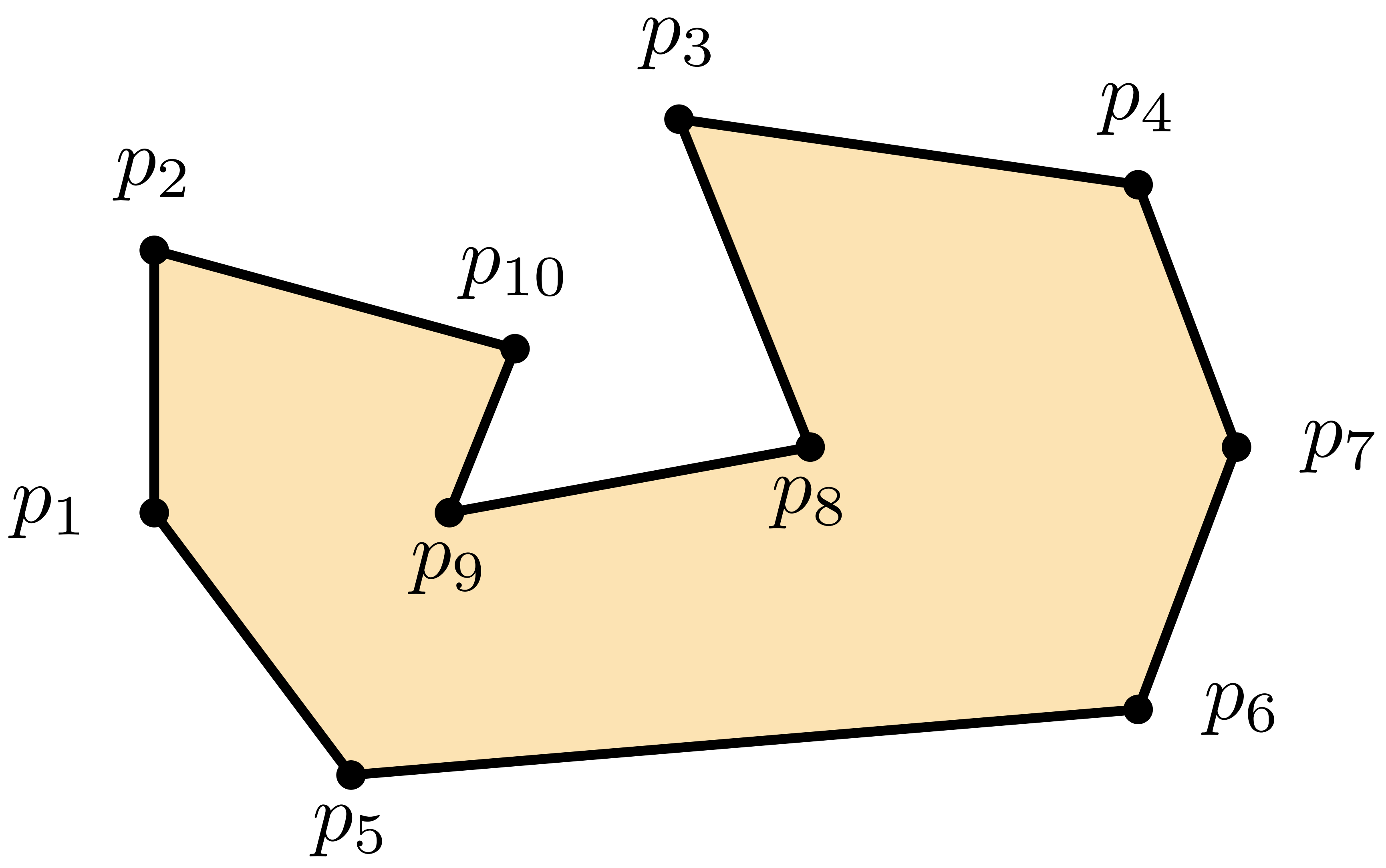
Peter Kramer | November 7th, 2024

Convex hull

Computing the hull of a simple polygon

Theorem E2.1

Let $P := p_1, \ldots, p_n$ be a simple polygon in the euclidean plane $\mathbb{R}^2$, in CCW order. The convex hull of P can be computed in $\mathcal{O}(n)$.



Polygons & Curves

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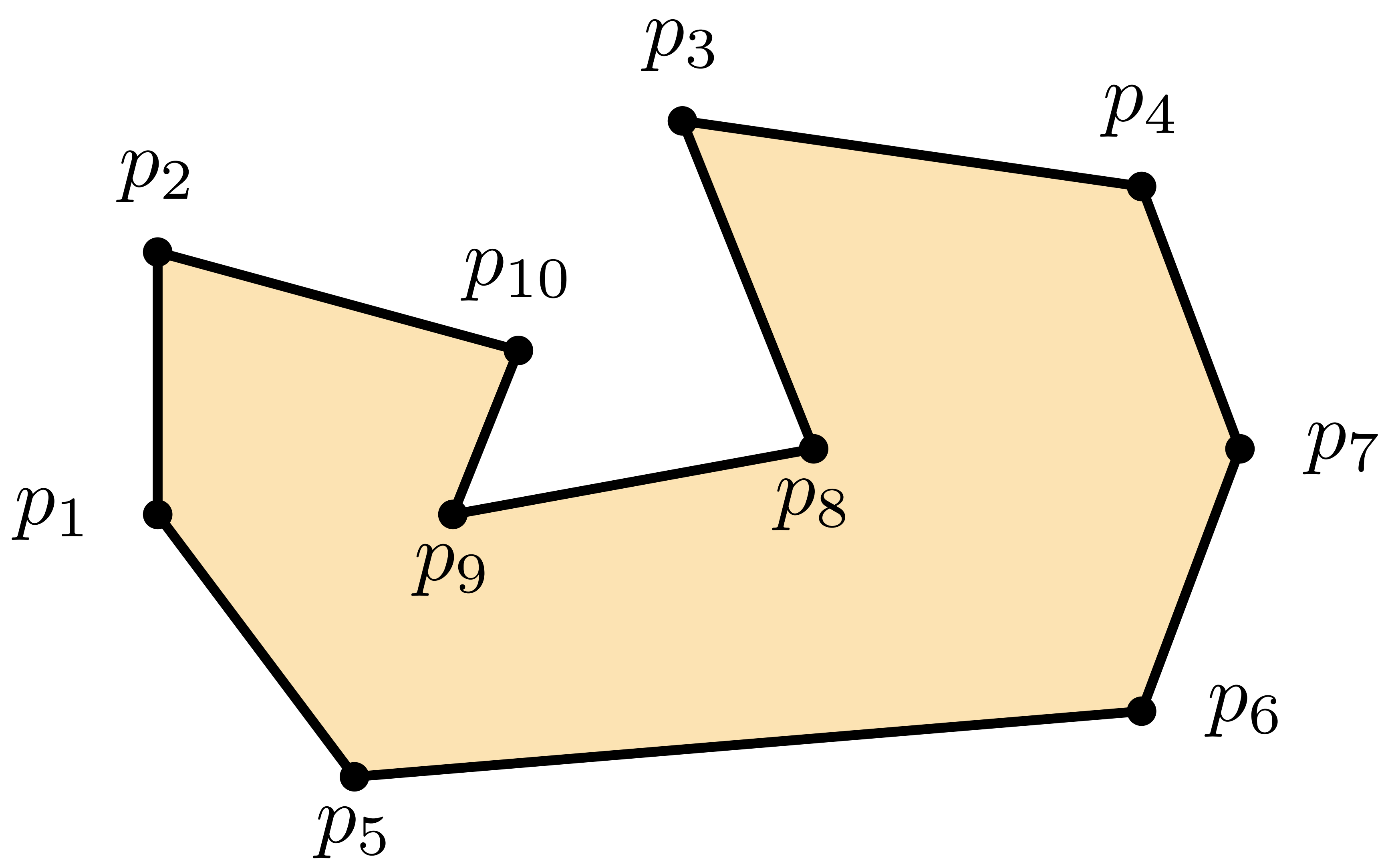
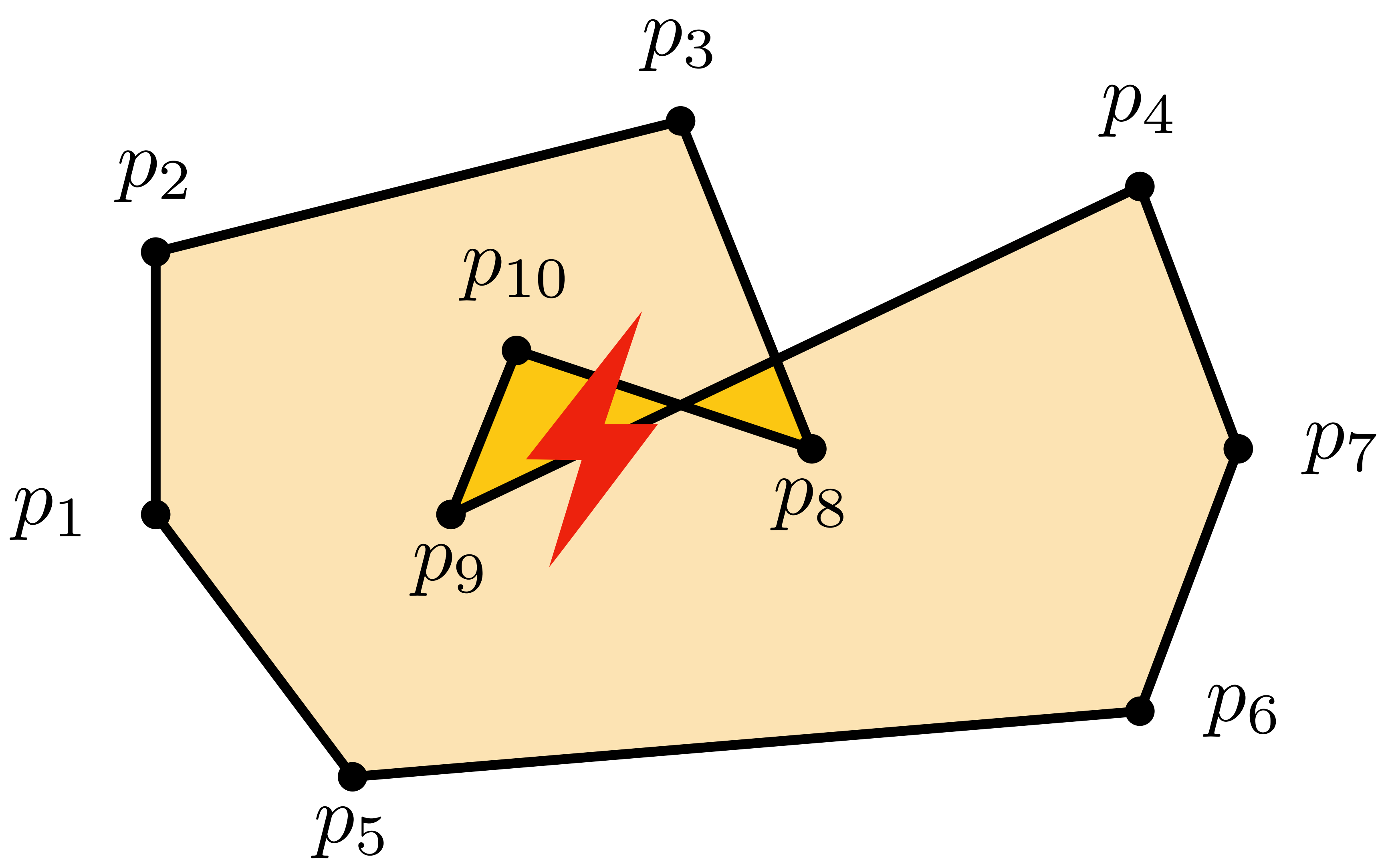
Point sets & Polygons
What's the difference?

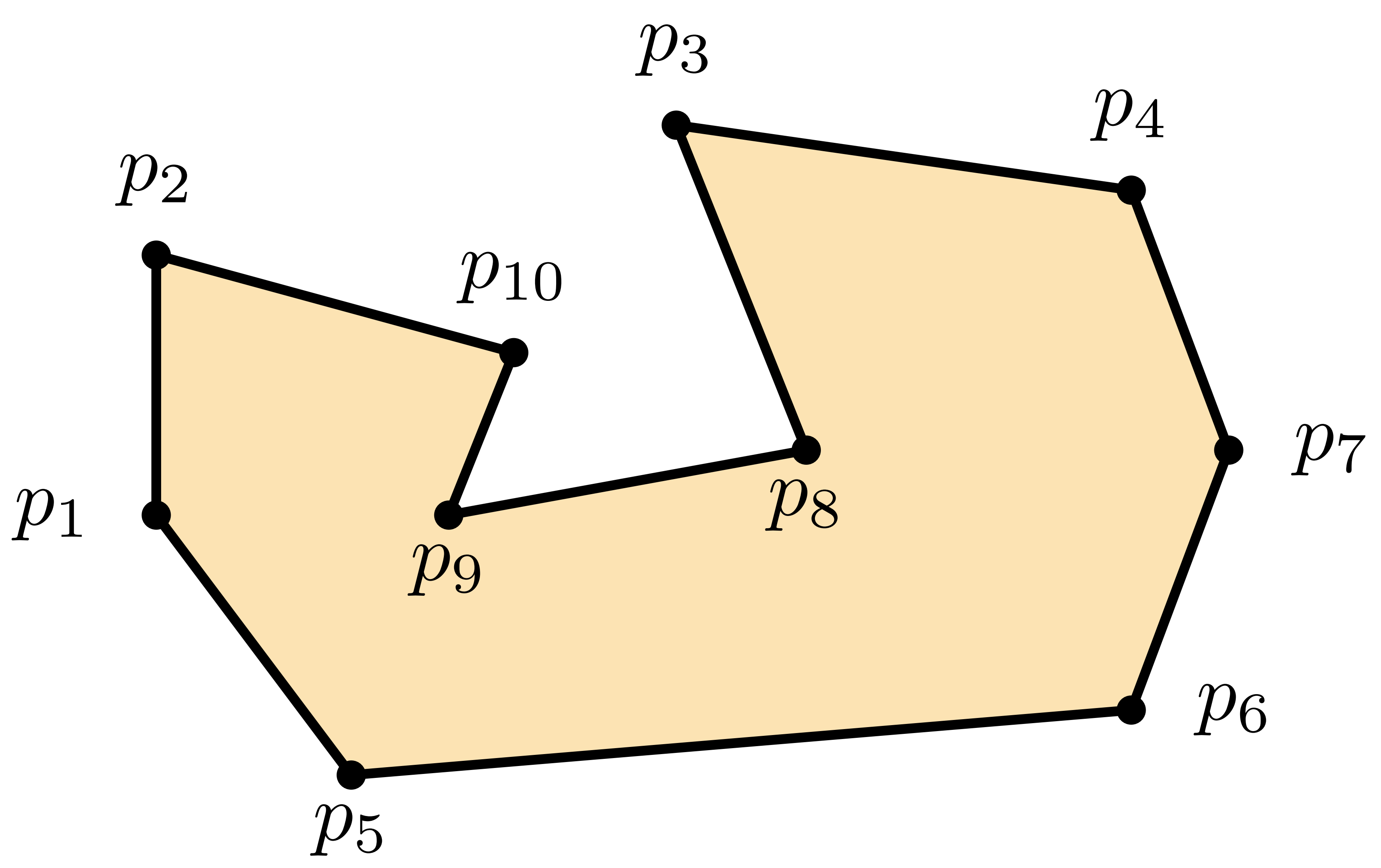
- Consider a finite point set $\mathscr{P} := \{p_1, p_2, \ldots, p_n\} \subset \mathbb{R}^2$.
- A *simple* polygon $P \in \mathscr{P}^n$ is a *permutation (i.e., an ordering)* of $\mathscr{P}$ that induces a simple closed *curve*.

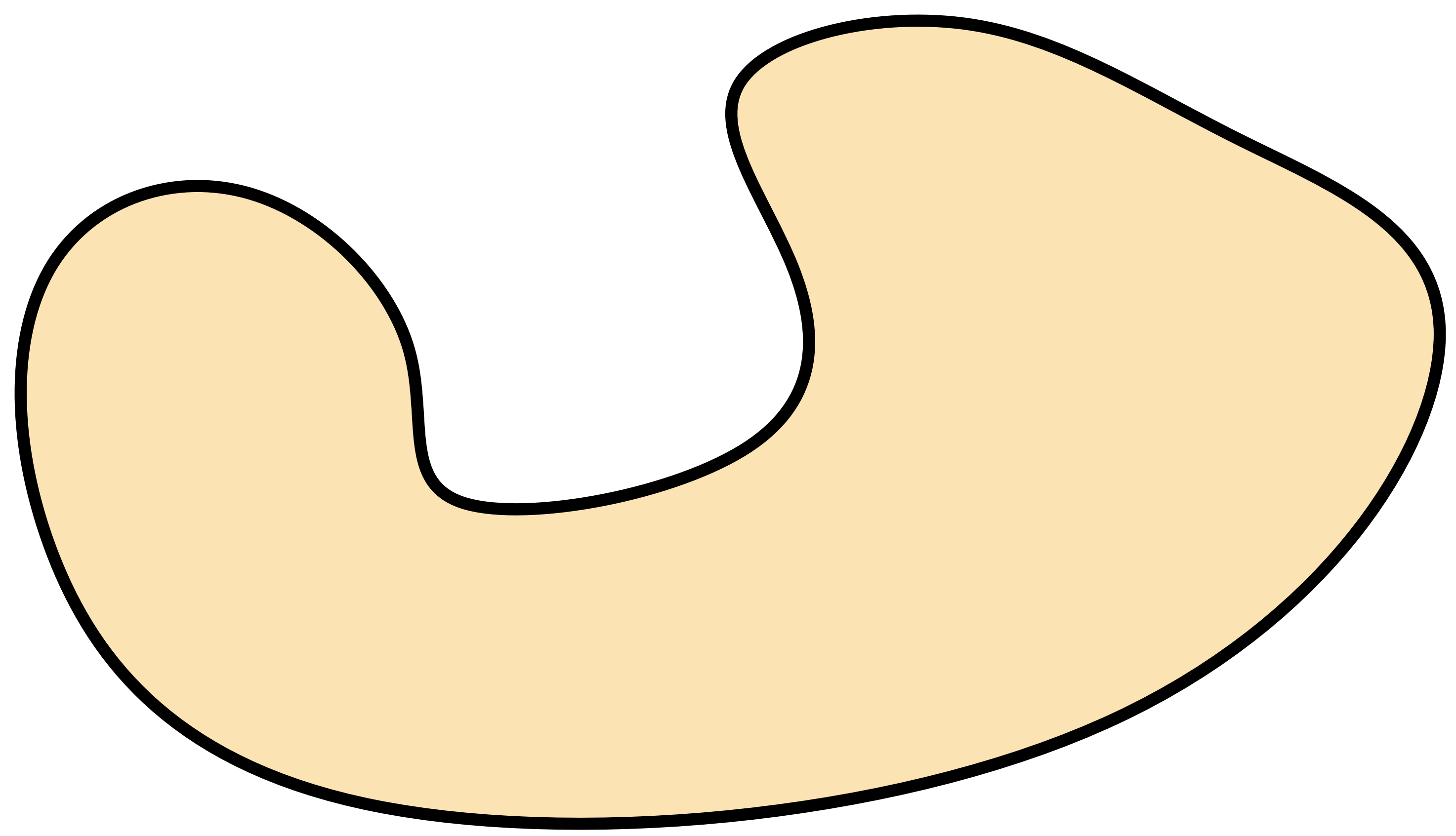
Typically, we describe polygons in CCW order.





Generalising Polygons
Curves in the plane

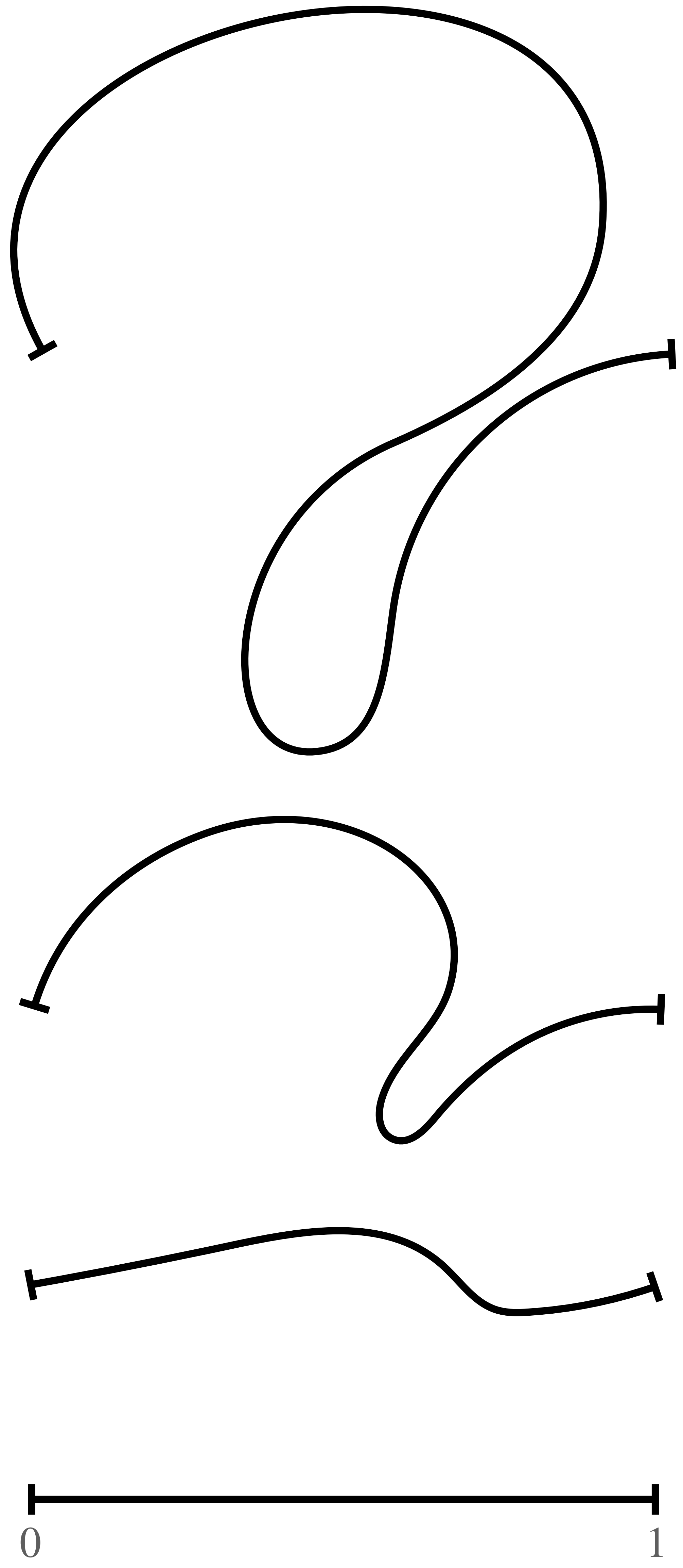




Simple paths in the plane

- Two sets are *homeomorphic* if we can define a continuous, bijective mapping $h : X \to Y$ between the two.
- A *simple path* is a subset of the plane that is homeomorphic to the interval $[0,1] \subset \mathbb{R}$, i.e., $h : [0,1] \to L \subset \mathbb{R}^2$.

Geometrically, imagine any shape that can be continuously "morphed" from a straight line.



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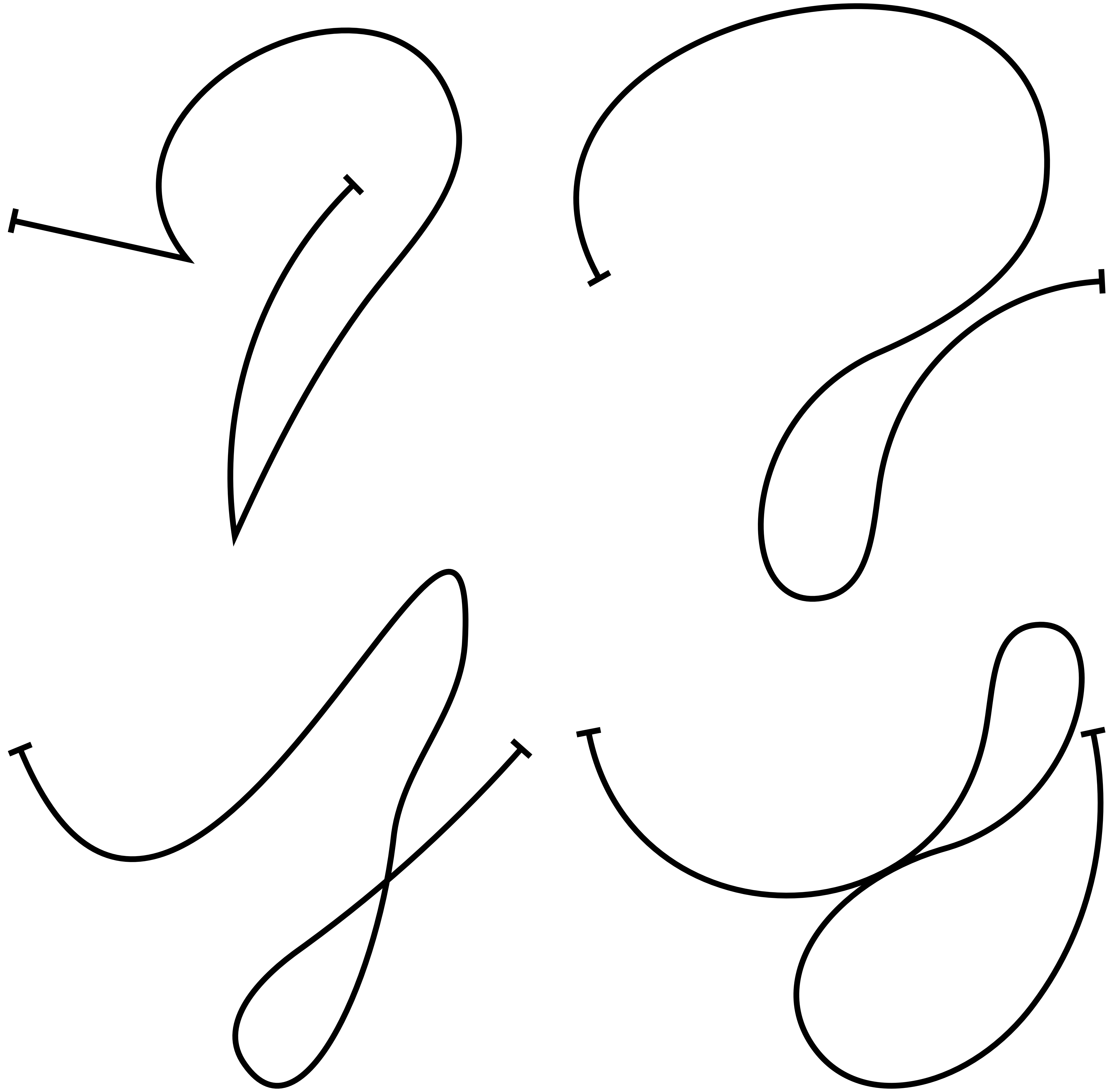

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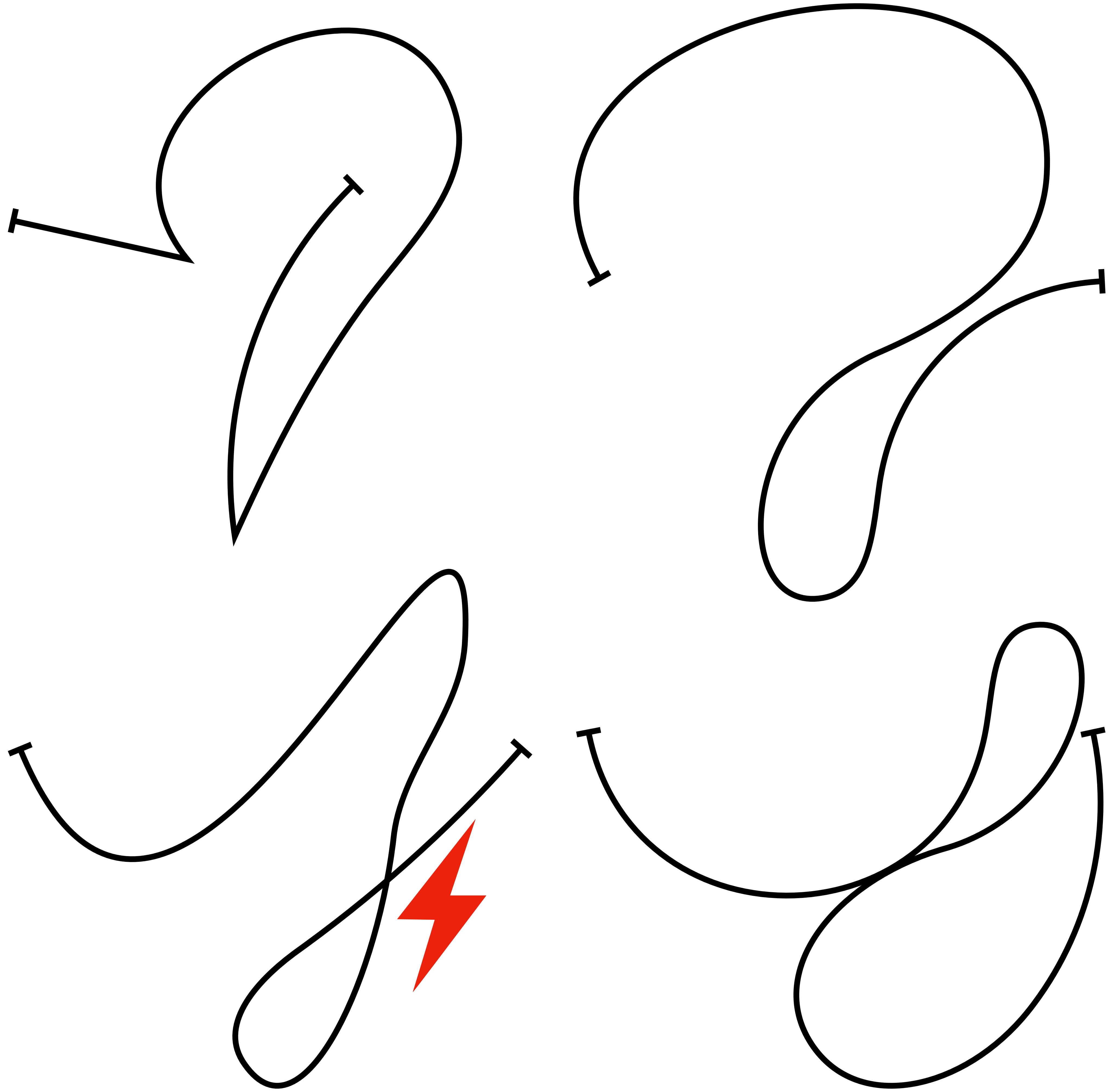

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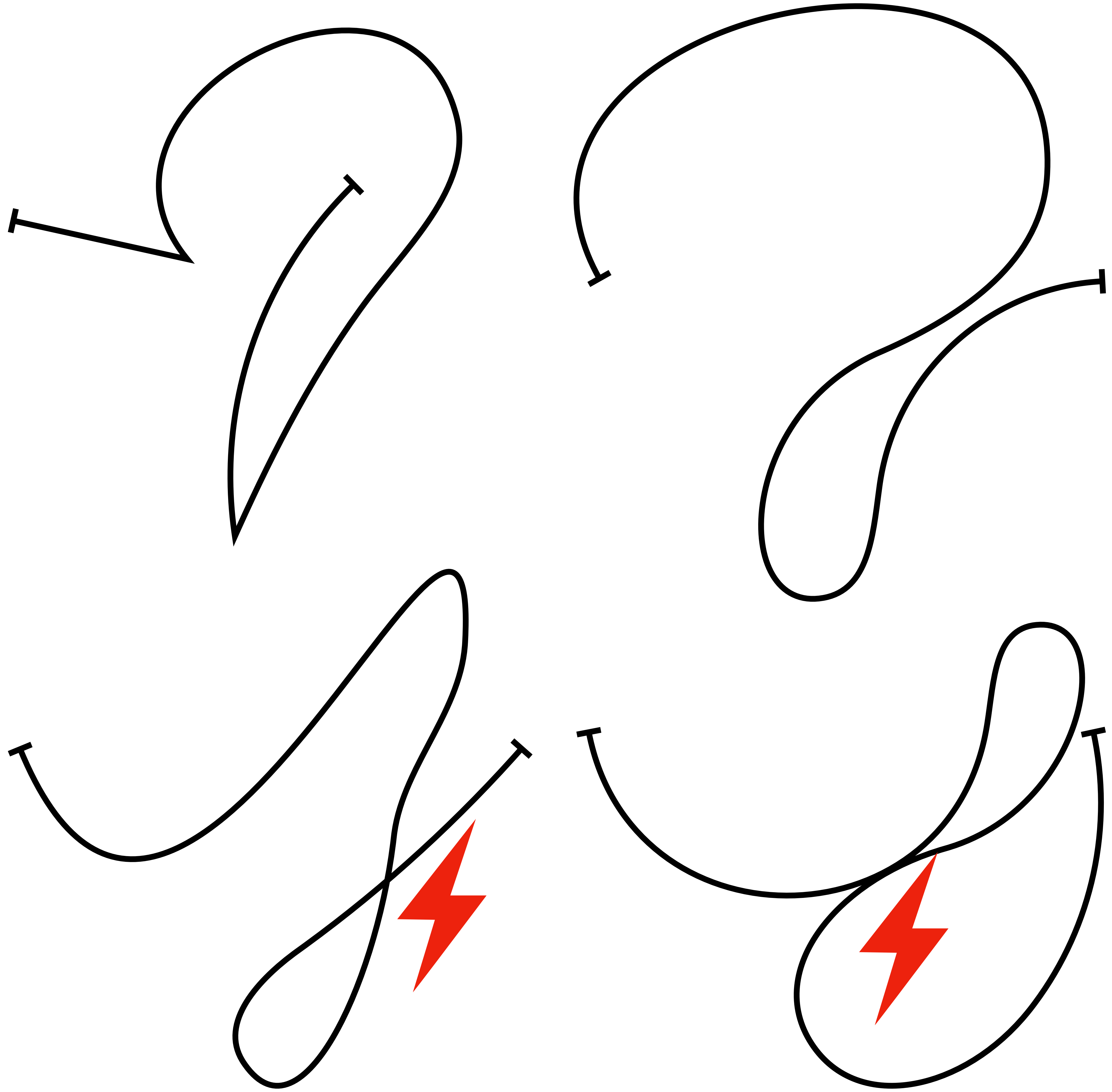


Simple paths in the plane

Example: Straight line segments

- The line segment between two points $p, q \in \mathbb{R}^2$ is a simple curve

$$\overline{pq} = \{\, x \in \mathbb{R}^2 \quad | \quad \exists\ s \in [0,1] :\ x = p \cdot s + q \cdot (1 - s) \,\}.$$

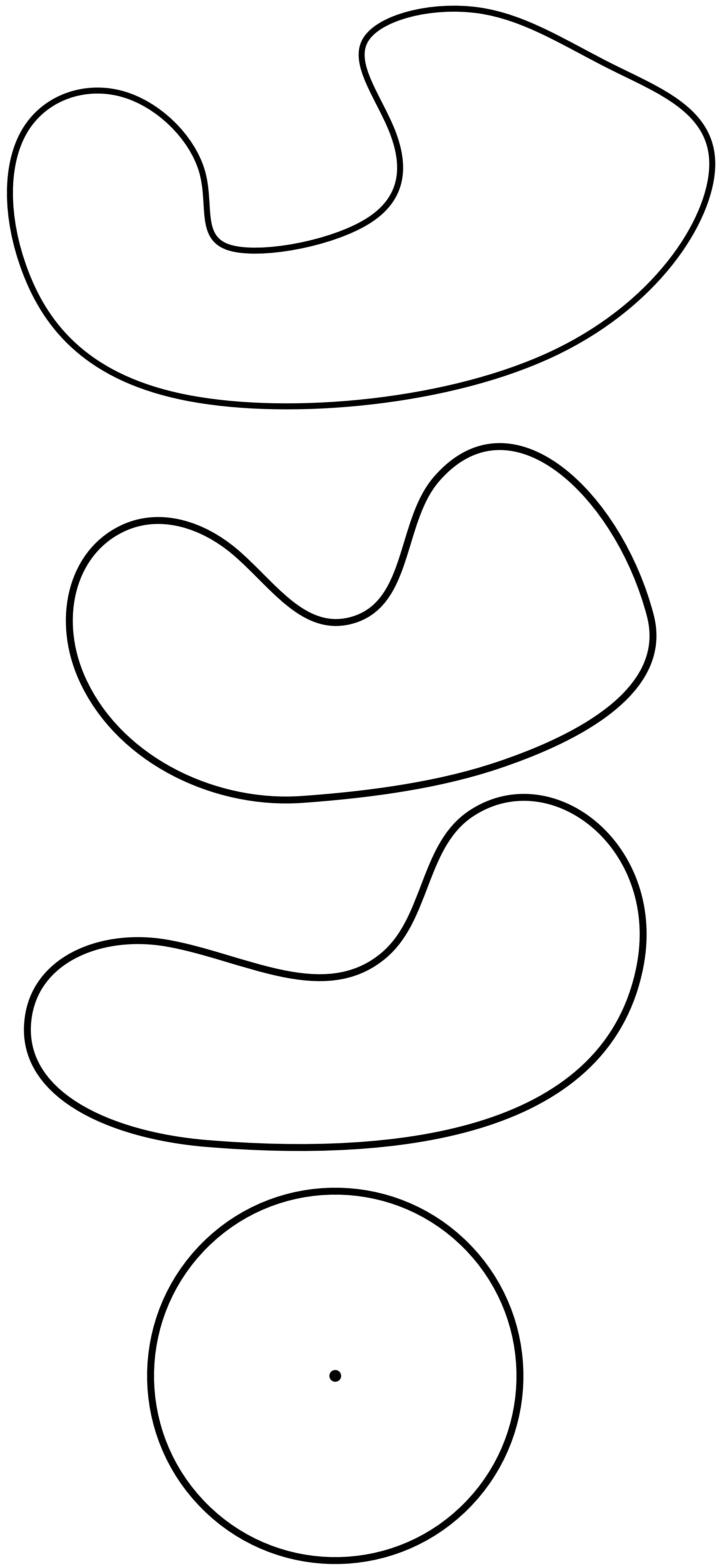
- We can define a homeomorphism between $[0,1] \subset \mathbb{R}$ and $\overline{pq}$:

$$h_{\overline{pq}} : [0,1] \to \overline{pq}, \quad s \mapsto p \cdot s + p \cdot (1 - s)$$

Closed curves in the plane

- Consider the (Euclidean) unit circle $S_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.
- A *simple closed curve* is a point set in the plane that is homeomorphic to S_1.
- Then: $h : S_1 \to C \subset \mathbb{R}^2$.

Geometrically, imagine any shape that can be continuously "morphed" from the circle.



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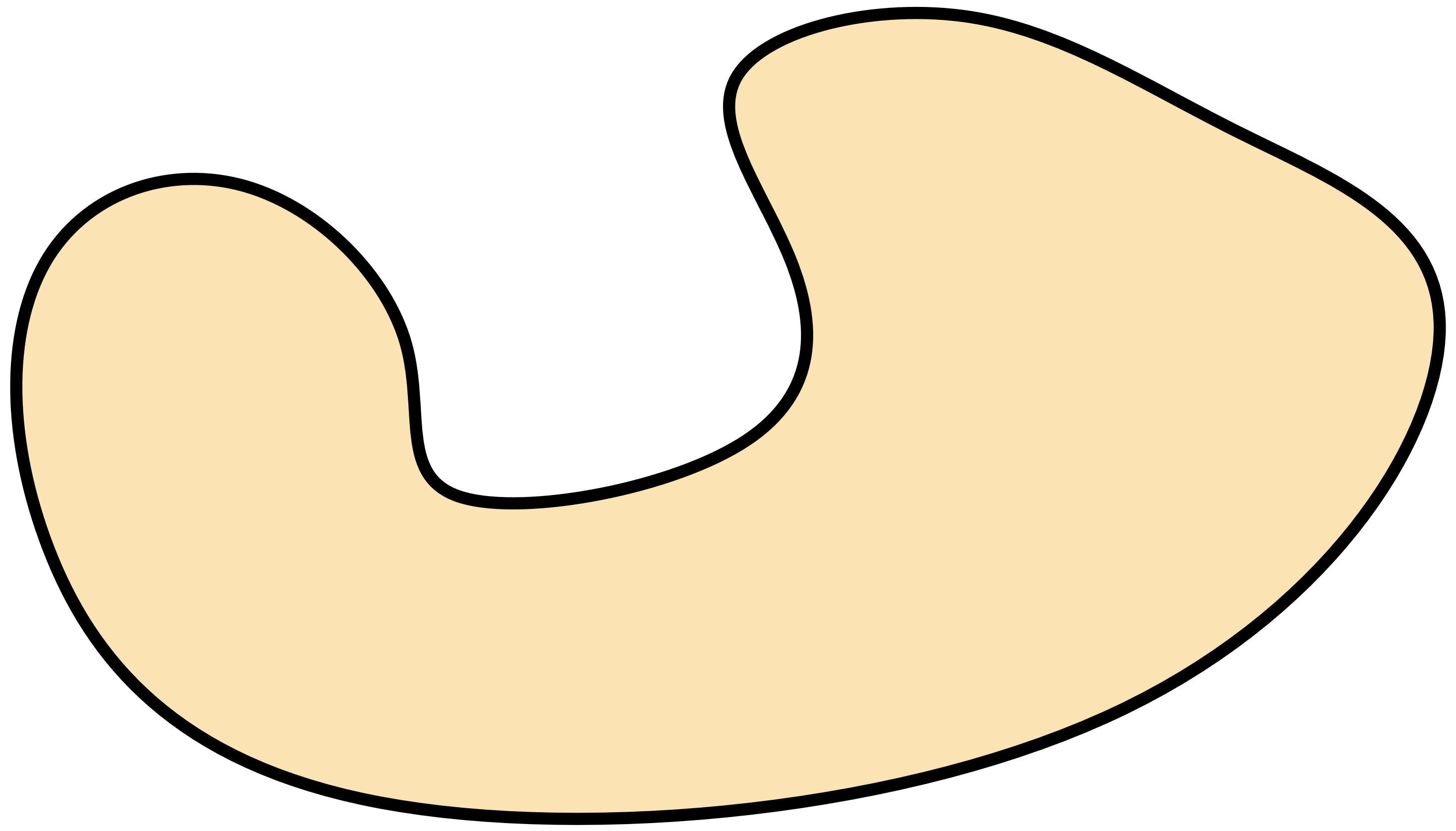
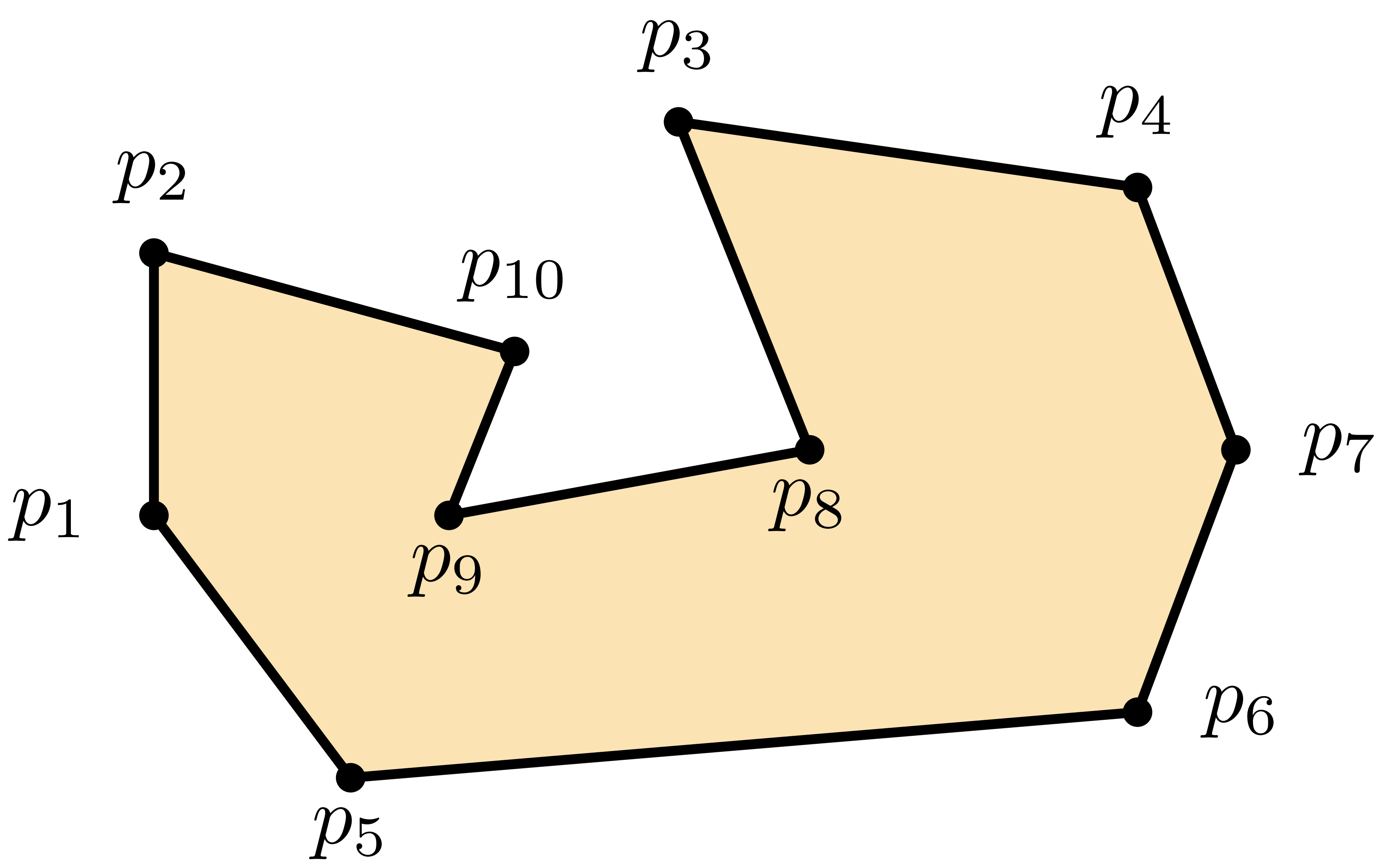


- A *simple* polygon P is a *closed curve* composed of a finite number of line segments.



Closed curves in the plane

Example: Simple polygons

- Recall that line segments are simple paths.
- If two simple paths meet only in a common endpoint, we can join them into a longer path.
- By induction, we c an join the line segments that define a simple polygon P into a closed curve by joining them in CCW order.

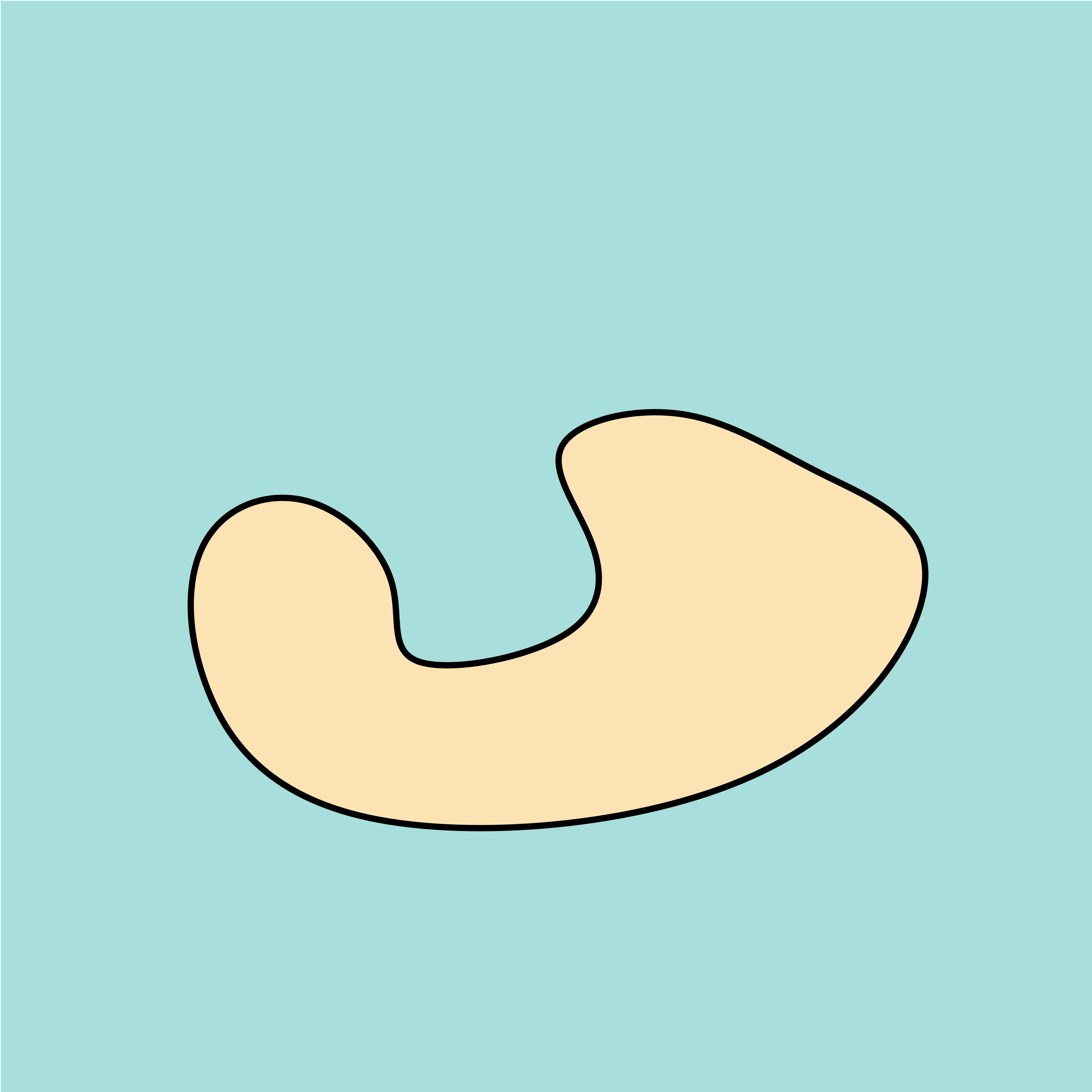
Jordan Curves

Peter Kramer | November 7th, 2024

Jordan Curve Theorem

Theorem E2.2 (Jordan Curves)

Let C be a Jordan curve in the plane $\mathbb{R}^2$. Then its complement $\mathbb{R}^2 \setminus C$ consists of exactly two connected components: the bounded ***interior*** *and the unbounded* ***exterior***.



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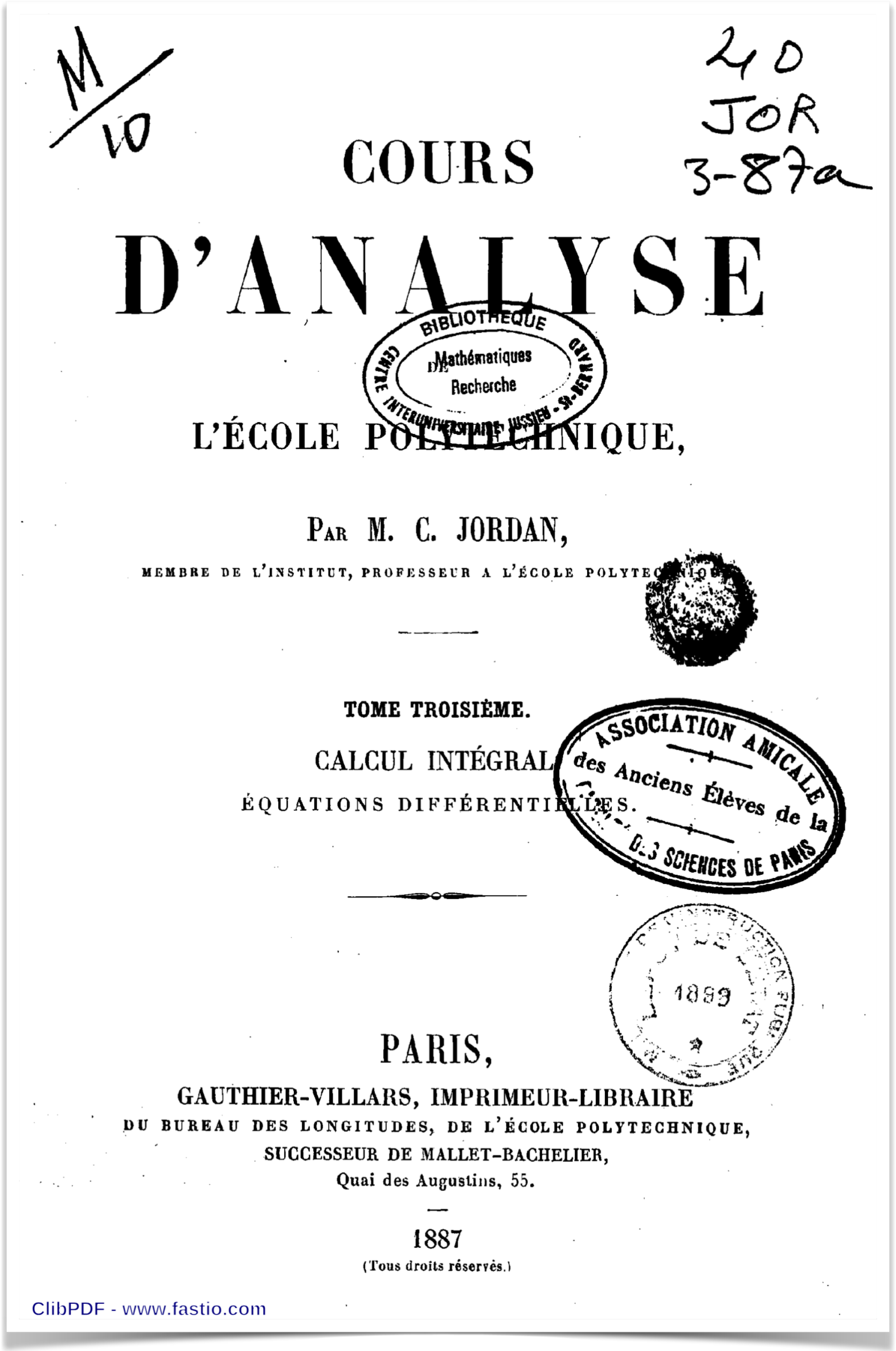

COURS

D'ANALYSE

DE

L'ÉCOLE POLYTECHNIQUE,

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TOME TROISIÈME.

CALCUL INTÉGRAL.

ÉQUATIONS DIFFÉRENTIELLES.

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DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,

SUCCESSEUR DE MALLET-BACHELIER,

Quai des Augustins, 55.

1887

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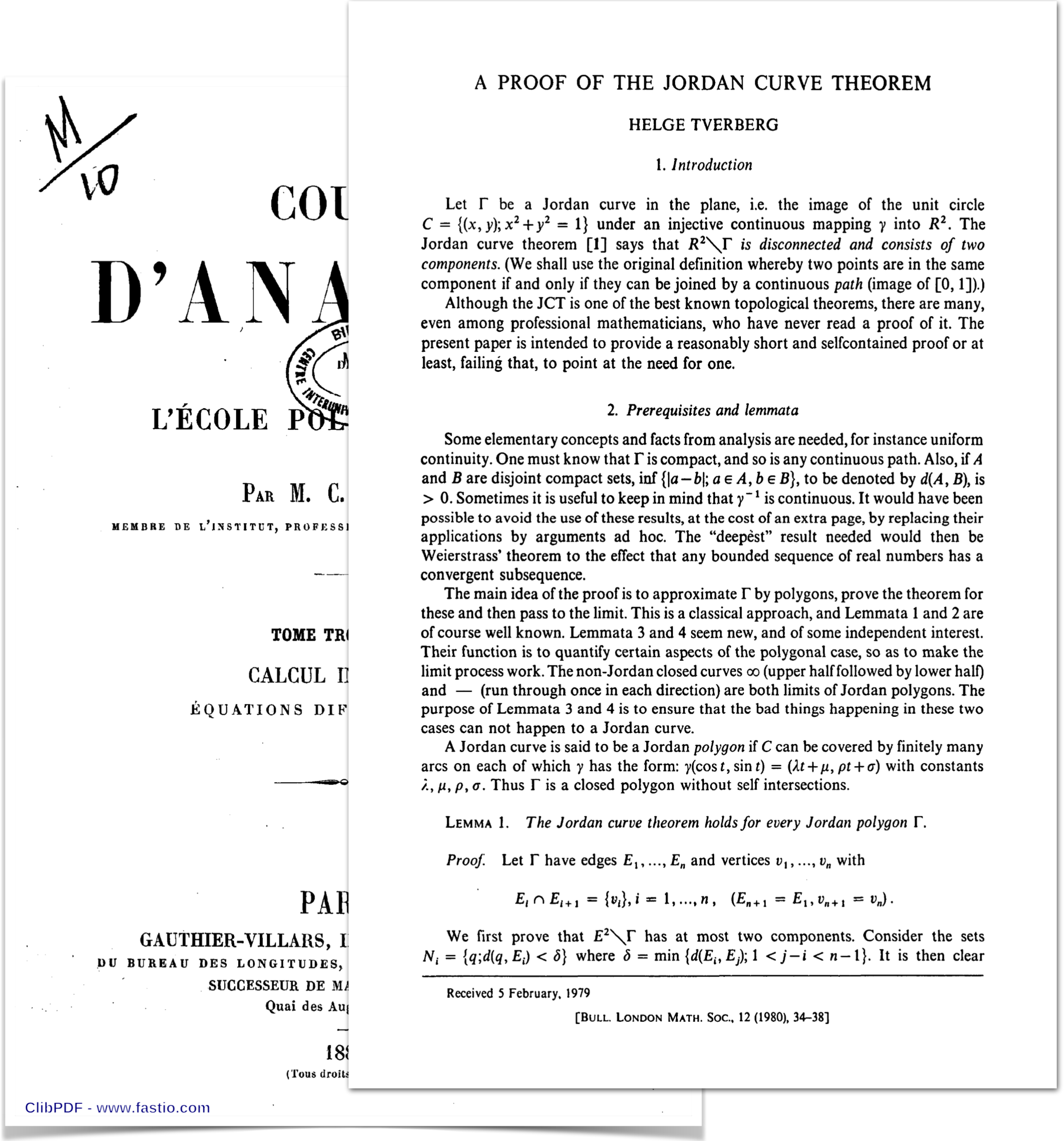

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A PROOF OF THE JORDAN CURVE THEOREM

HELGE TVERBERG

1. *Introduction*

Let Γ be a Jordan curve in the plane, i.e. the image of the unit circle $C = \{(x, y); x^2+y^2 = 1\}$ under an injective continuous mapping γ into R^2. The Jordan curve theorem [1] says that $R^2 \setminus \Gamma$ *is disconnected and consists of two components*. (We shall use the original definition whereby two points are in the same component if and only if they can be joined by a continuous *path* (image of [0, 1]).)

Although the JCT is one of the best known topological theorems, there are many, even among professional mathematicians, who have never read a proof of it. The present paper is intended to provide a reasonably short and selfcontained proof or at least, failing that, to point at the need for one.

2. *Prerequisites and lemmata*

Some elementary concepts and facts from analysis are needed, for instance uniform continuity. One must know that Γ is compact, and so is any continuous path. Also, if A and B are disjoint compact sets, $\inf\{|a-b|; a \in A, b \in B\}$, to be denoted by $d(A, B)$, is > 0. Sometimes it is useful to keep in mind that γ^{-1} is continuous. It would have been possible to avoid the use of these results, at the cost of an extra page, by replacing their applications by arguments ad hoc. The "deepest" result needed would then be Weierstrass' theorem to the effect that any bounded sequence of real numbers has a convergent subsequence.

The main idea of the proof is to approximate Γ by polygons, prove the theorem for these and then pass to the limit. This is a classical approach, and Lemmata 1 and 2 are of course well known. Lemmata 3 and 4 seem new, and of some independent interest. Their function is to quantify certain aspects of the polygonal case, so as to make the limit process work. The non-Jordan closed curves ∞ (upper half followed by lower half) and — (run through once in each direction) are both limits of Jordan polygons. The purpose of Lemmata 3 and 4 is to ensure that the bad things happening in these two cases can not happen to a Jordan curve.

A Jordan curve is said to be a Jordan *polygon* if C can be covered by finitely many arcs on each of which γ has the form: $\gamma(\cos t, \sin t) = (\lambda t+\mu, \rho t+\sigma)$ with constants $\lambda, \mu, \rho, \sigma$. Thus Γ is a closed polygon without self intersections.

LEMMA 1. *The Jordan curve theorem holds for every Jordan polygon* Γ.

Proof. Let Γ have edges $E_1, \ldots, E_n$ and vertices $v_1, \ldots, v_n$ with

$$E_i \cap E_{i+1} = \{v_i\}, i = 1, \ldots, n, \quad (E_{n+1} = E_1, v_{n+1} = v_n).$$

We first prove that $E^2 \setminus \Gamma$ has at most two components. Consider the sets $N_i = \{q; d(q, E_i) < \delta\}$ where $\delta = \min\{d(E_i, E_j); 1 < j-i < n-1\}$. It is then clear

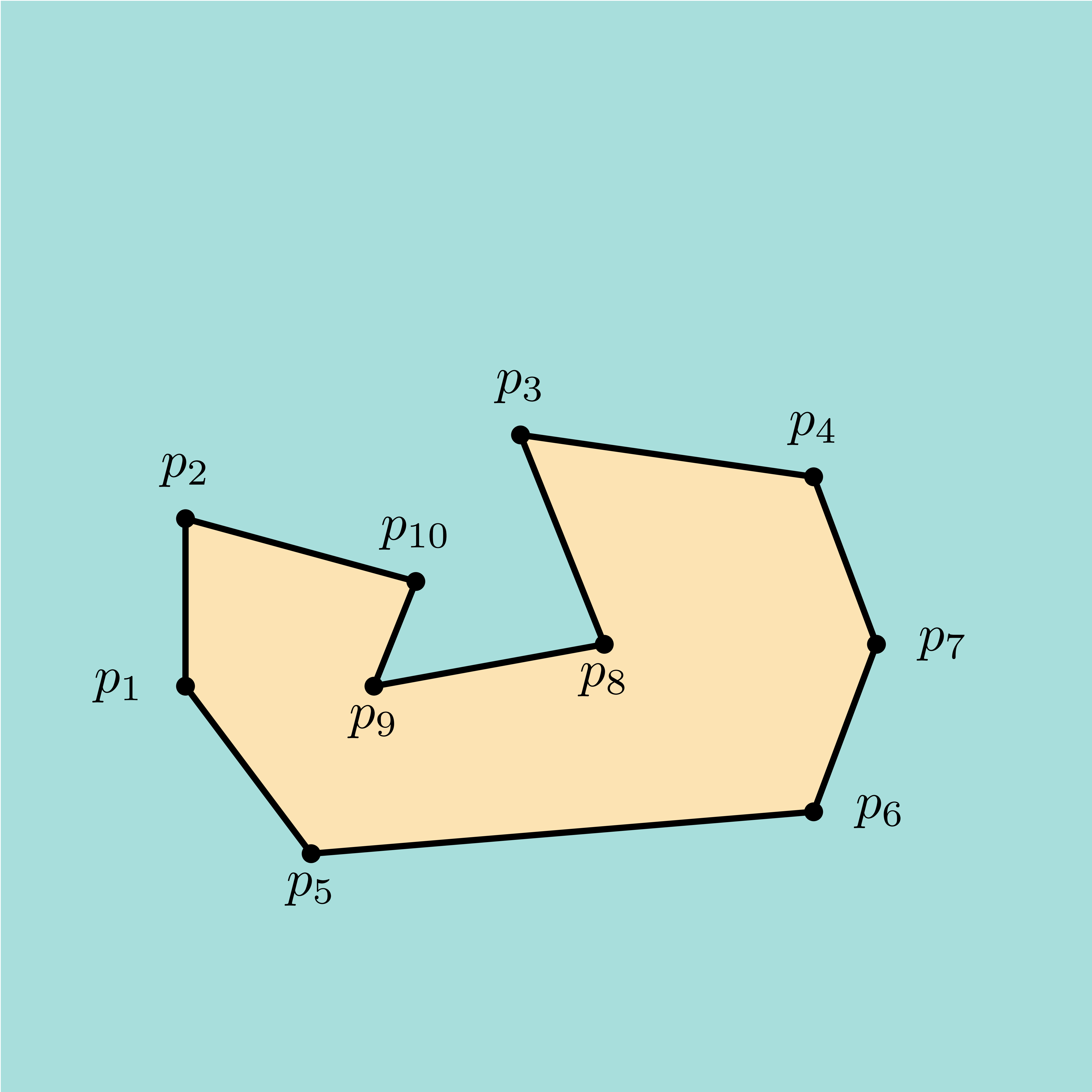
Received 5 February, 1979

[BULL. LONDON MATH. SOC., 12 (1980), 34–38]

Jordan Curve Theorem

Theorem E2.3 (Jordan Polygons)

Let P be a simple polygon in the plane $\mathbb{R}^2$. Then its complement $\mathbb{R}^2 \setminus P$ consists of exactly two connected components: the bounded ***interior*** *and the unbounded* ***exterior****.*



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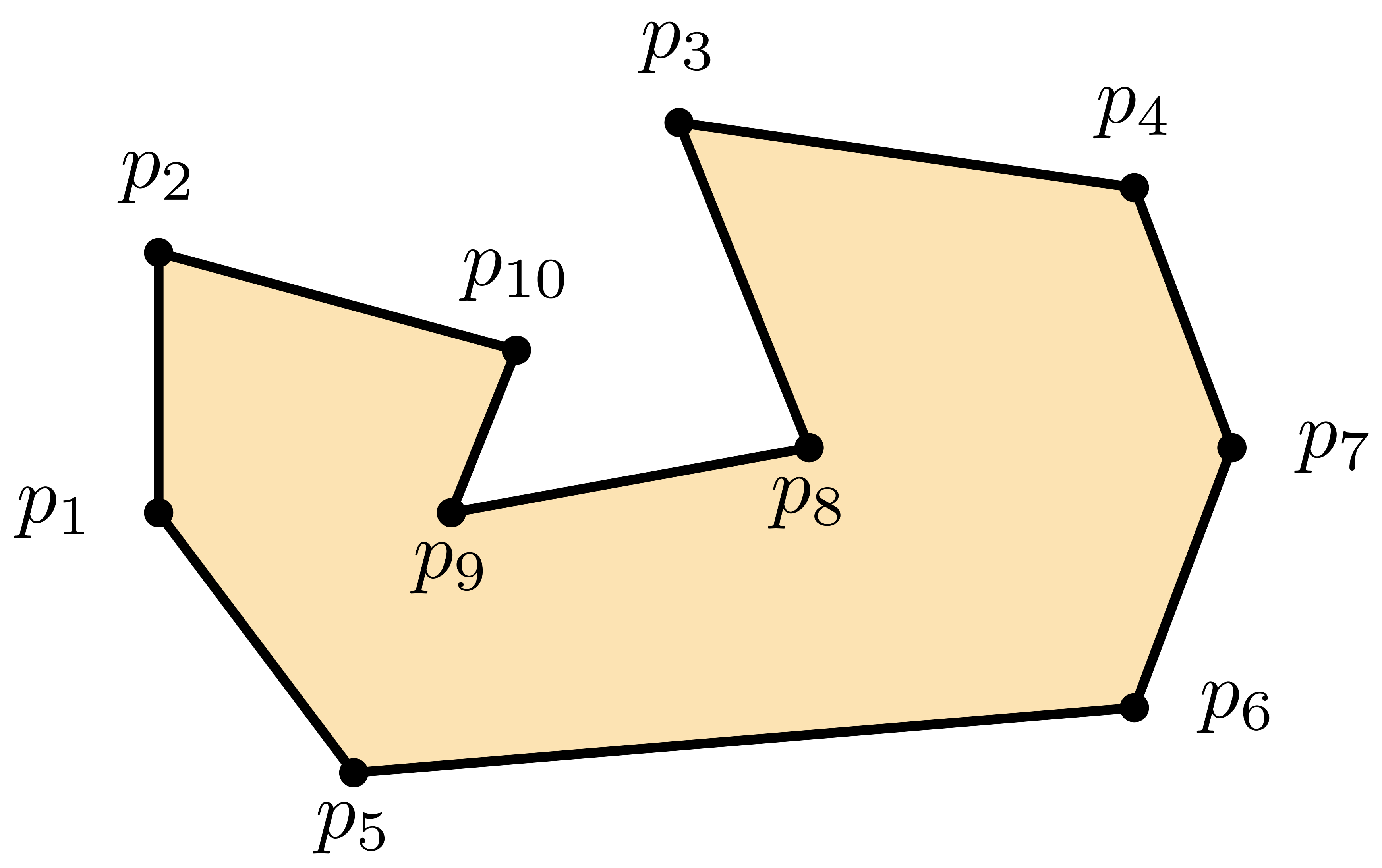

Convex hulls of polygons

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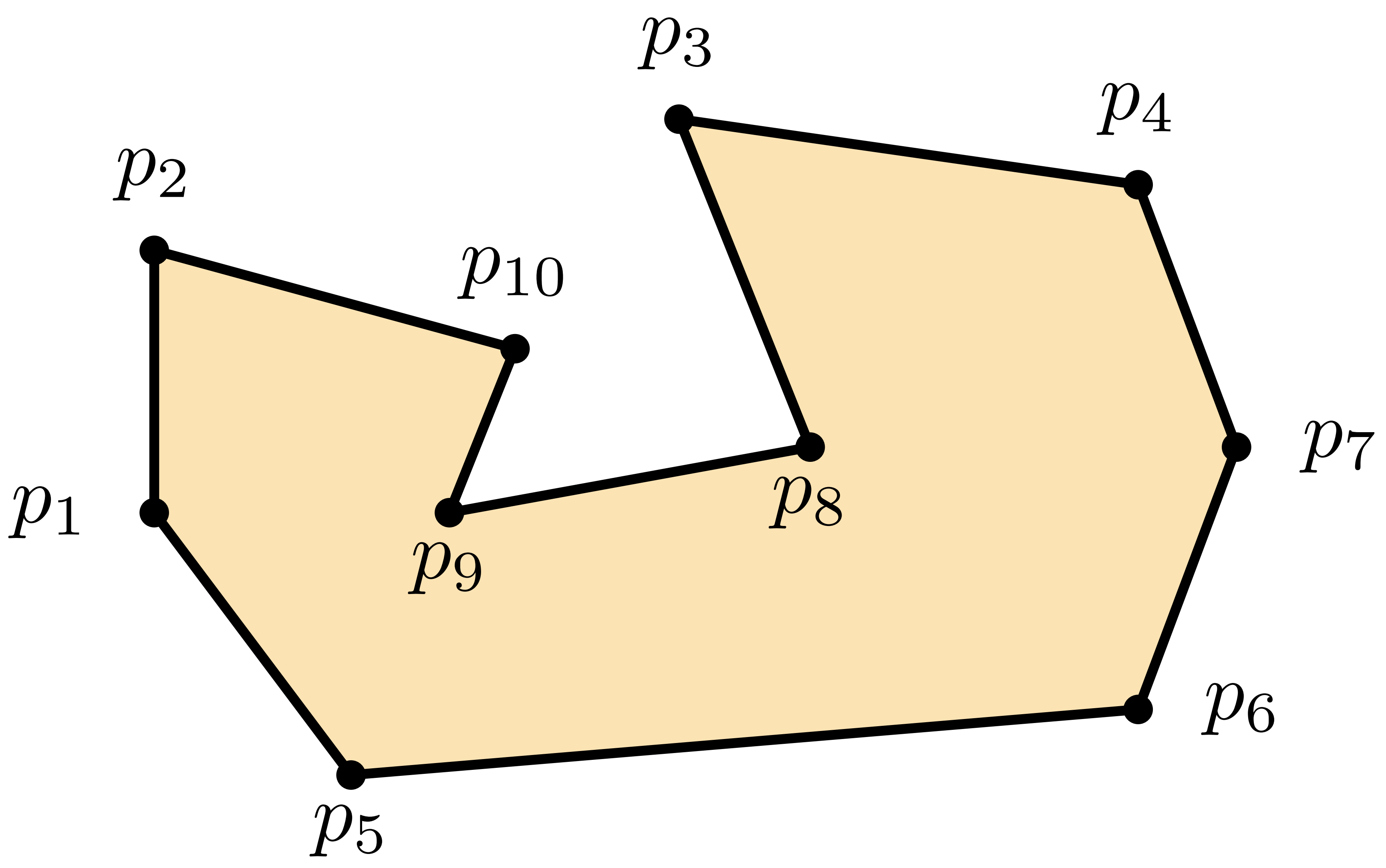
Convex hull

Computing the hull of a simple polygon

Theorem E2.1

Let $P := p_1, \ldots, p_n$ be a simple polygon in the euclidean plane $\mathbb{R}^2$, in CCW order. The convex hull of P can be computed in $\mathcal{O}(n)$.



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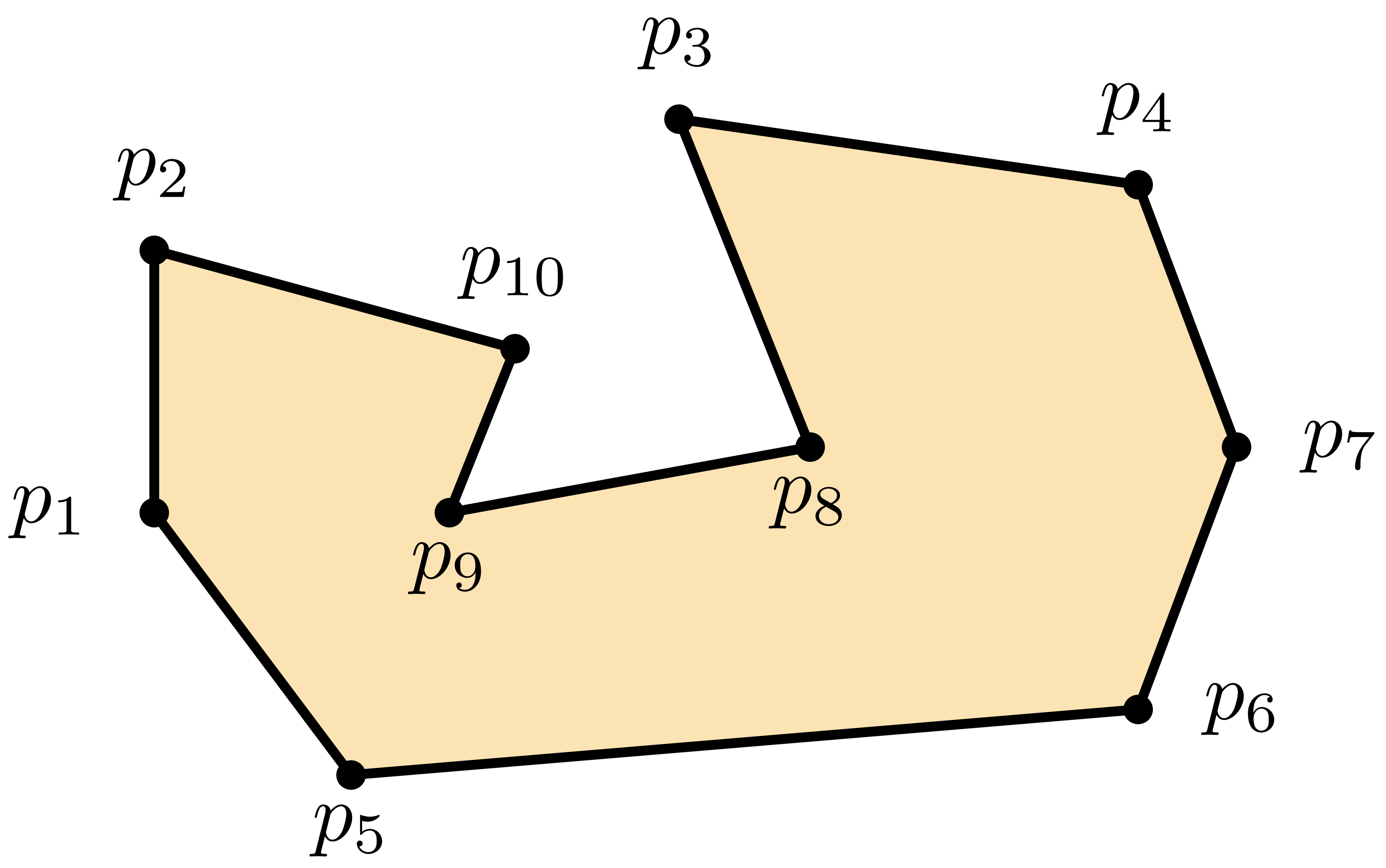


Idea:

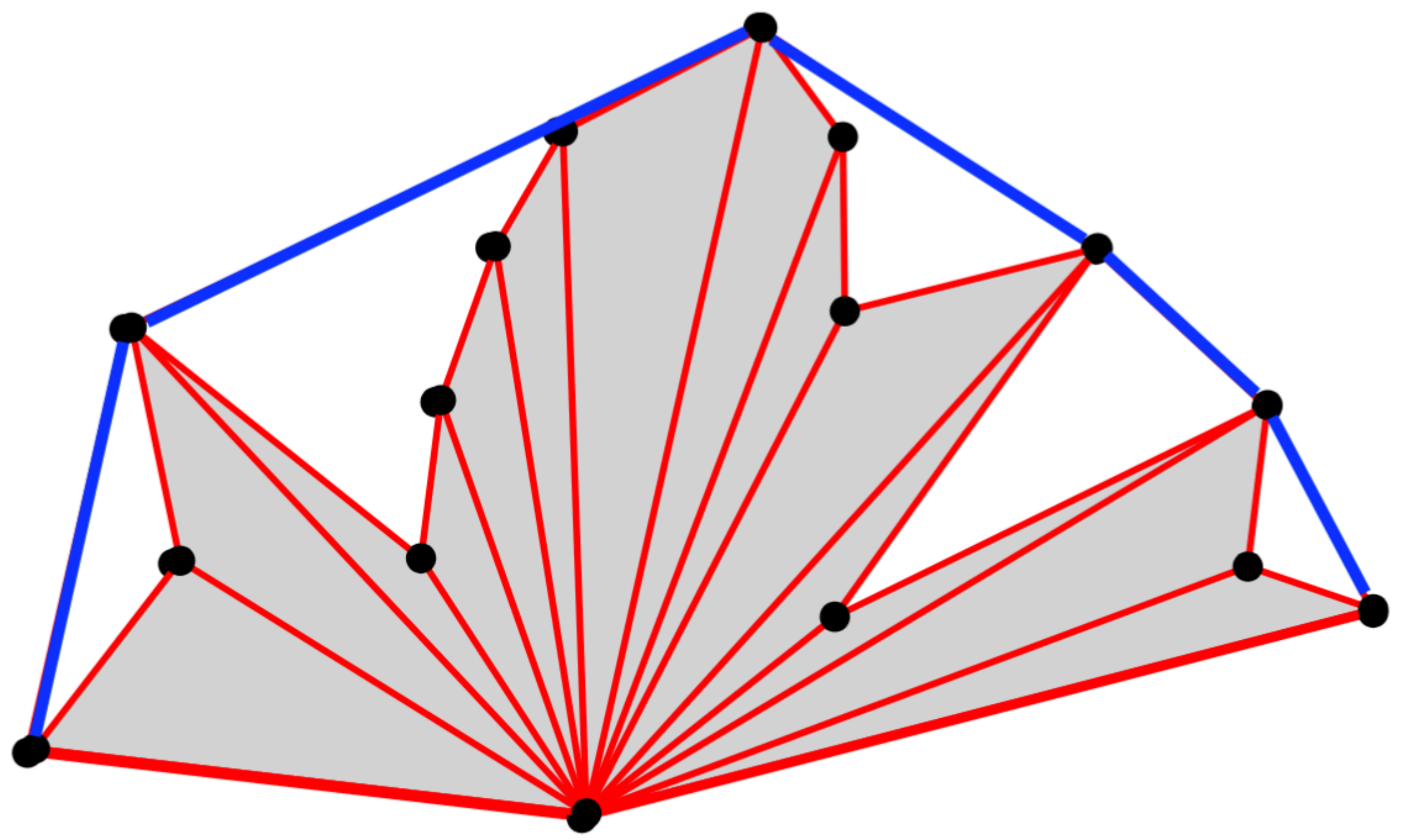
Theorem E2.3 implies that vertices of P *that are also on* $\mathrm{conv}(P)$ *appear in the exact same CCW order on both bounding polygons.*
We can perform Graham Scan without sorting.

Goal:

- Finding a sequence of „left“ turns

$$\Updownarrow$$

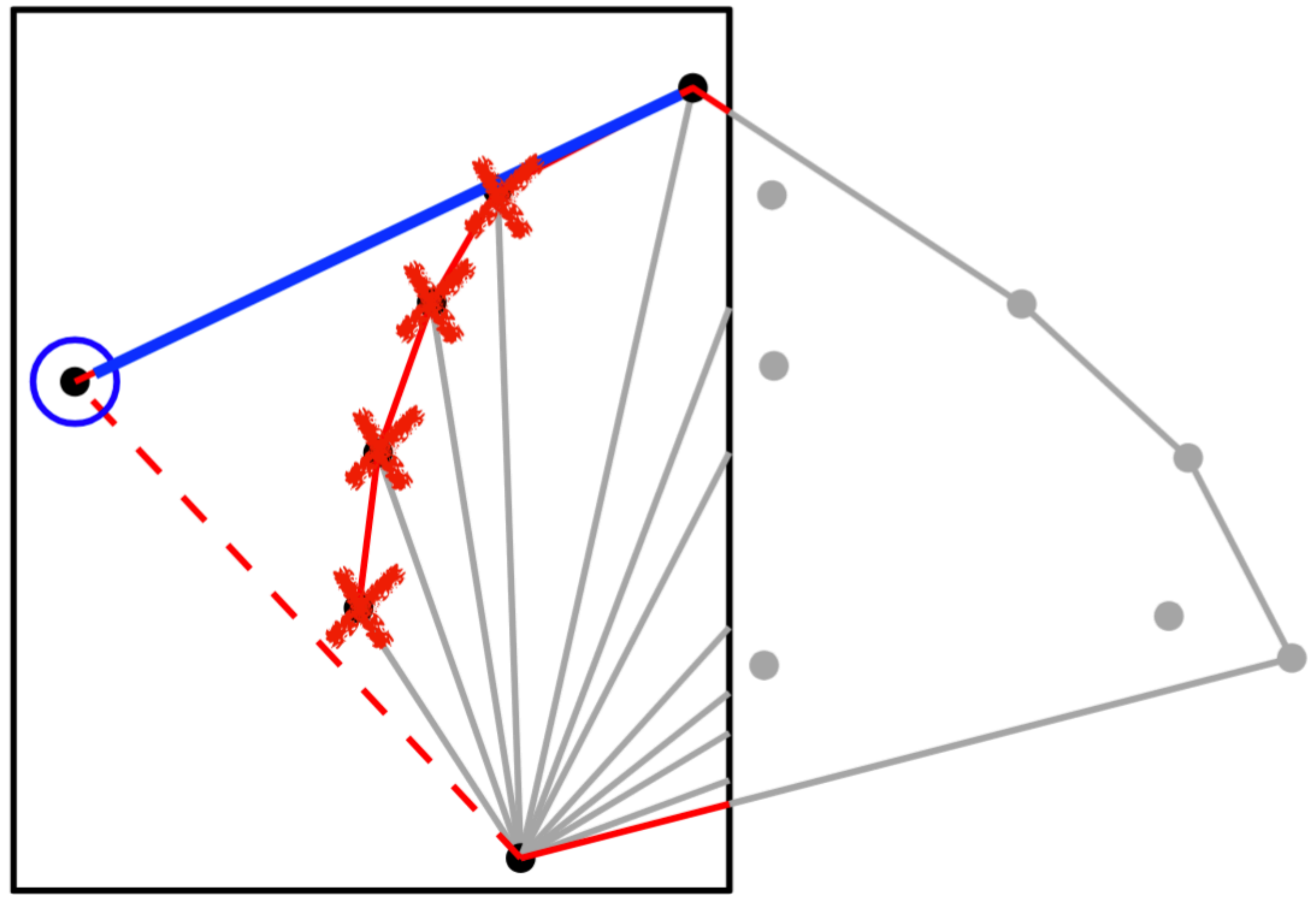
Knowing $conv(\mathcal{P})$



Approach:

- Maintain stack of vertices
- In case of a „right“ turn:

  Pop vertex off stack



Algorithm 2.23: Compute $conv(\mathcal{P})$ with Graham's Scan.

```
let points be the list of points
let stack = empty_stack()

find the lowest y-coordinate and leftmost
point, called P0
sort points by polar angle with P0, if
several points have the same polar angle
then only keep the farthest

for point in points:
    # pop the last point from the stack if
we turn clockwise to reach this point
    while count stack > 1 and
ccw(next_to_top(stack), top(stack), point)
<= 0:
        pop stack
    push point to stack
end
```

$\mathcal{O}(n)$

$\mathcal{O}(n \log n)$

$\mathcal{O}(n)$