Computational Geometry

Tutorial #3 — Farthest Pairs

Institut für Betriebssysteme
und Rechnerverbund
Algorithmik

Peter Kramer | *November 14th, 2024*

Farthest point pairs
What we know

Exercise 3 (Farthest Point Pairs). **(5+10 points)**

Given a set $\mathcal{P}$ of n points in the Euclidean plane, two points $p, q \in \mathcal{P}$ are a *farthest pair* in $\mathcal{P}$ if

$$\forall\, u, v \in \mathcal{P}: \quad |p - q| \;\geq\; |u - v|.$$

The Euclidean distance between p and q is then also called the *diameter* of $\mathcal{P}$.

a) Prove that all farthest pairs in $\mathcal{P}$ are vertices of the convex hull $\mathrm{conv}(\mathcal{P})$.

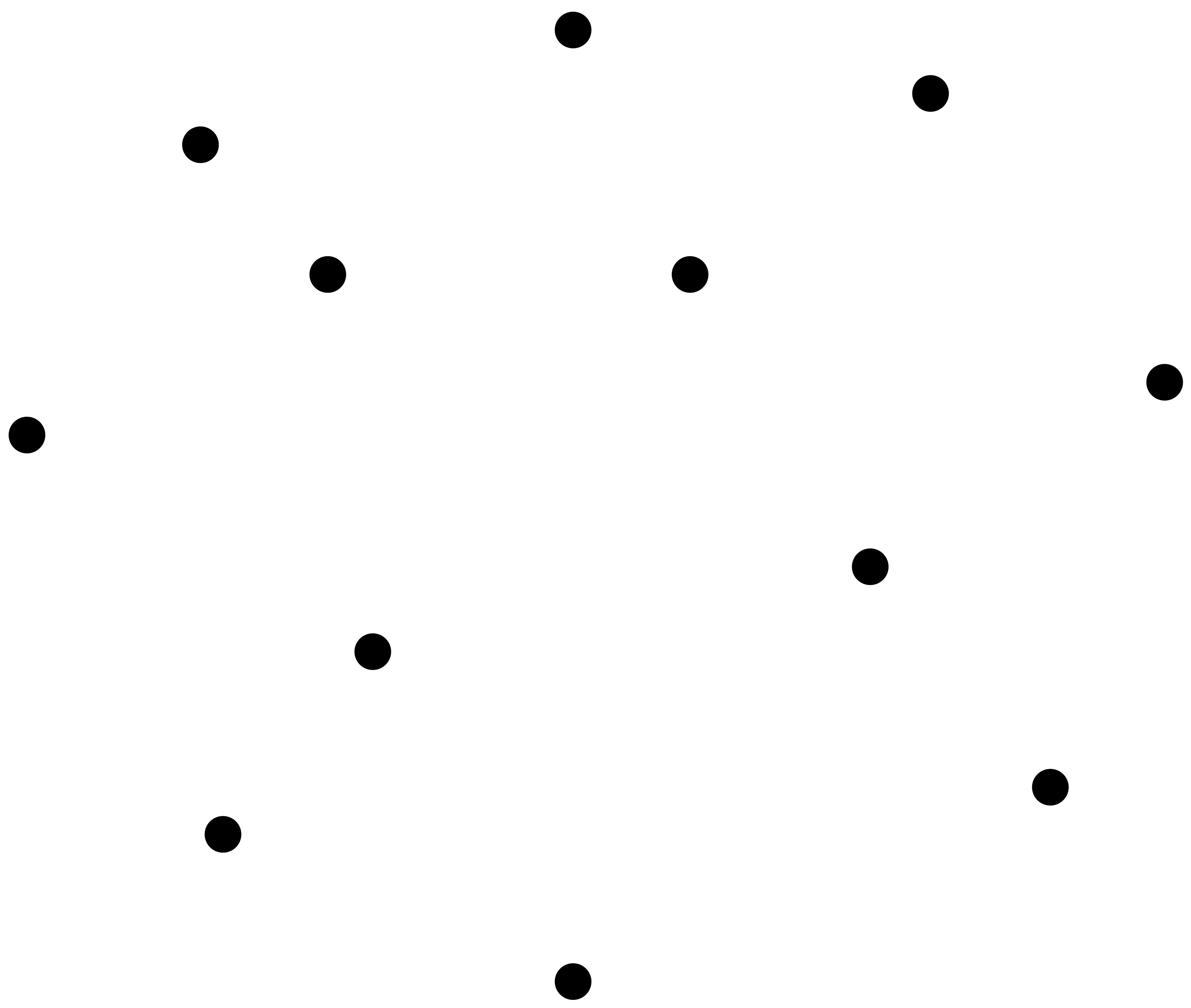
b) Design an $\mathcal{O}(n)$ algorithm that approximates the diameter of $\mathcal{P}$ up to a constant factor. Argue its correctness, approximation factor, and runtime!

- We will assume that **a)** is true — Discussion next week :)
- An exact $\mathcal{O}(n^2)$ time algorithm is trivial — can we do better?

Farthest point pairs
Convex hull

Let $\mathscr{P}$ be set of n points in the Euclidean plane $\mathbb{R}^2$, in general position*.

Lemma E3.1 All farthest pairs of $\mathscr{P}$ consist of two vertices of the convex hull $\mathrm{conv}(\mathscr{P})$.



** no three points in $\mathscr{P}$ are collinear.*

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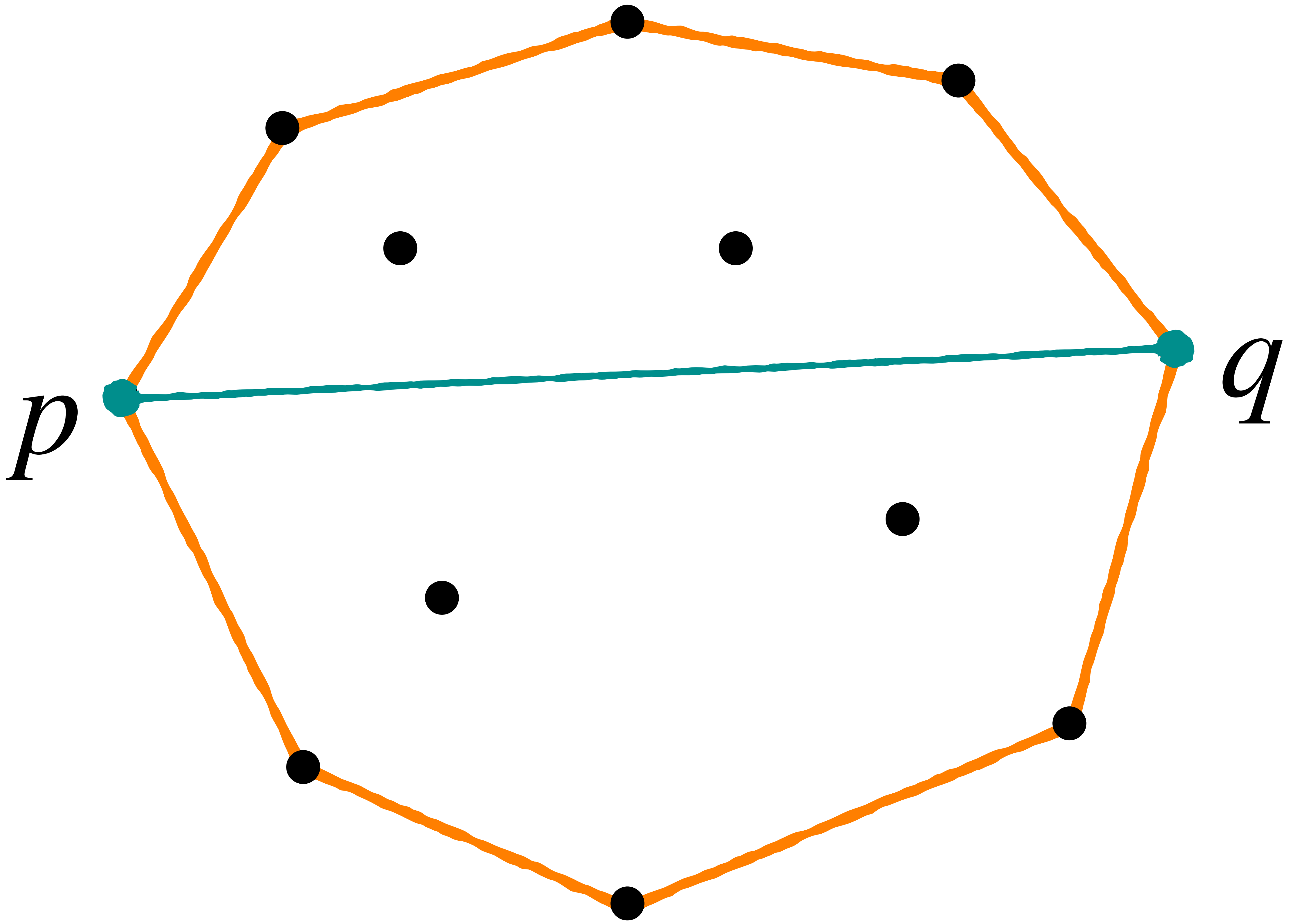

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Farthest point pairs
Antipodal pairs

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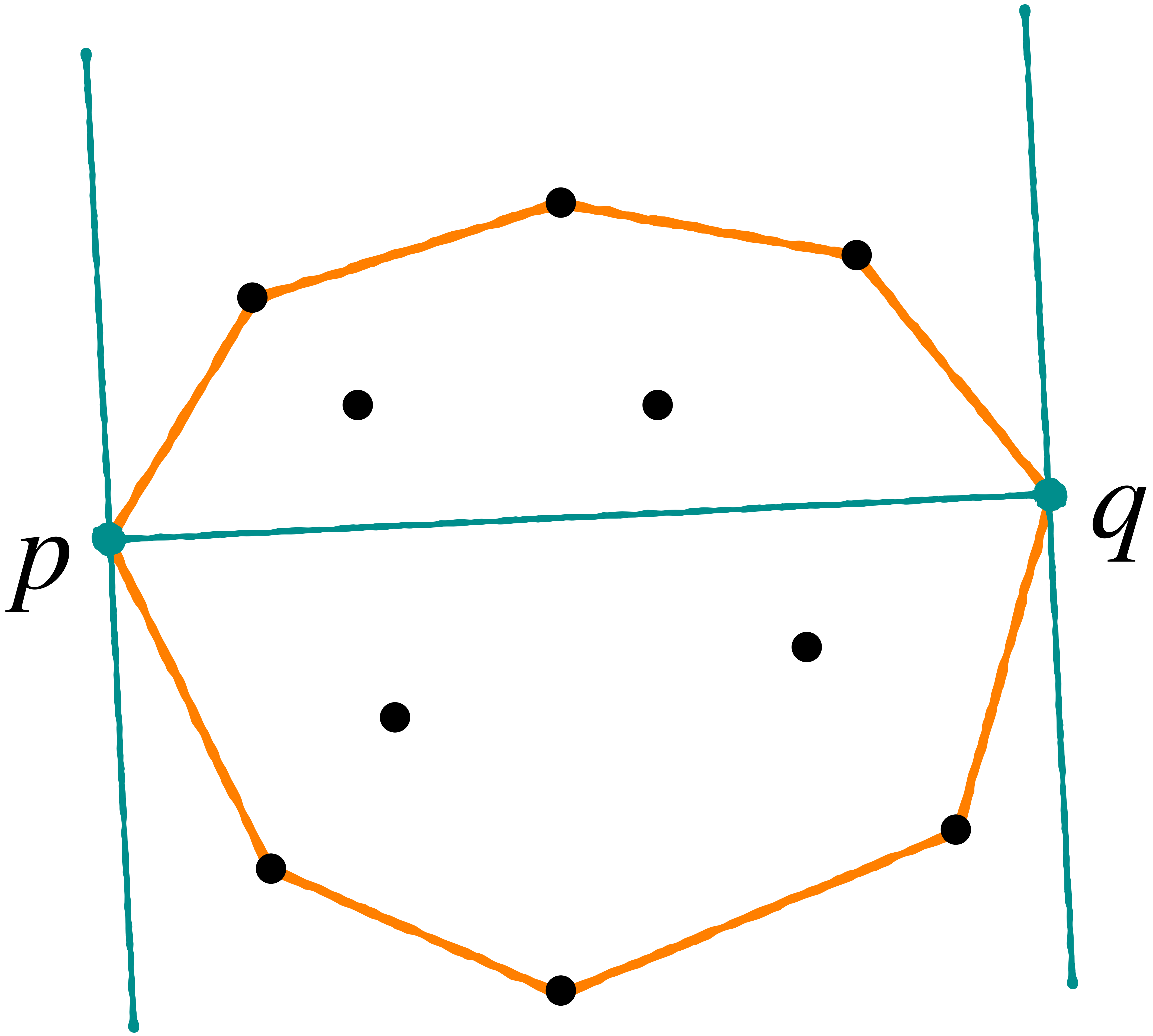
Lemma E3.1 All farthest pairs of $\mathscr{P}$ consist of two vertices of the convex hull $\text{conv}(\mathscr{P})$.

Definition. Two points $p, q \in \mathscr{P}$ are **antipodal** if there exist parallel **supporting lines** through them which touch, but do not cut the convex hull.



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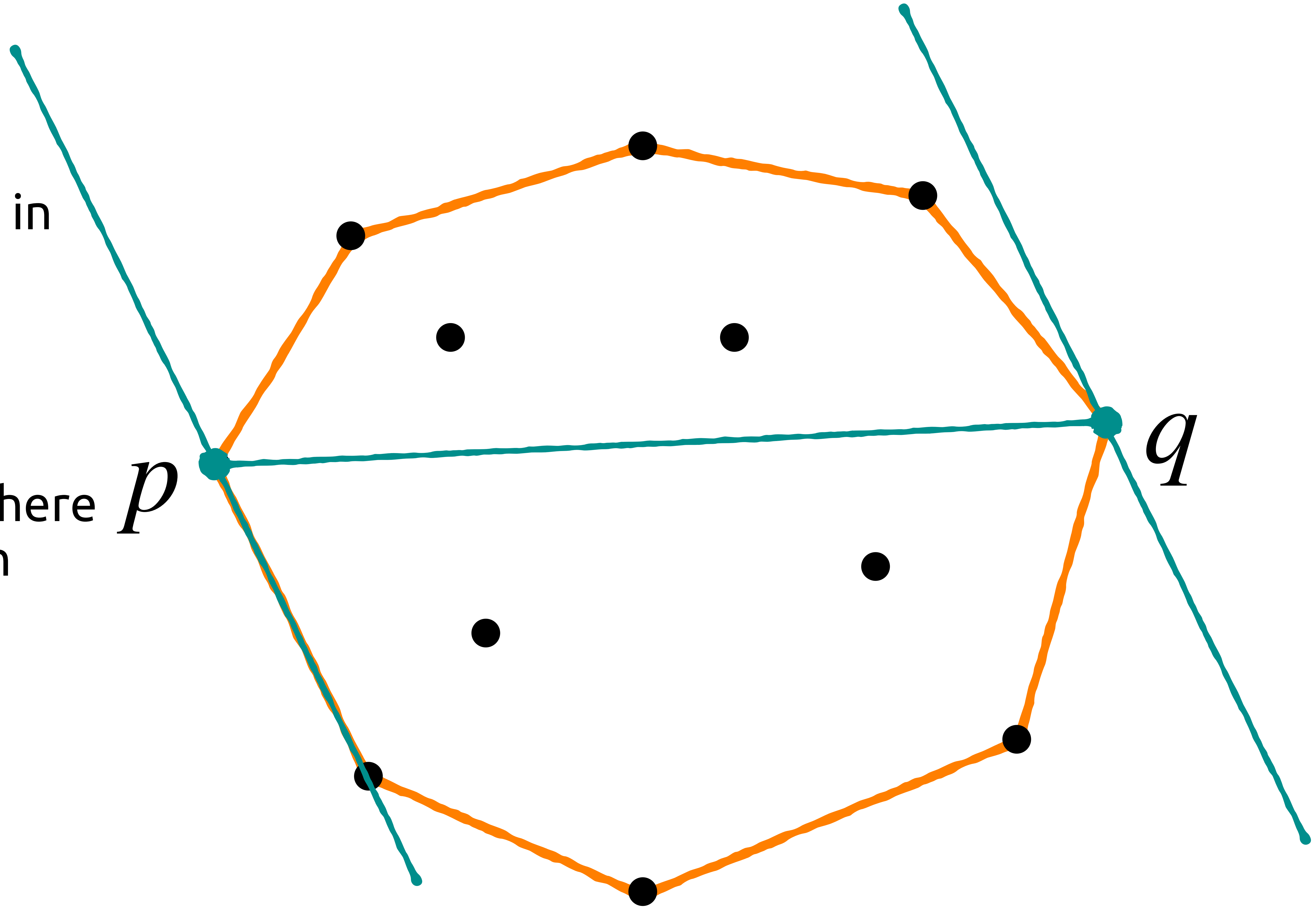

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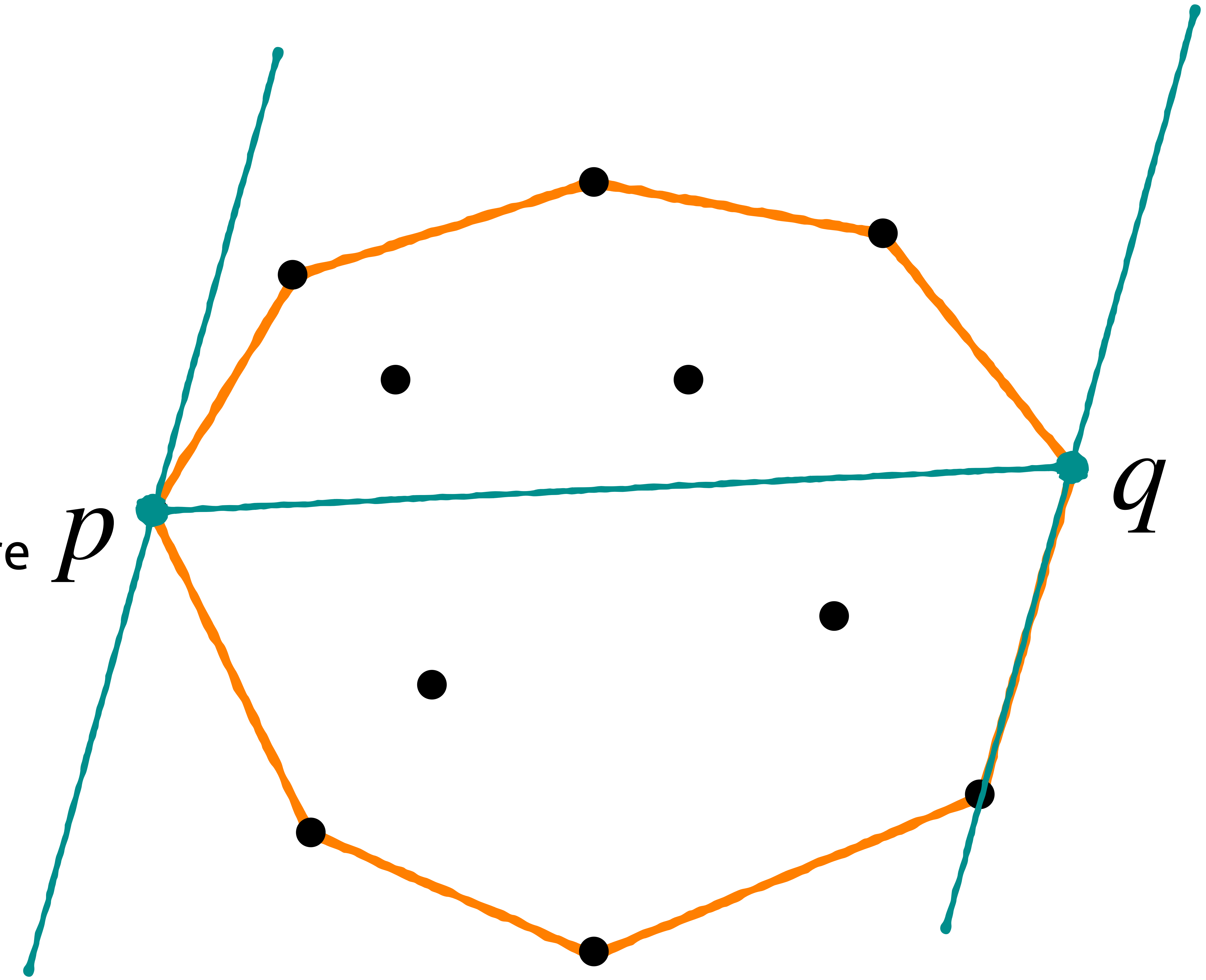

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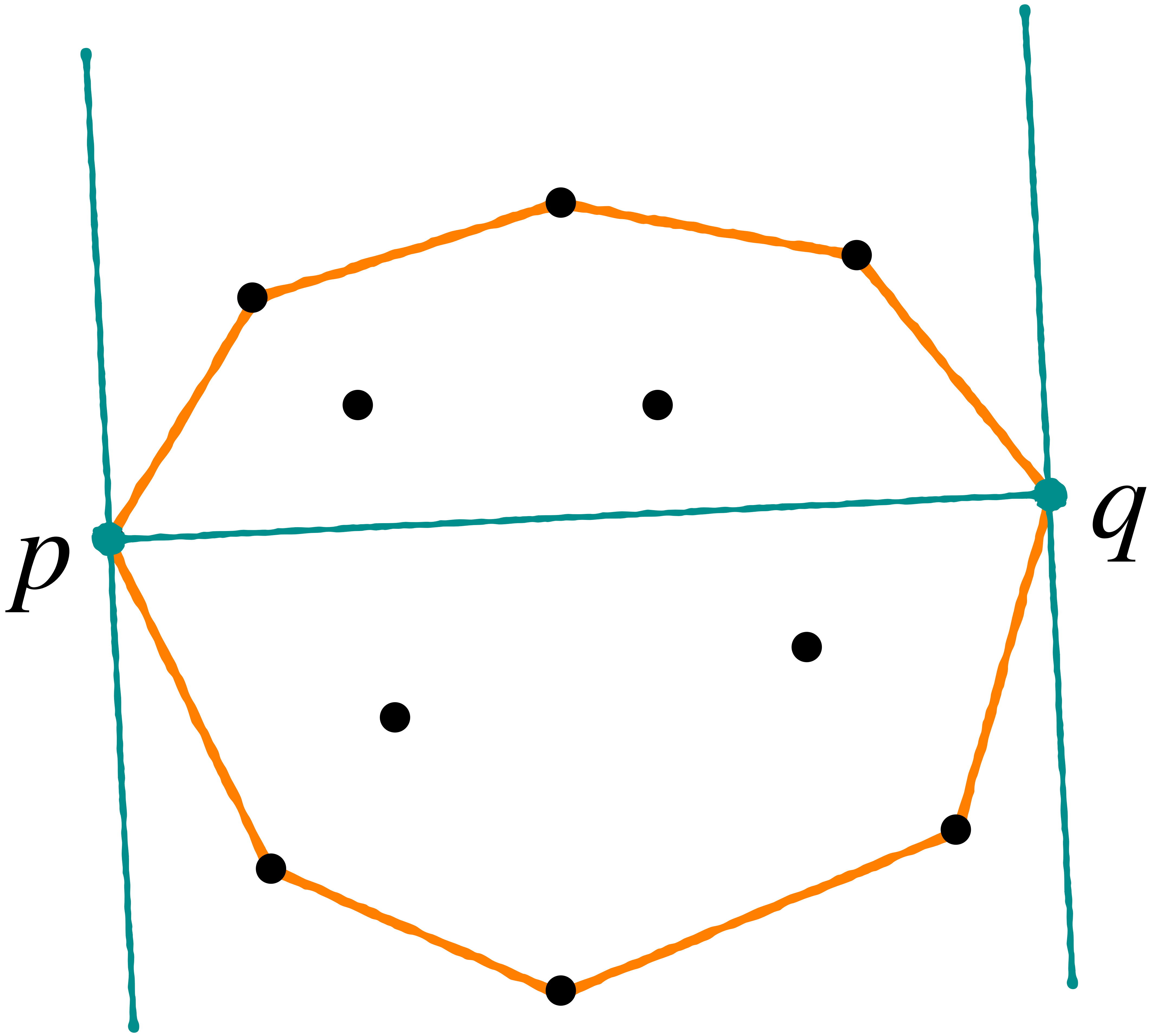

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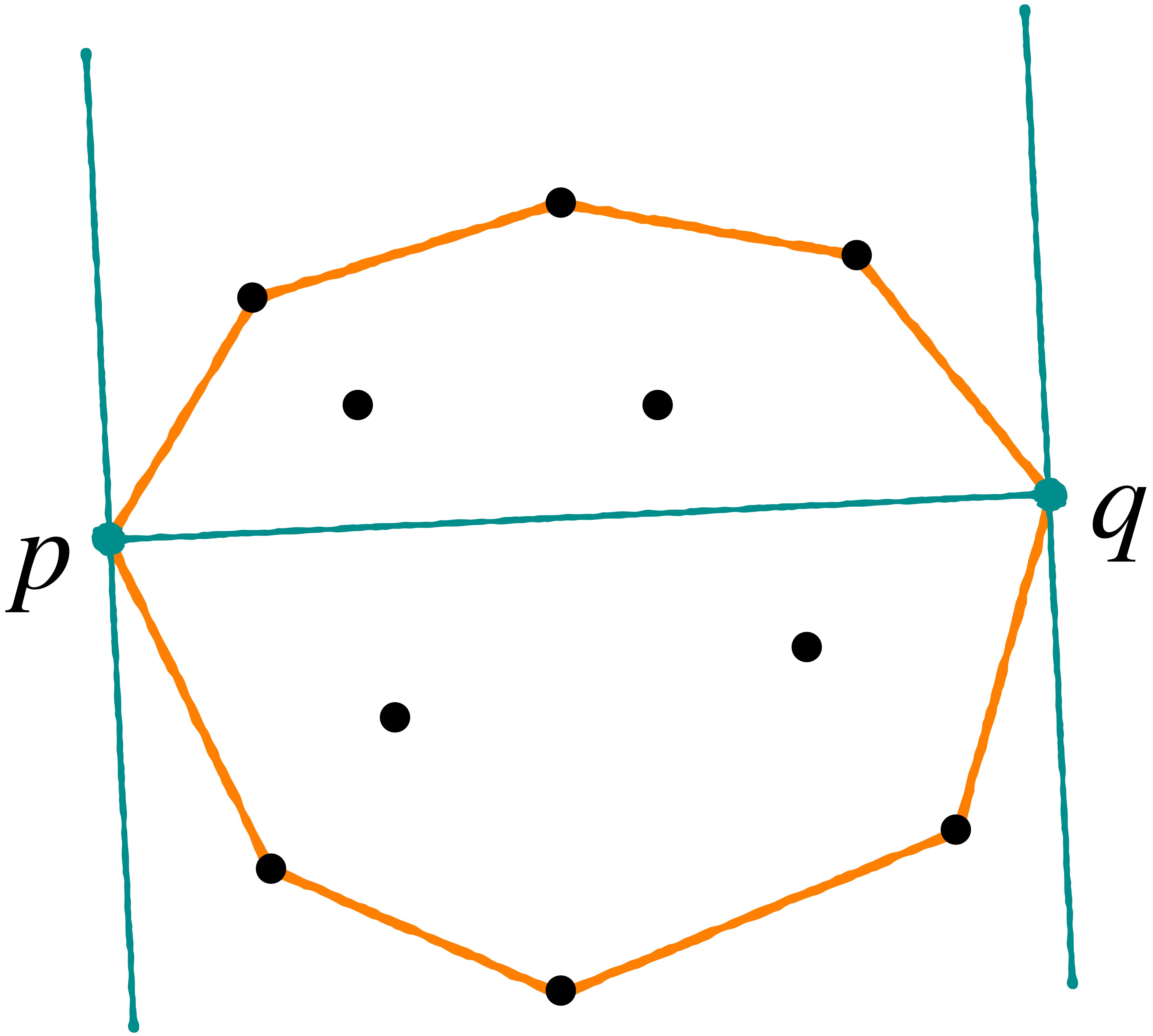
Lemma E3.2 All farthest pairs of $\mathscr{P}$ are also antipodal.



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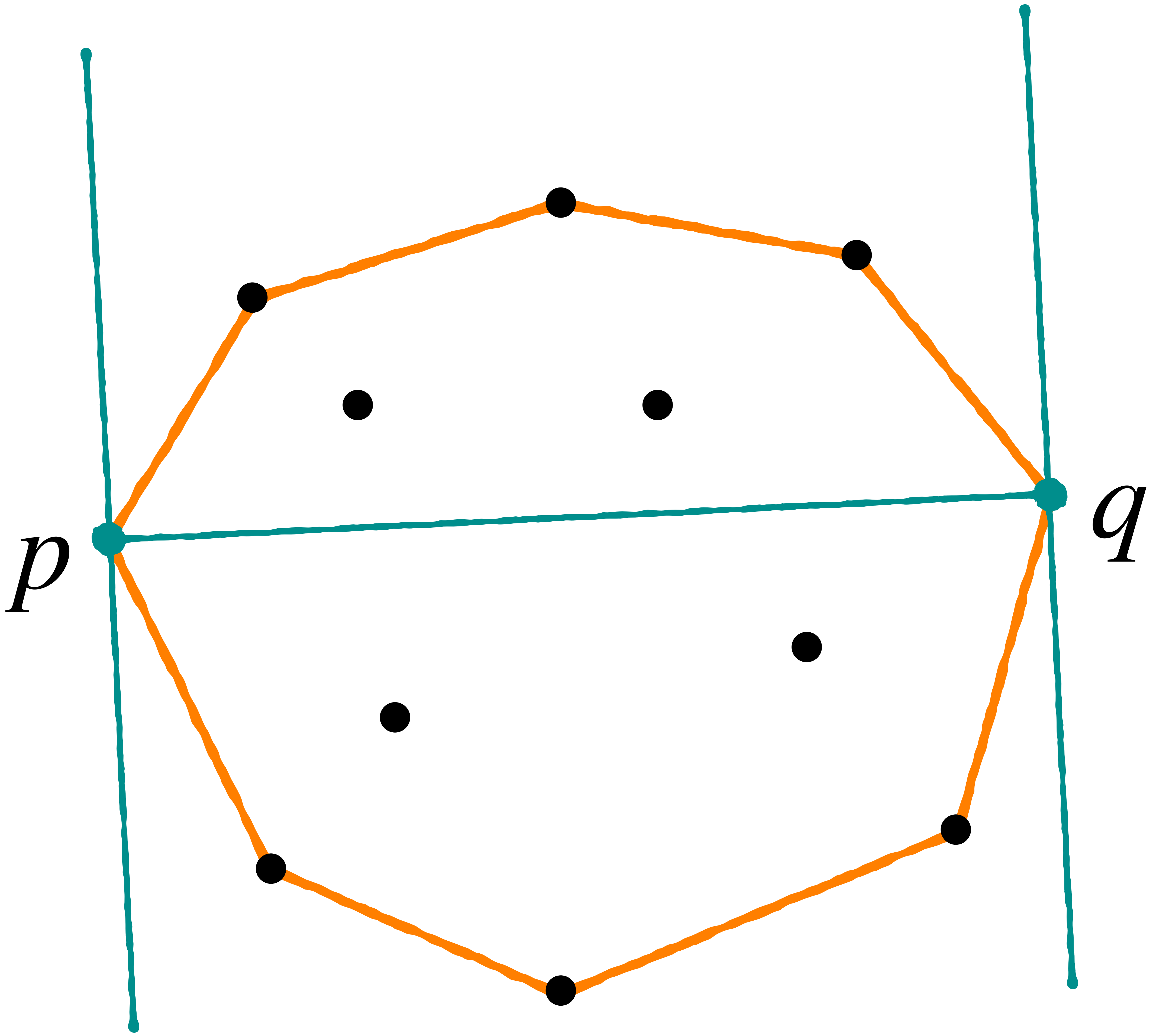
Take 10 minutes to think about this and discuss :)



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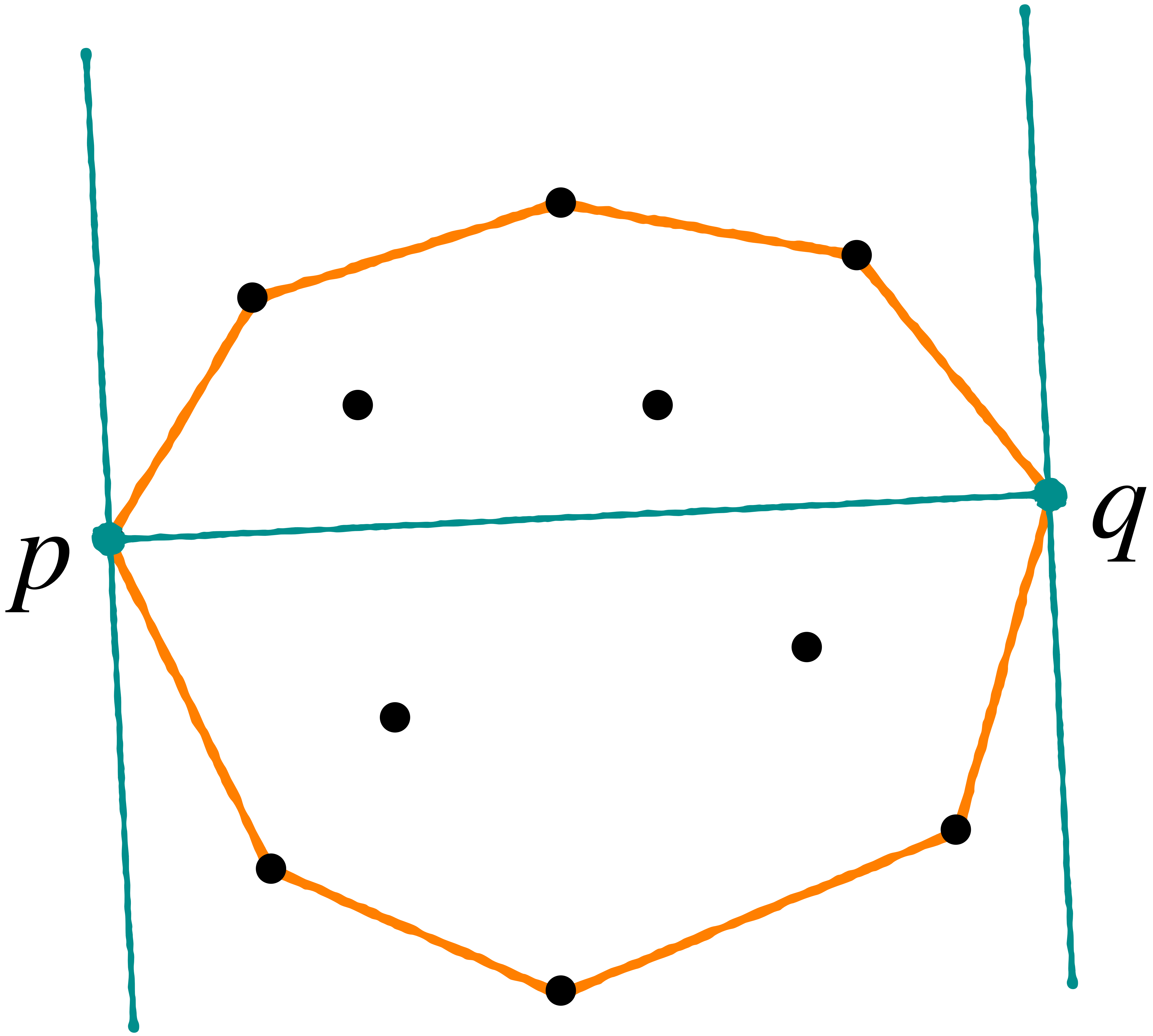

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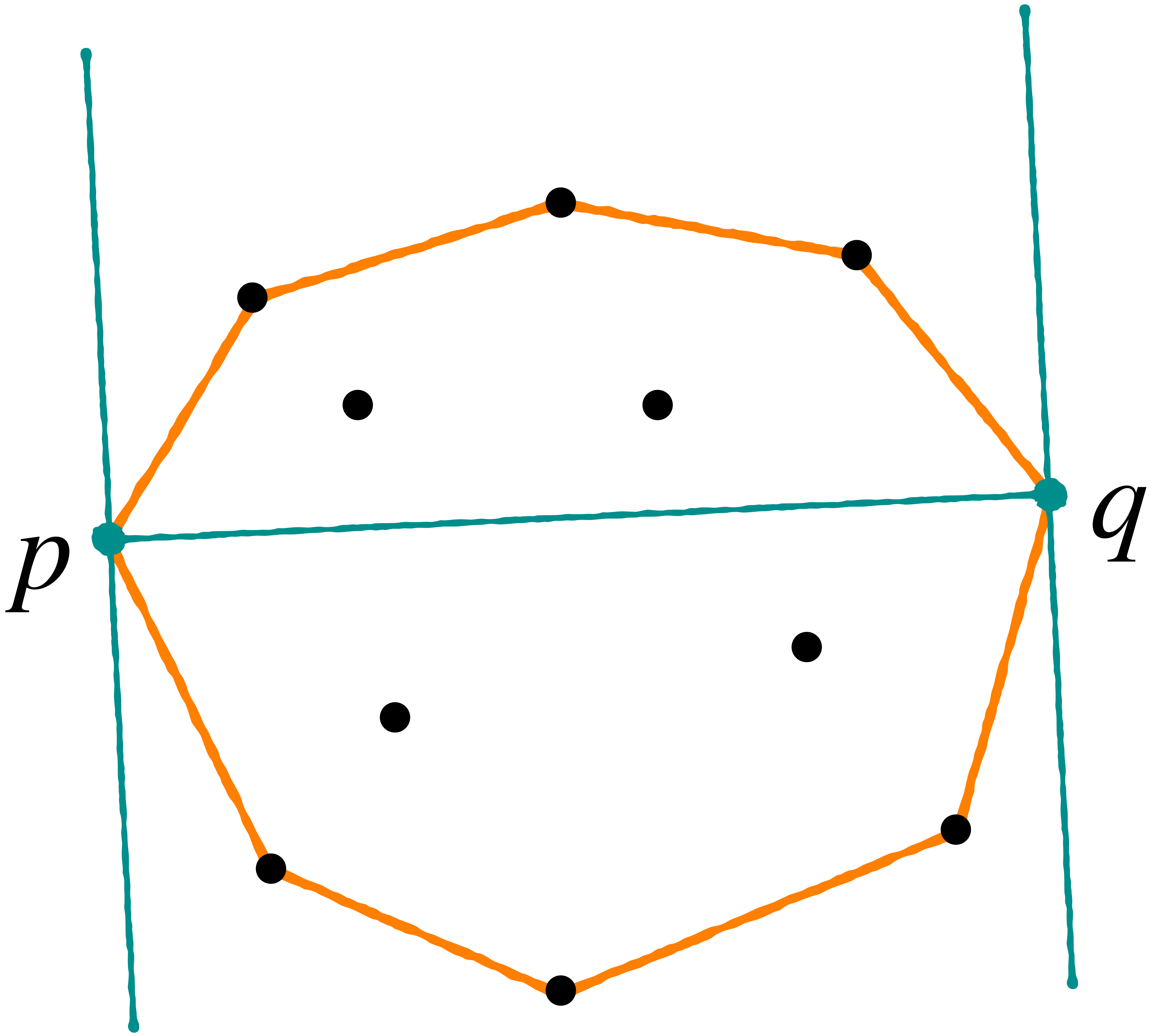
Lemma E3.3 There are $\mathcal{O}(n)$ antipodal pairs in $\mathscr{P}$.



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Michael Shamos, 1978

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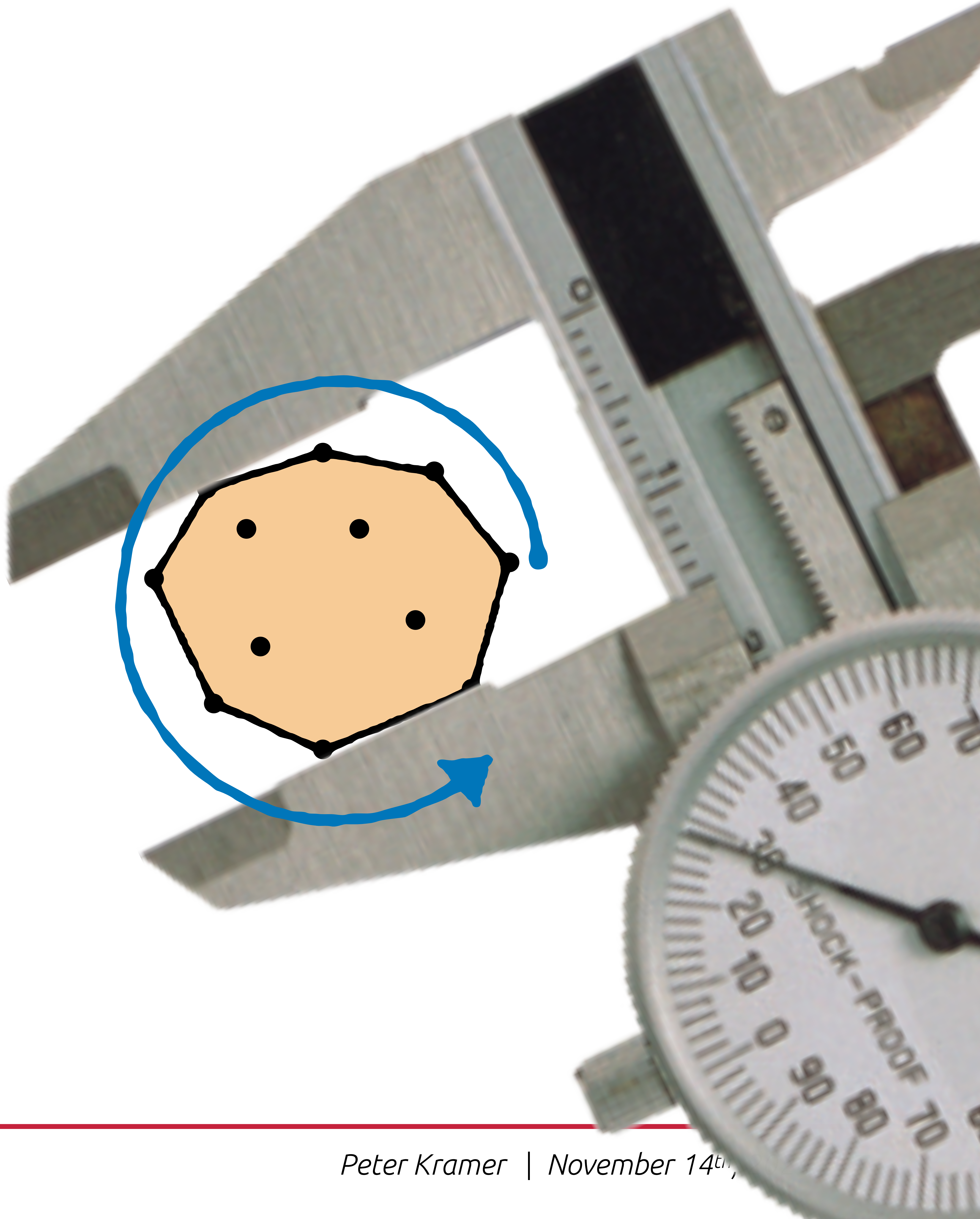

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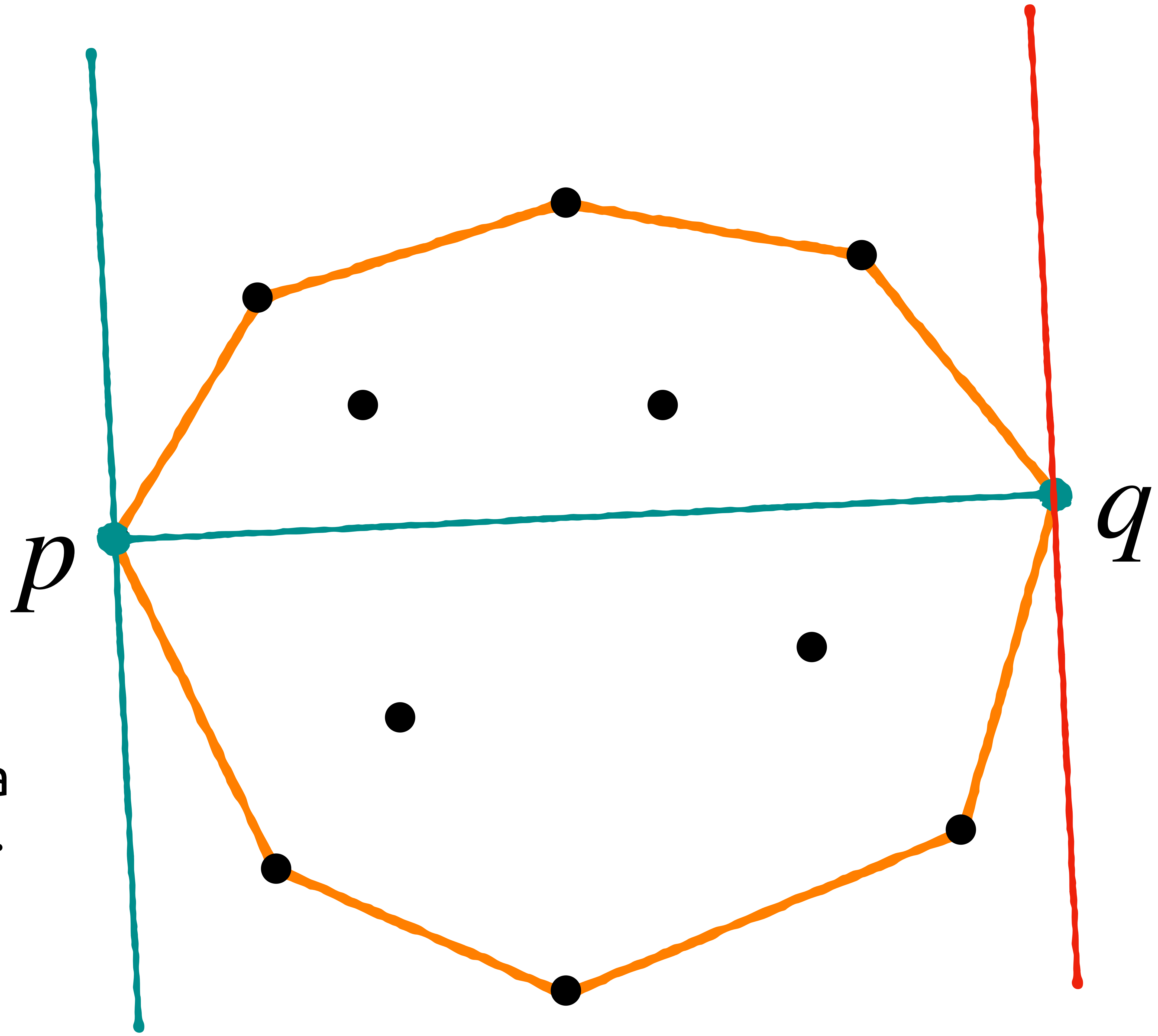

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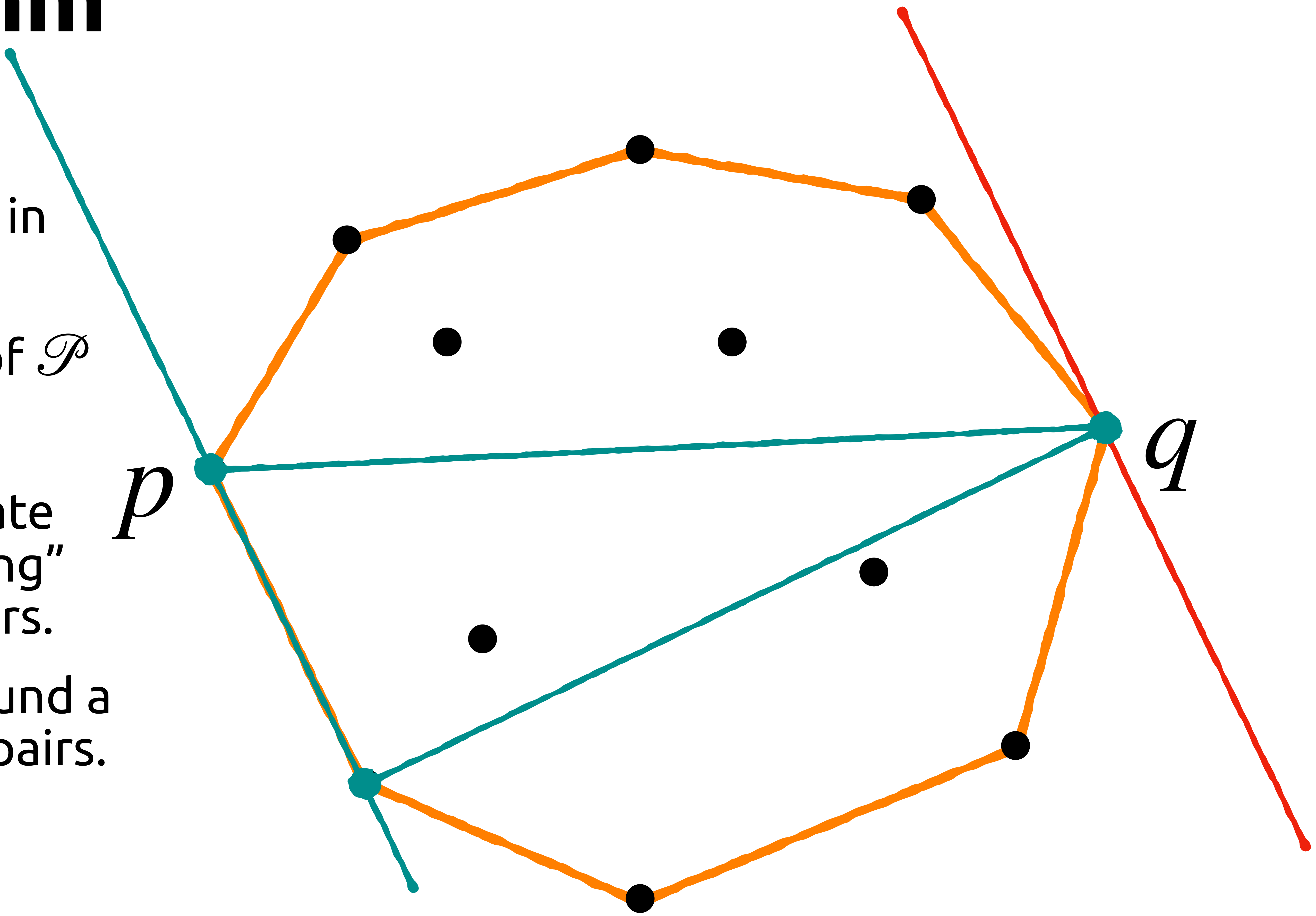

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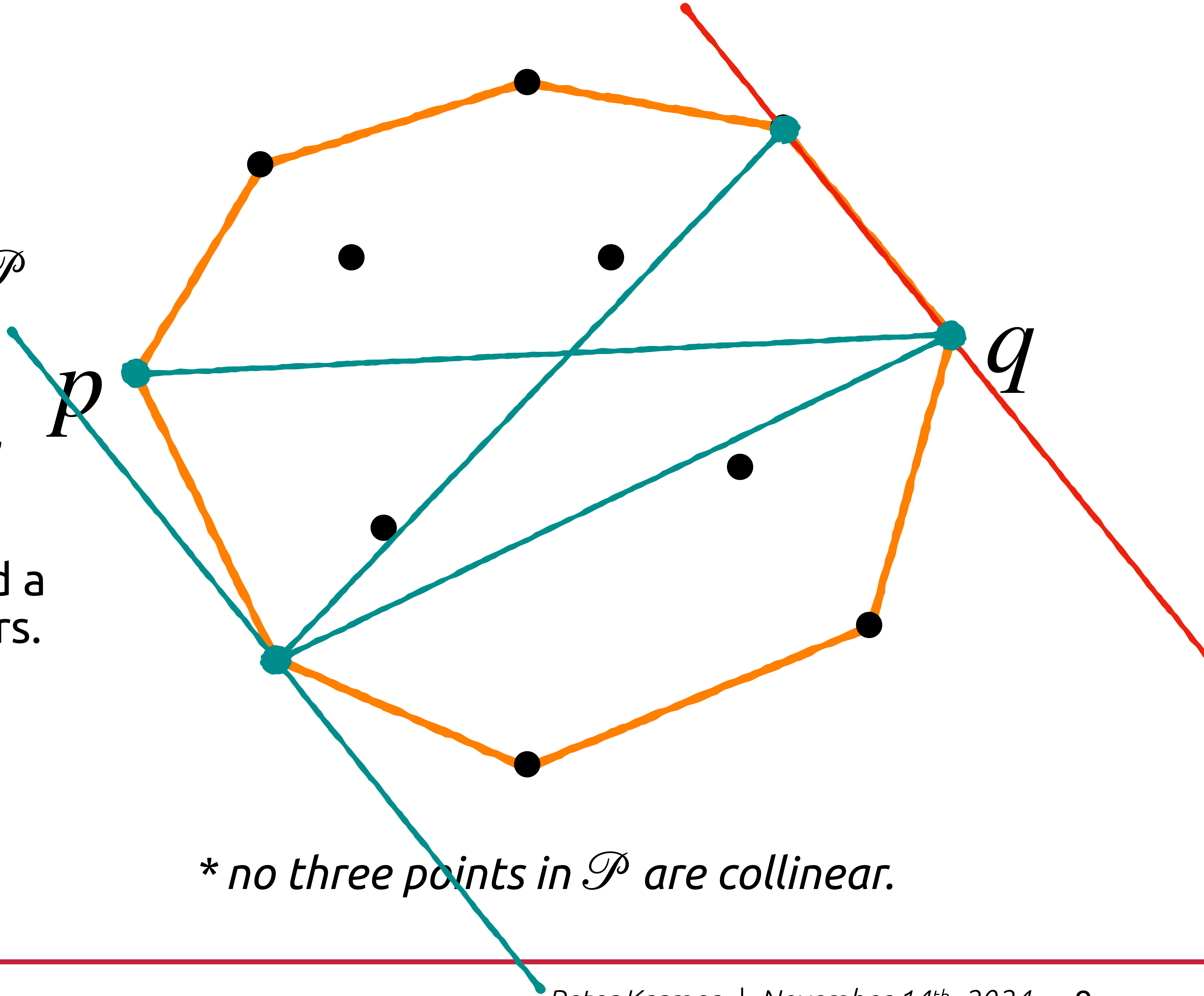

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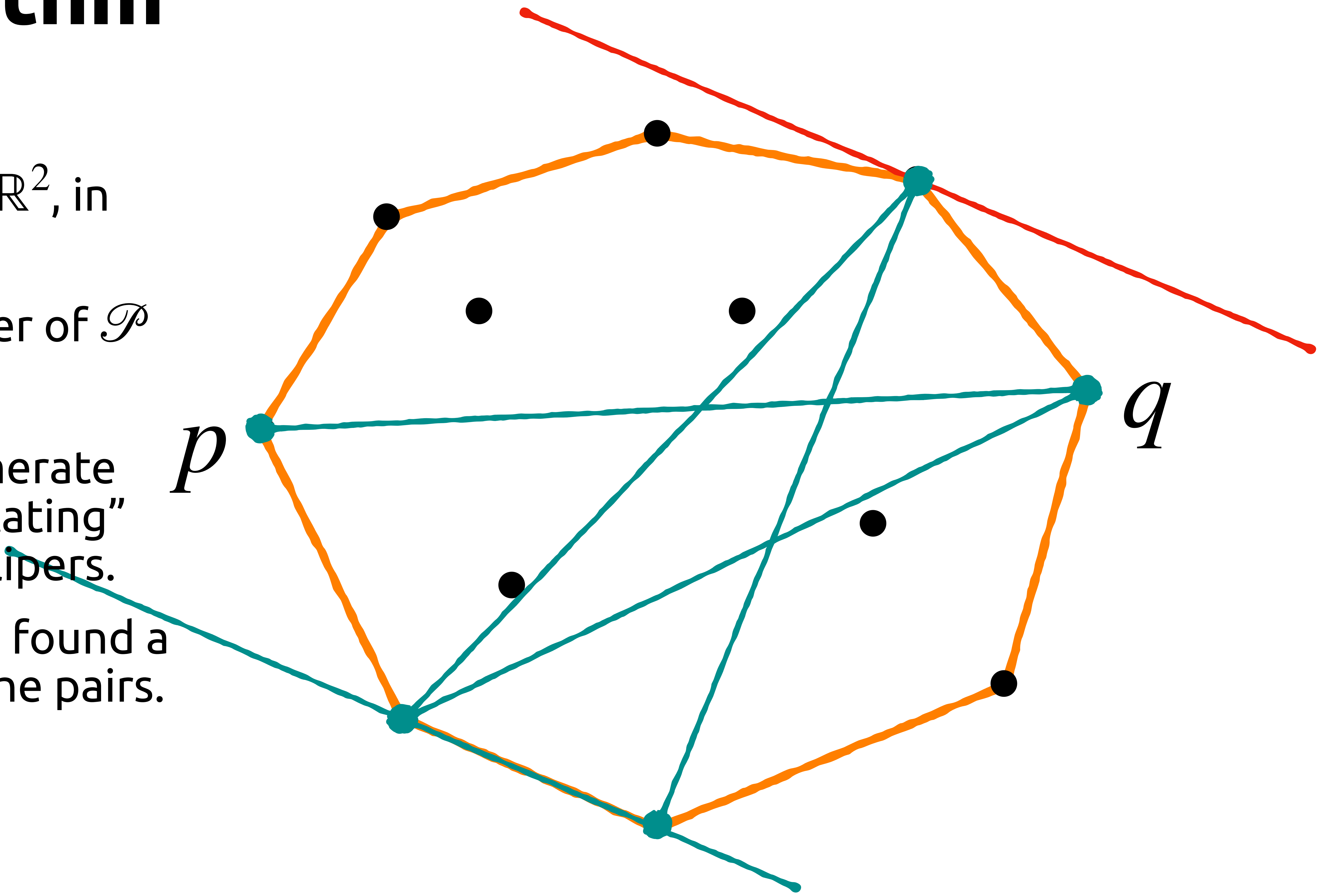

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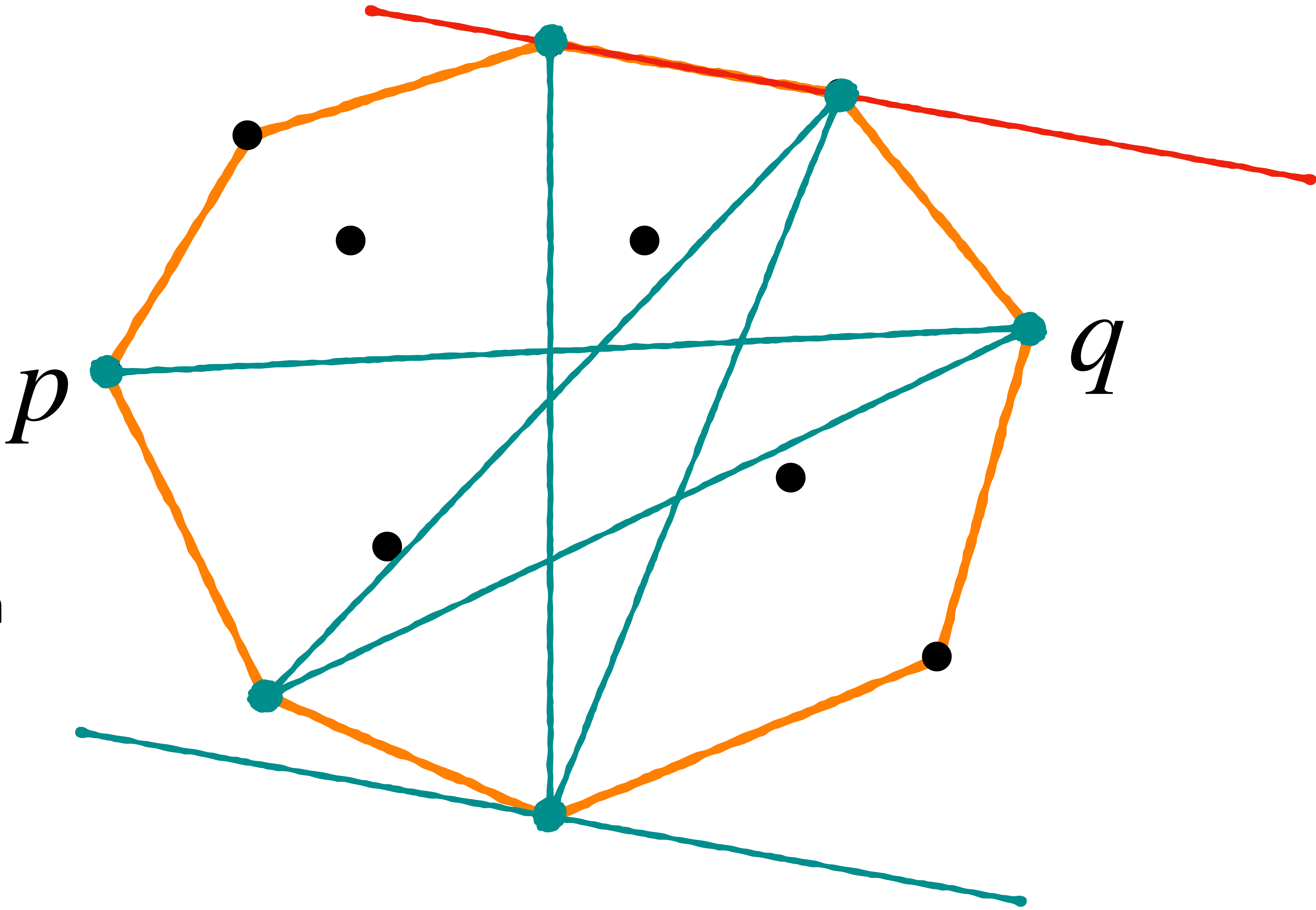

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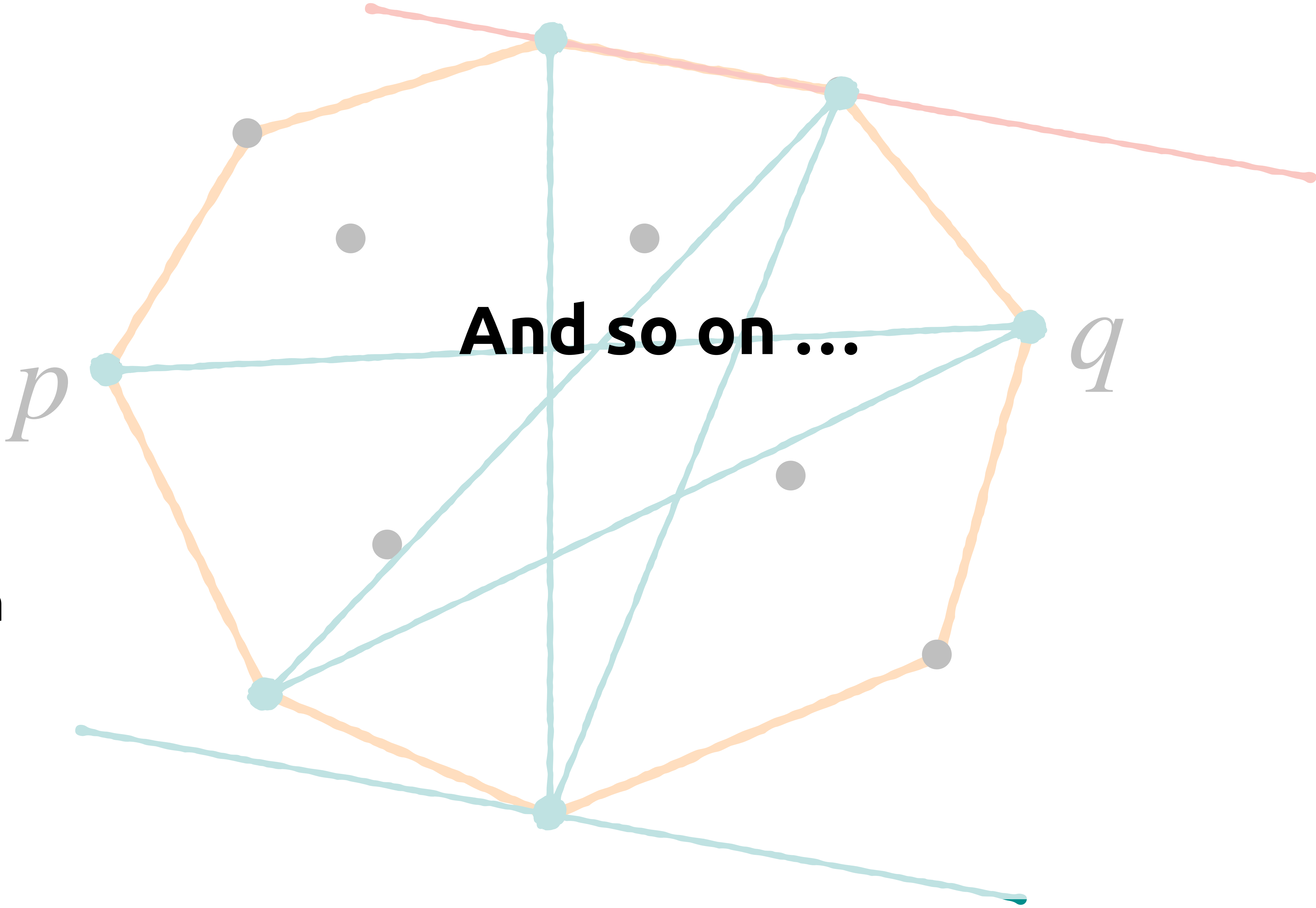

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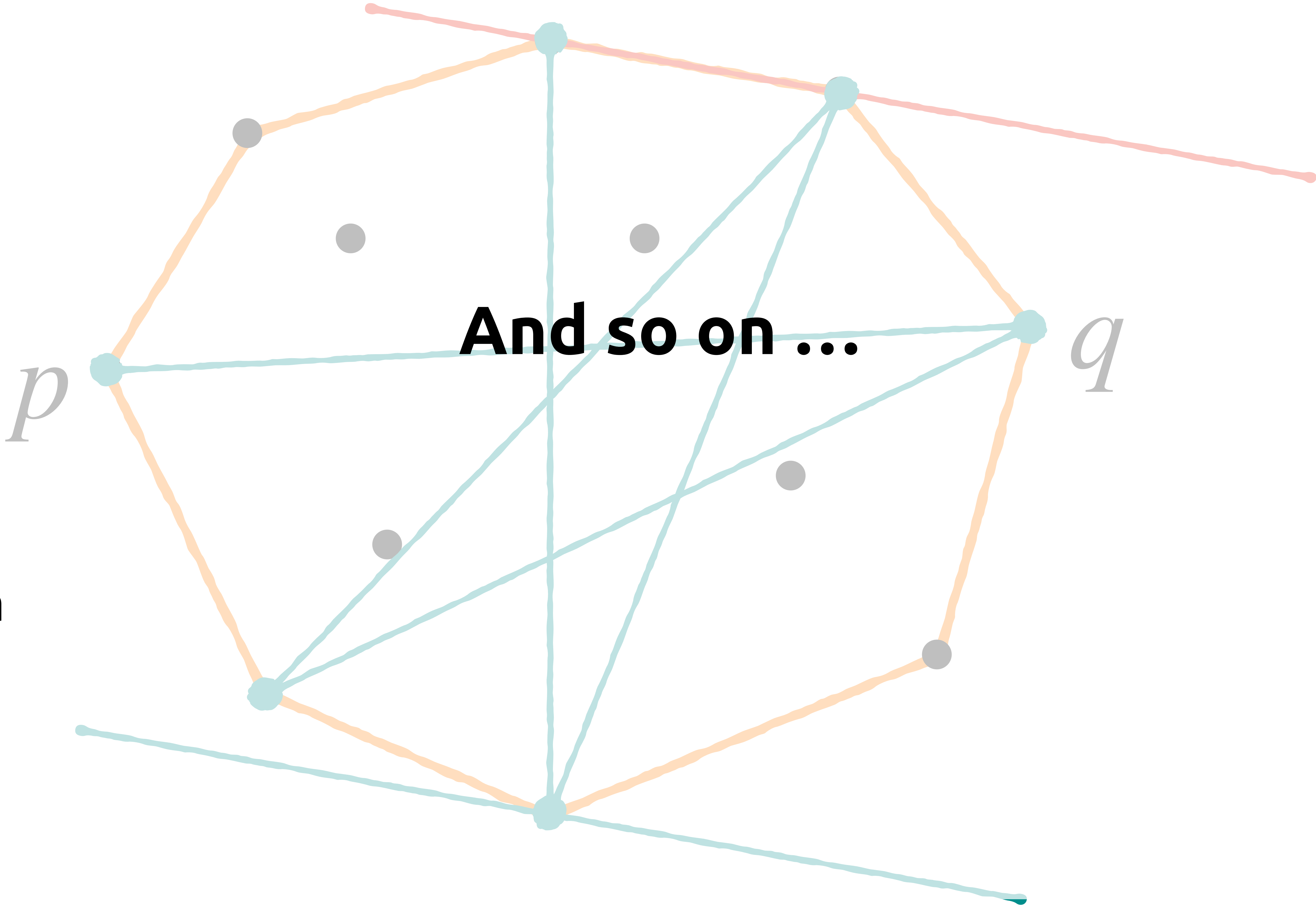

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Diameter(n: number, (p₁, …, pₙ): points) : number {
   // Linear probing / brute force – implicitly, i = 1
   find first (i,j) such that (pᵢ,pⱼ) is antipodal
   let diameter = 0
   while (j != n) {
      // Which edge do we hit?
      if A(Δ(pᵢ, pᵢ₊₁, pⱼ₊₁)) > A(Δ(pᵢ, pᵢ₊₁, pⱼ)) {
         ++j
      } else {
         ++i
      }
      // pᵢ,pⱼ is a farthest pair!
      diameter = max(diameter, d(pᵢ,pⱼ))
      // [… edge case handling for parallel lines: Up to 3 more pairs]
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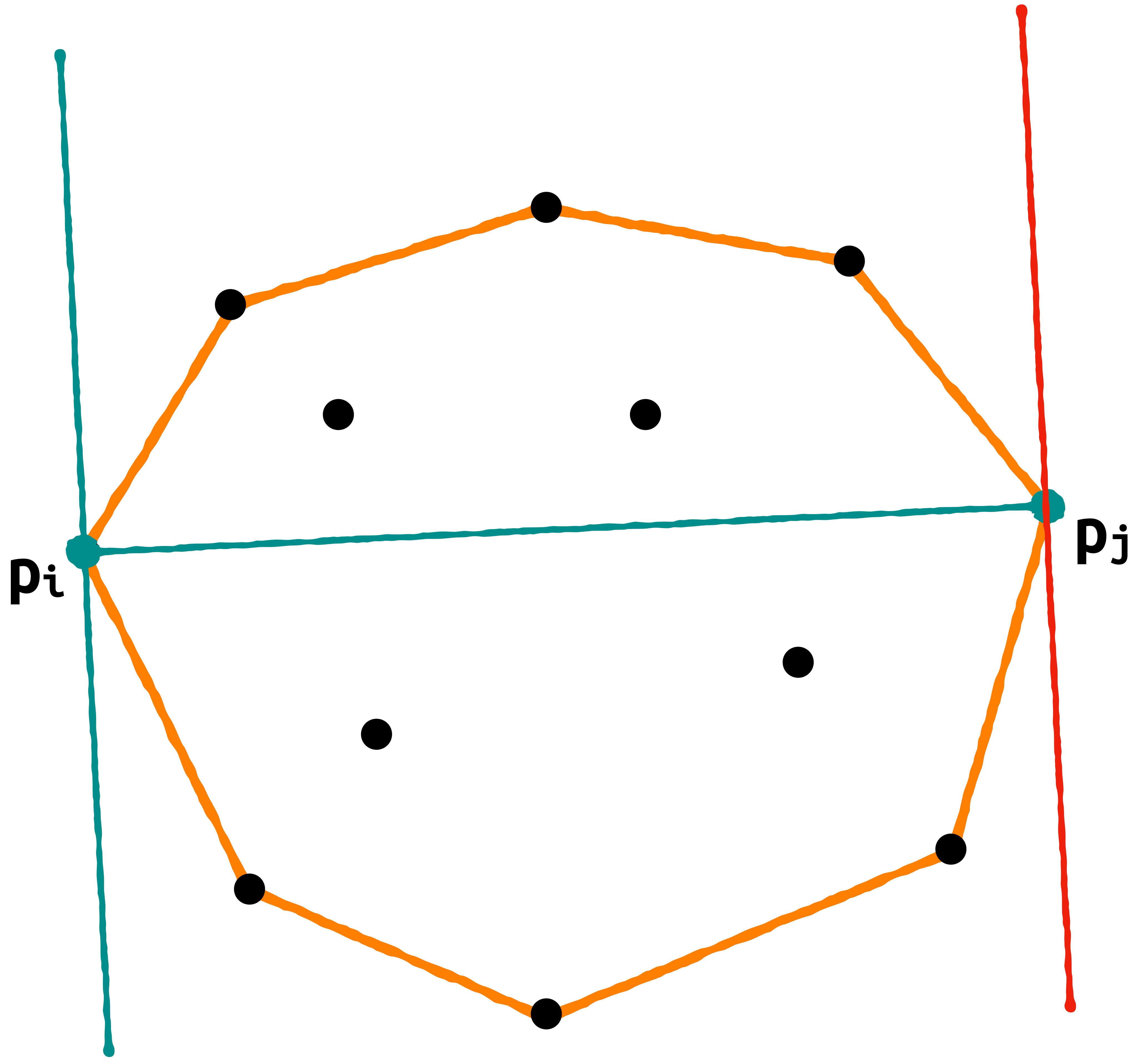

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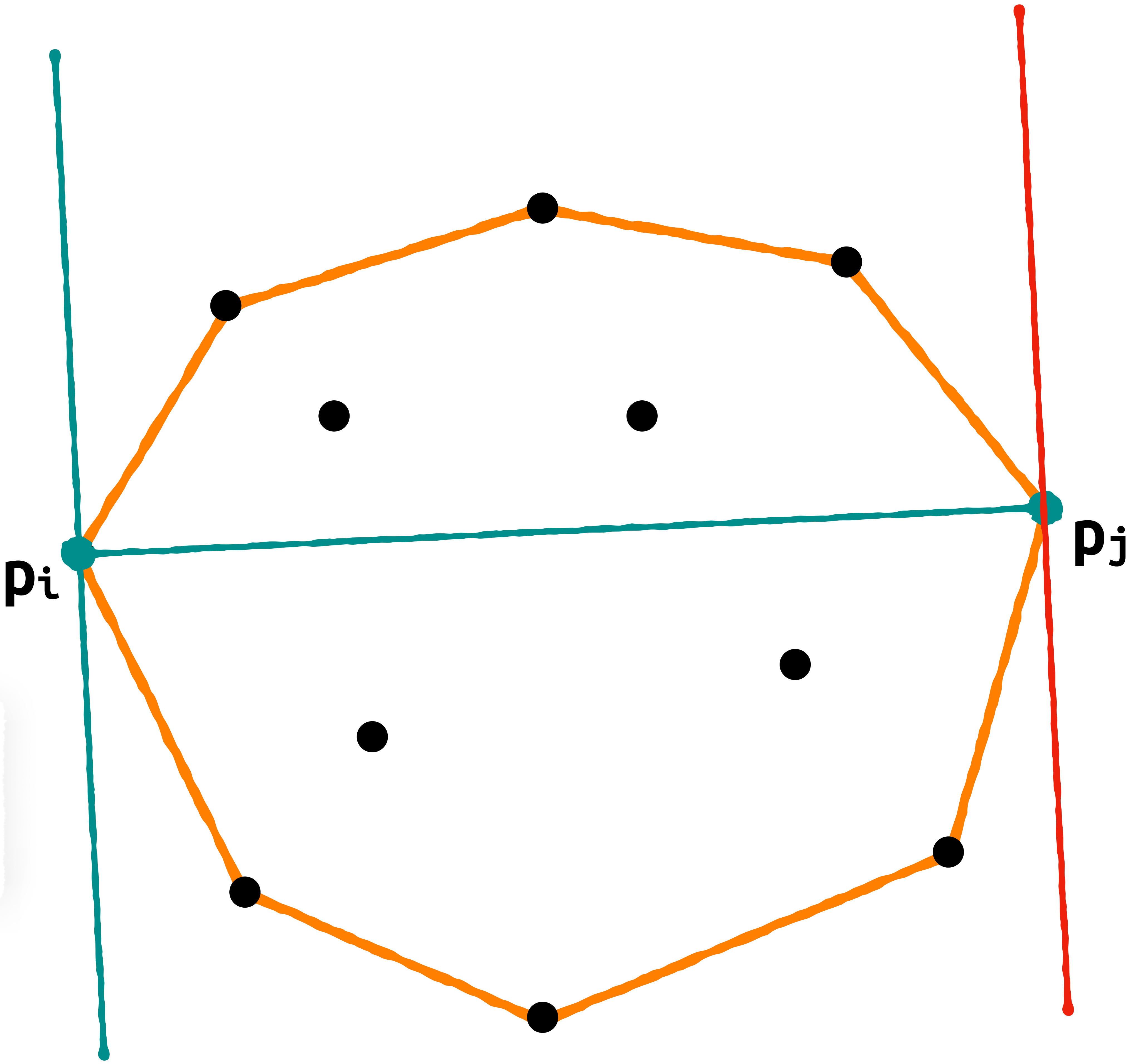
$A(\triangle(p,q,r)) > 0 \Leftrightarrow p,q,r$ oriented in counterclockwise (CCW) order

- $A(\triangle(p,q,r)) \begin{Bmatrix} > 0 \\ = 0 \\ < 0 \end{Bmatrix} \Leftrightarrow p,q,r \begin{Bmatrix} \text{Left turn} \\ \text{collinear} \\ \text{Right turn} \end{Bmatrix}$



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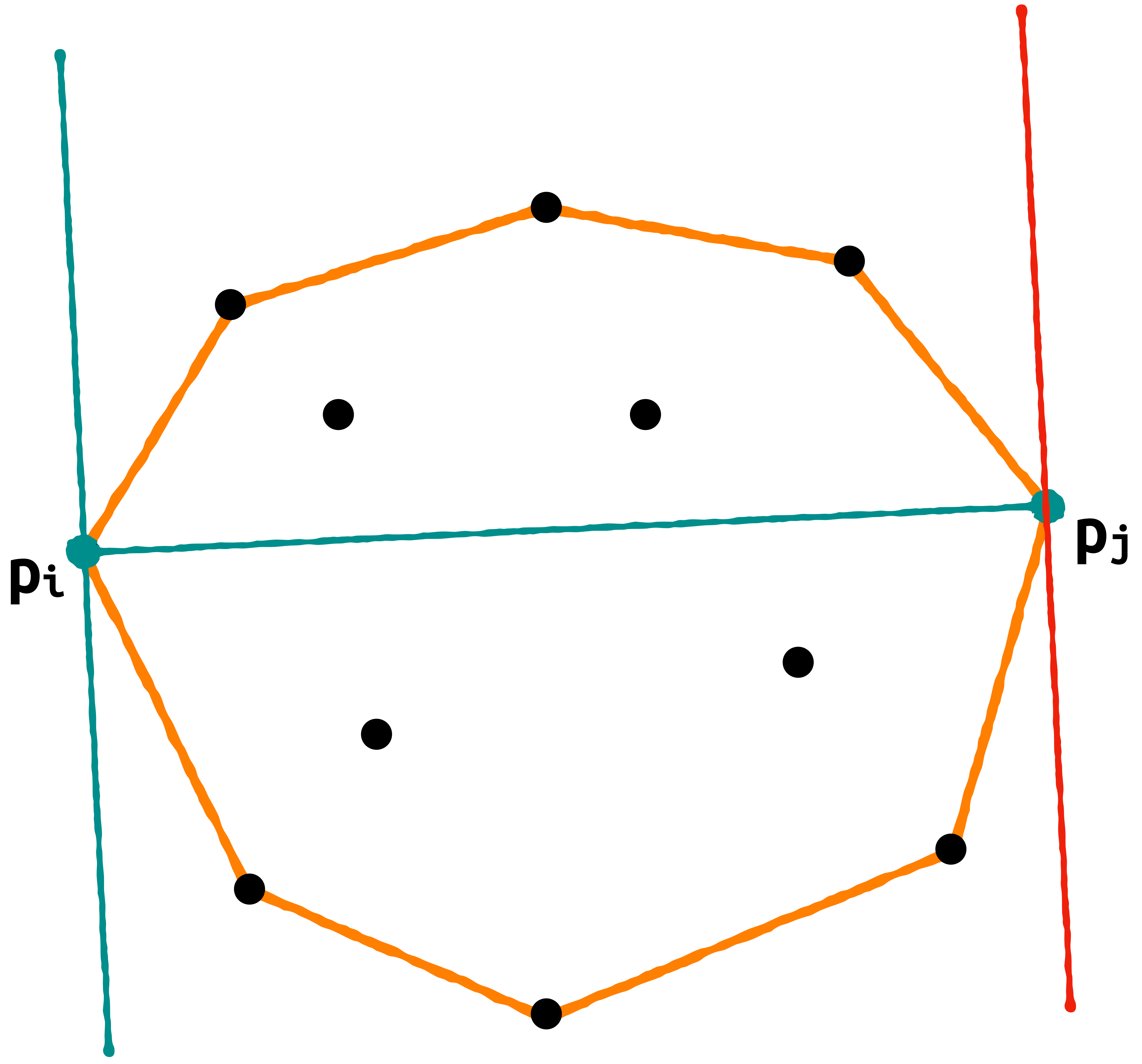

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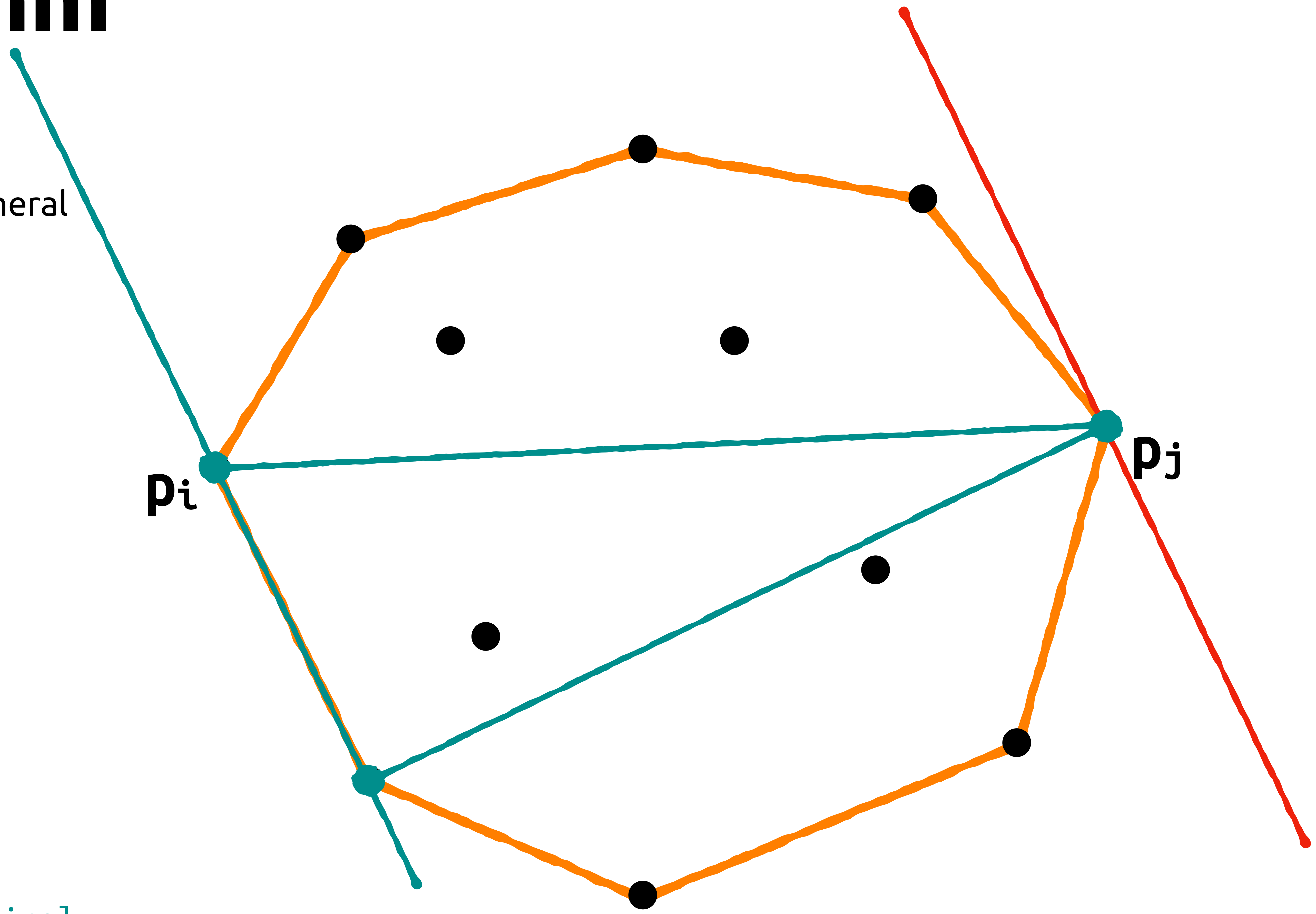

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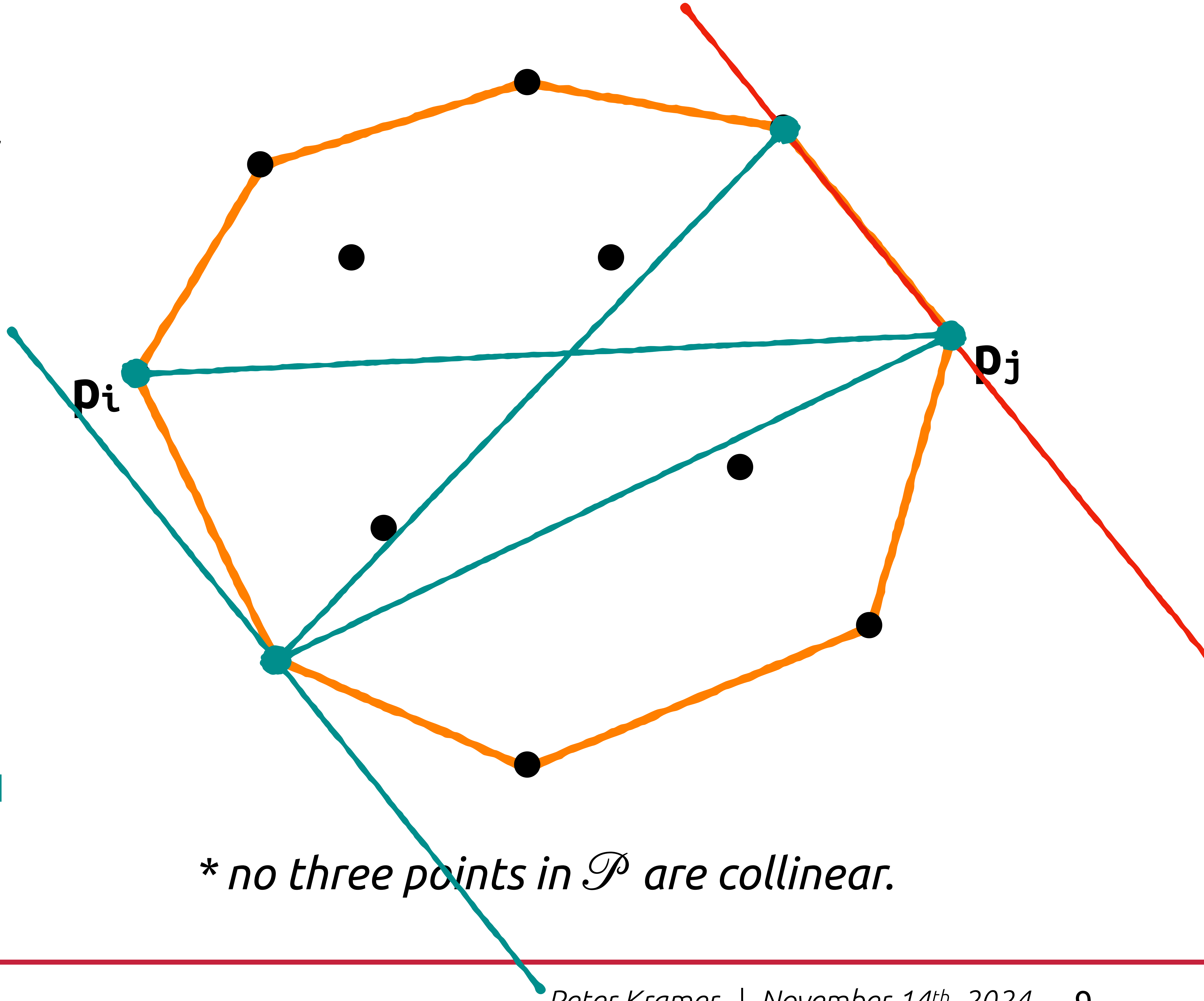

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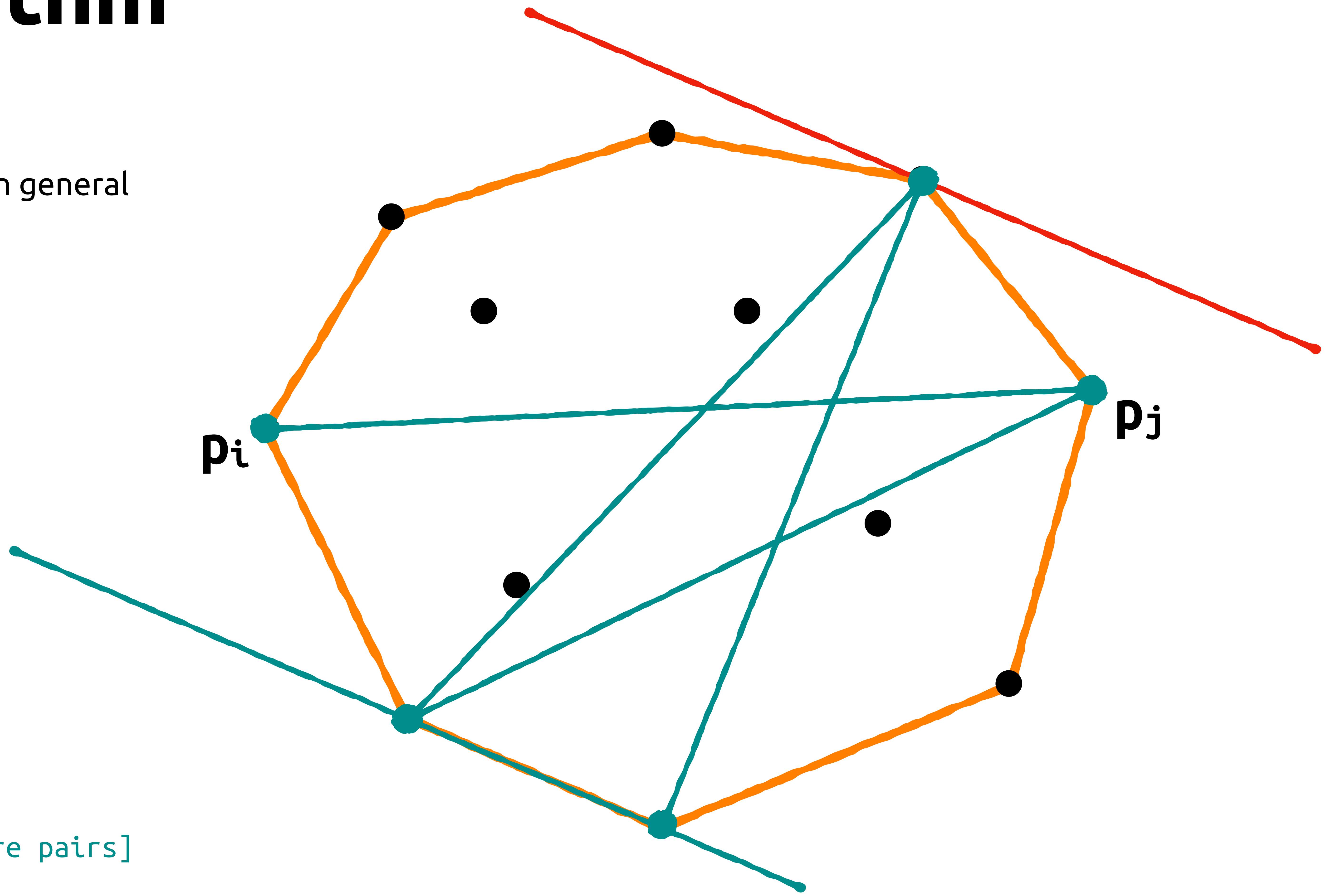

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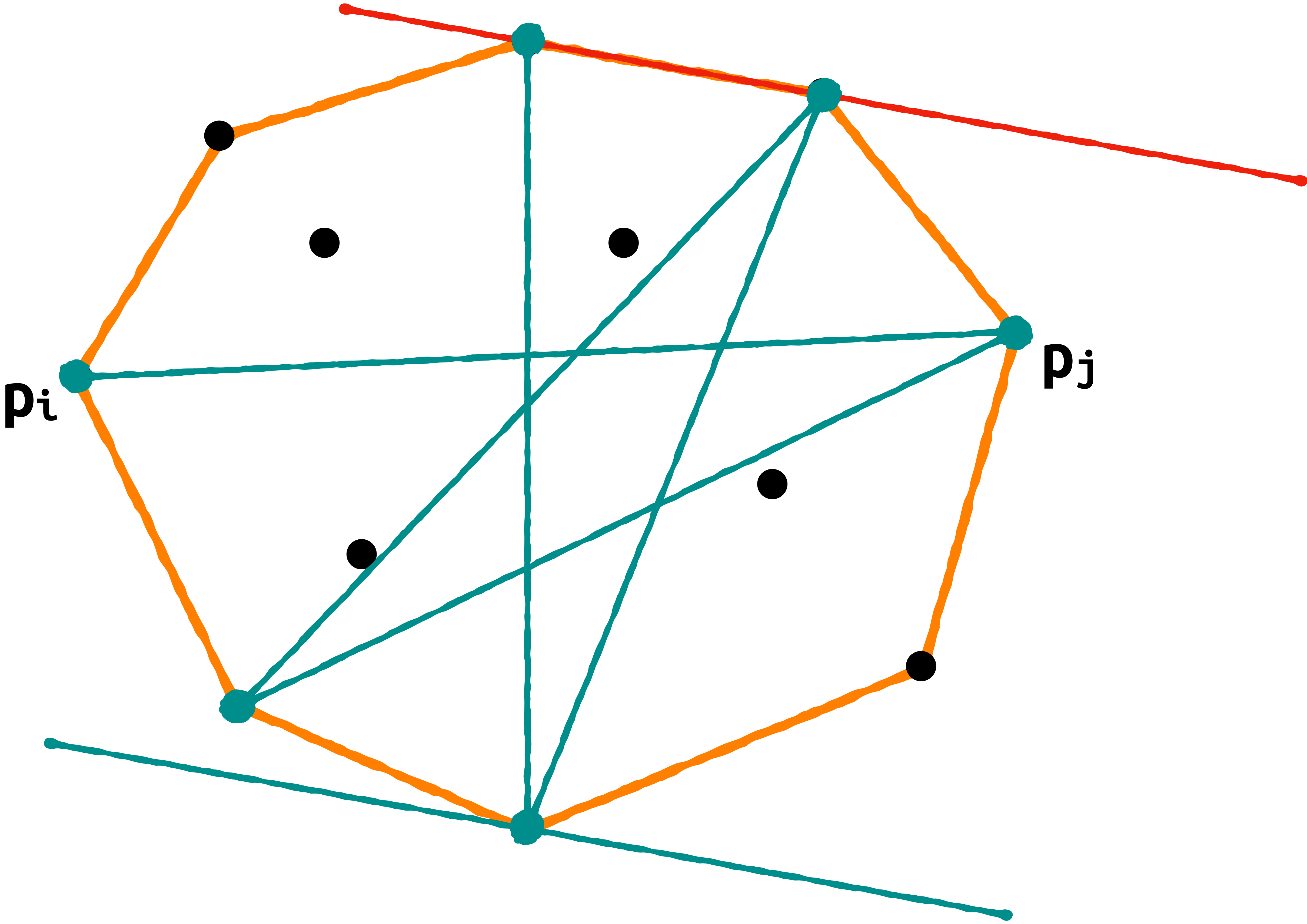
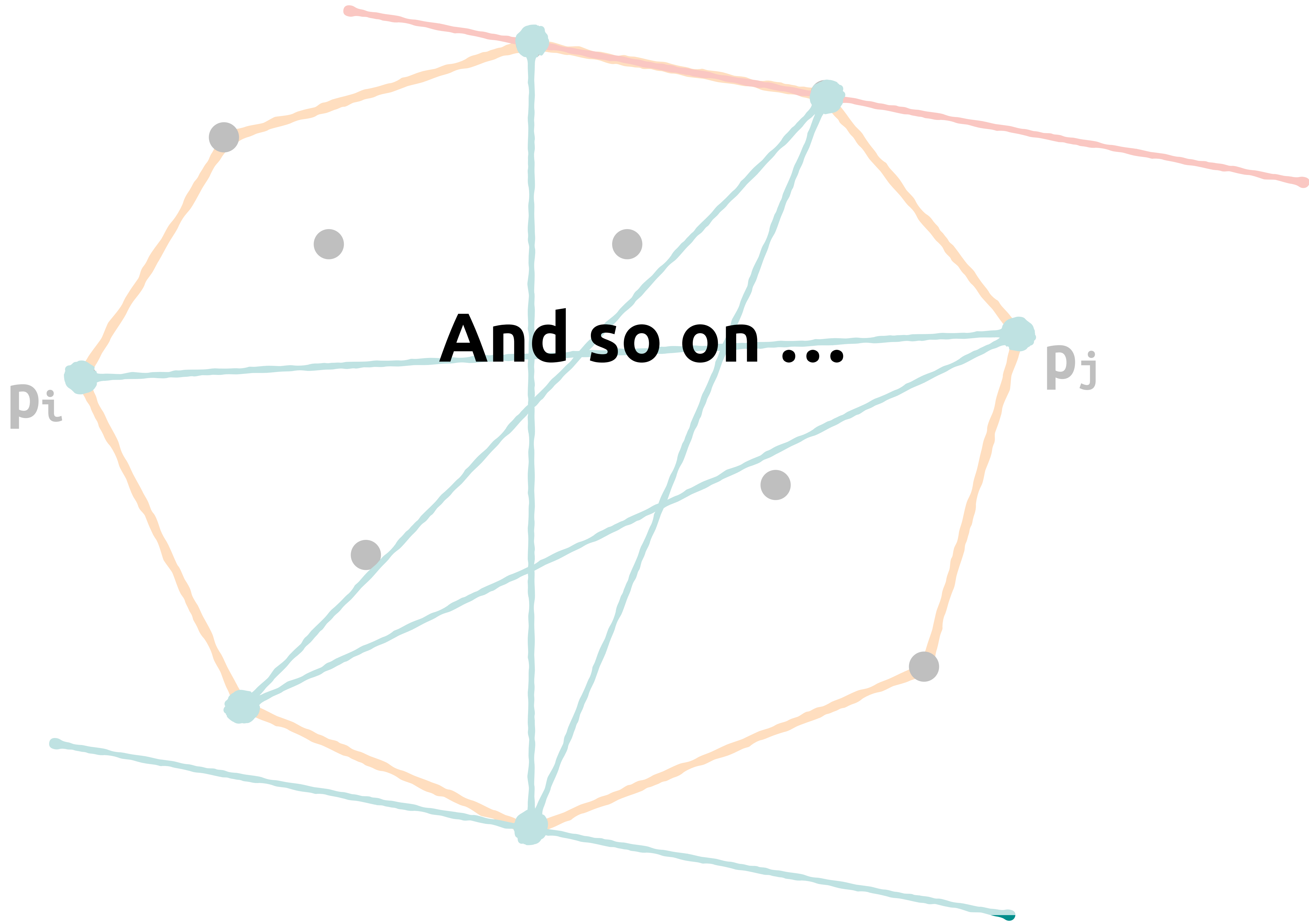


** no three points in $\mathscr{P}$ are collinear.*

Rotating Calipers Algorithm
Michael Shamos, 1978

Theorem E3.4 All farthest pairs and the diameter of of n points $\mathscr{P}$ in general position in the Euclidean plane $\mathbb{R}^2$ can be computed in $\mathcal{O}(n \log n)$.

```
Diameter(n: number, (p1, …, pn): points) : number {
   // Linear probing / brute force – implicitly, i = 1
   find first (i,j) such that (pi,pj) is antipodal
   let diameter = 0
   while (j != n) {
      // Which edge do we hit?
      if A(Δ(pi, pi+1, pj+1)) > A(Δ(pi, pi+1, pj)) {
         ++j
      } else {
         ++i
      }
      // pi,pj is a farthest pair!
      diameter = max(diameter, d(pi,pj))
      // [… edge case handling for parallel lines: Up to 3 more pairs]
   }
   return diameter
}
```



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Farthest point pairs
Convex hull

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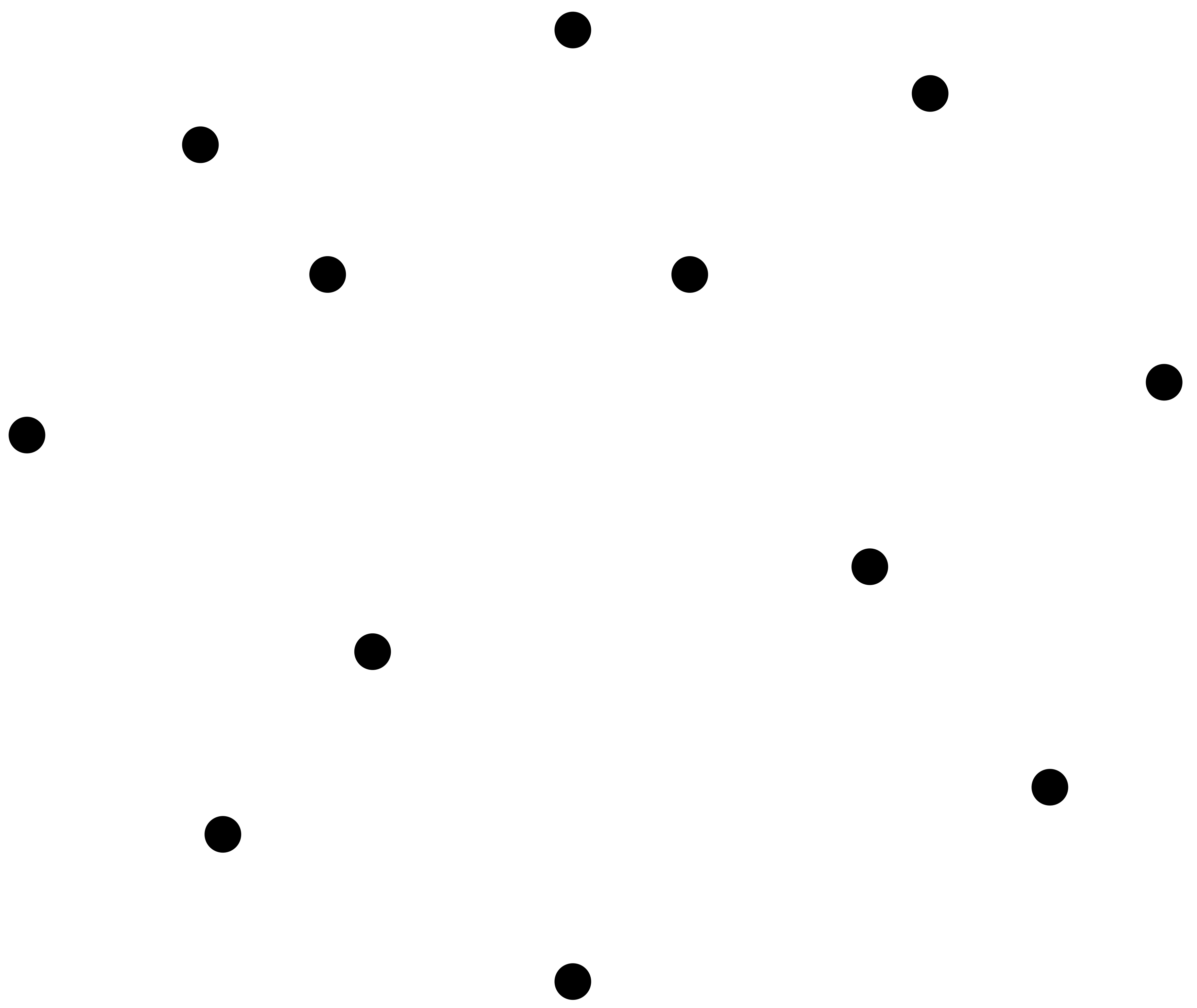

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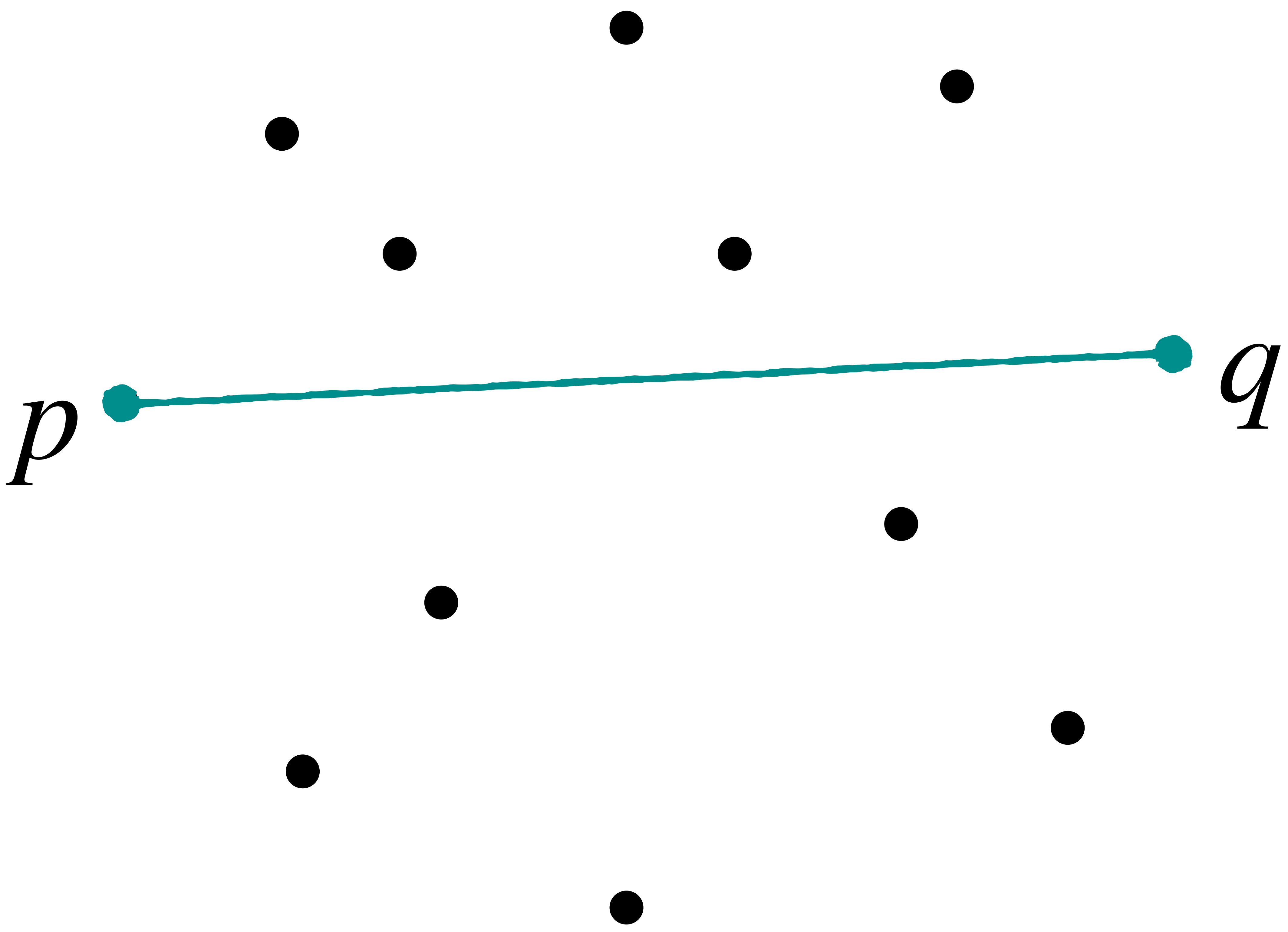

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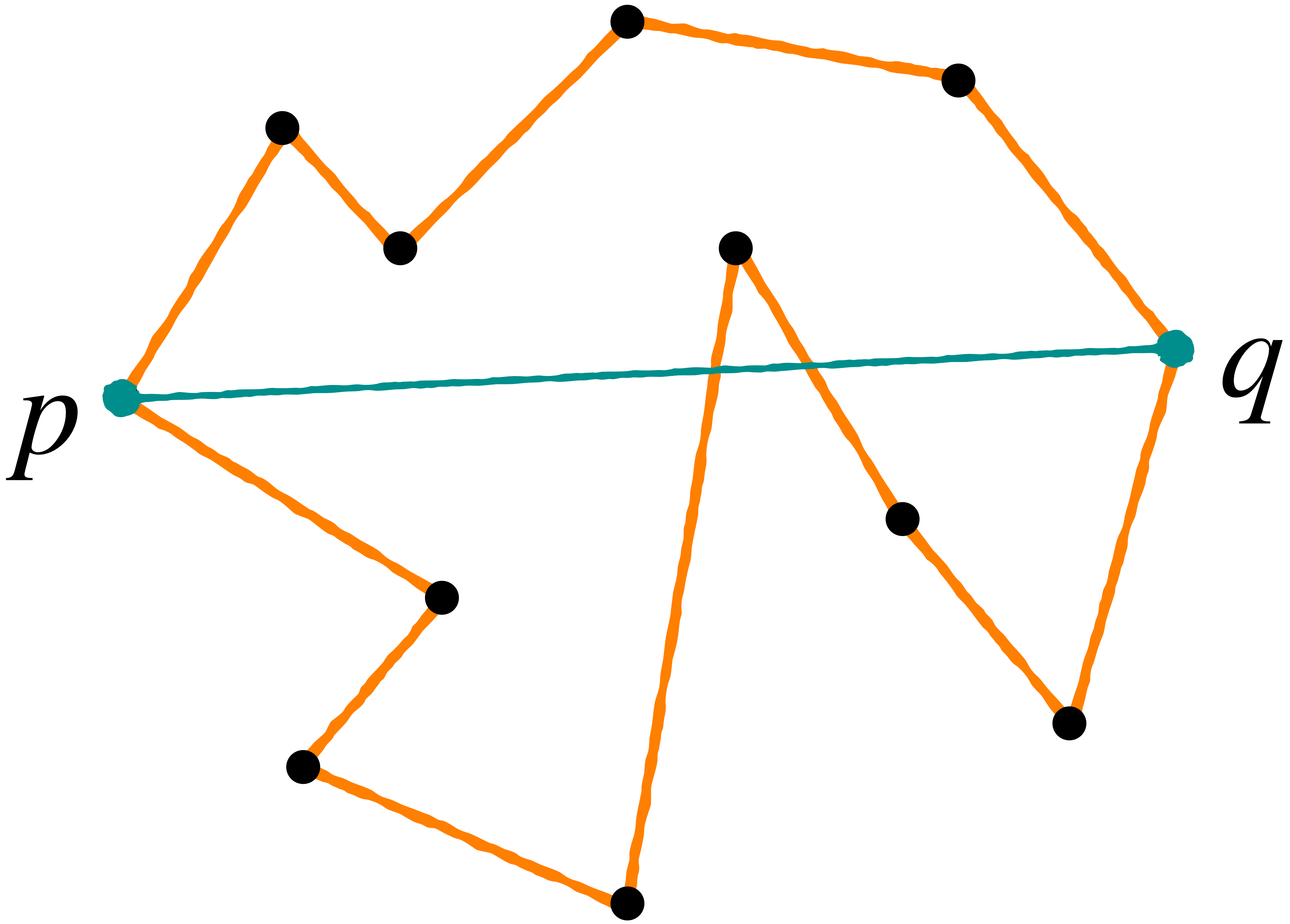
Theorem E3.5 All farthest pairs and the diameter of an n-vertex simple polygon P can be computed in …?



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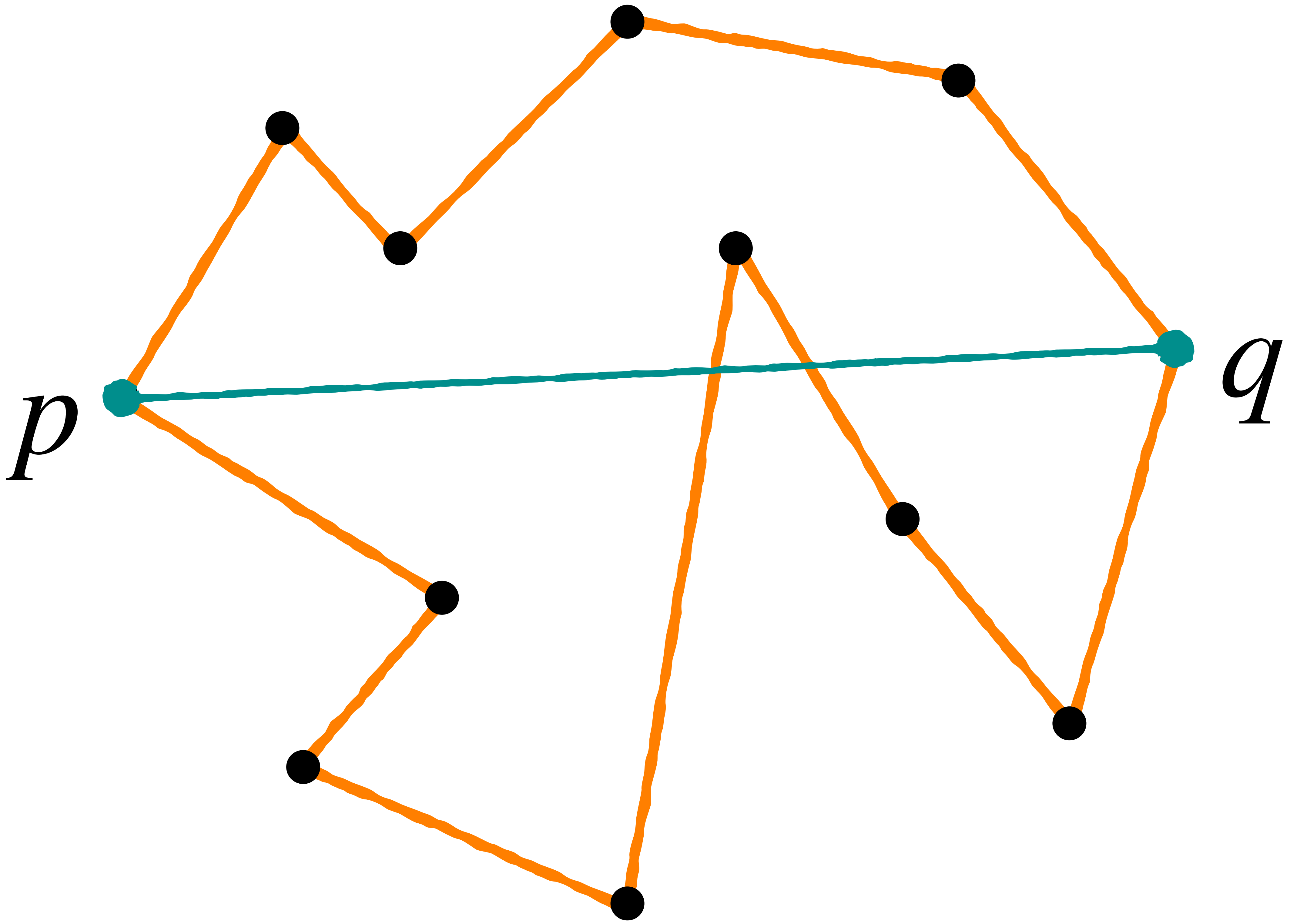

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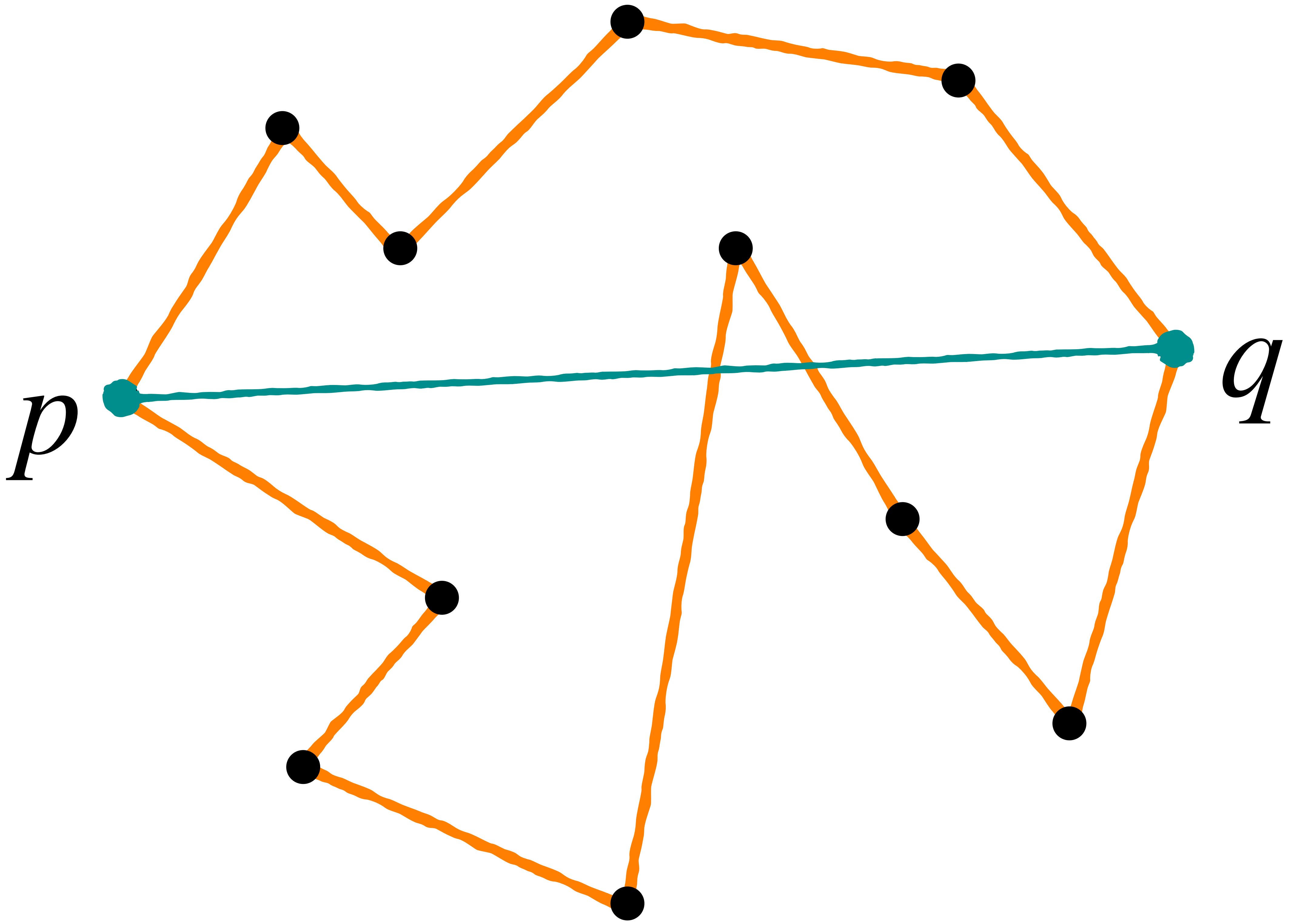
Reason: Convex hull of P in $\mathcal{O}(n)$, not $\Omega(n \log n)$.



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Rotating Calipers Algorithm
Michael Shamos, 1978

Distances [edit]

- Diameter (maximum width) of a convex polygon[6][7]
- Width (minimum width) of a convex polygon[8]
- Maximum distance between two convex polygons[9][10]
- Minimum distance between two convex polygons[11][12]
- Widest empty (or separating) strip between two convex polygons (a simplified low-dimensional variant of a problem arising in support vector machine based machine learning)
- Grenander distance between two convex polygons[13]
- Optimal strip separation (used in medical imaging and solid modeling)[14]

Bounding boxes [edit]

- Minimum area oriented bounding box
- Minimum perimeter oriented bounding box

Triangulations [edit]

- Onion triangulations
- Spiral triangulations
- Quadrangulation
- Nice triangulation
- Art gallery problem
- Wedge placement optimization problem[15]

Multi-polygon operations [edit]

- Union of two convex polygons
- Common tangents to two convex polygons
- Intersection of two convex polygons[16]
- Critical support lines of two convex polygons
- Vector sums (or Minkowski sum) of two convex polygons[17]
- Convex hull of two convex polygons

Traversals [edit]

- Shortest transversals[18][19]
- Thinnest-strip transversals[20]

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Homework Sheet #2

Peter Kramer | November 14th, 2024

Computational Geometry – Sheet 2 — **Winter 2024/2025**

Prof. Dr. Sándor P. Fekete — **Due** 28.11.2024

Peter Kramer — **Discussion** 05.12.2024

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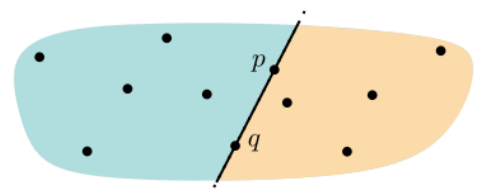

b) Design an algorithm that finds p and q in $\mathcal{O}(n)$ time for $n = |\mathcal{P}|$.

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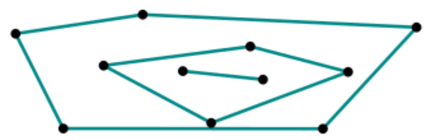

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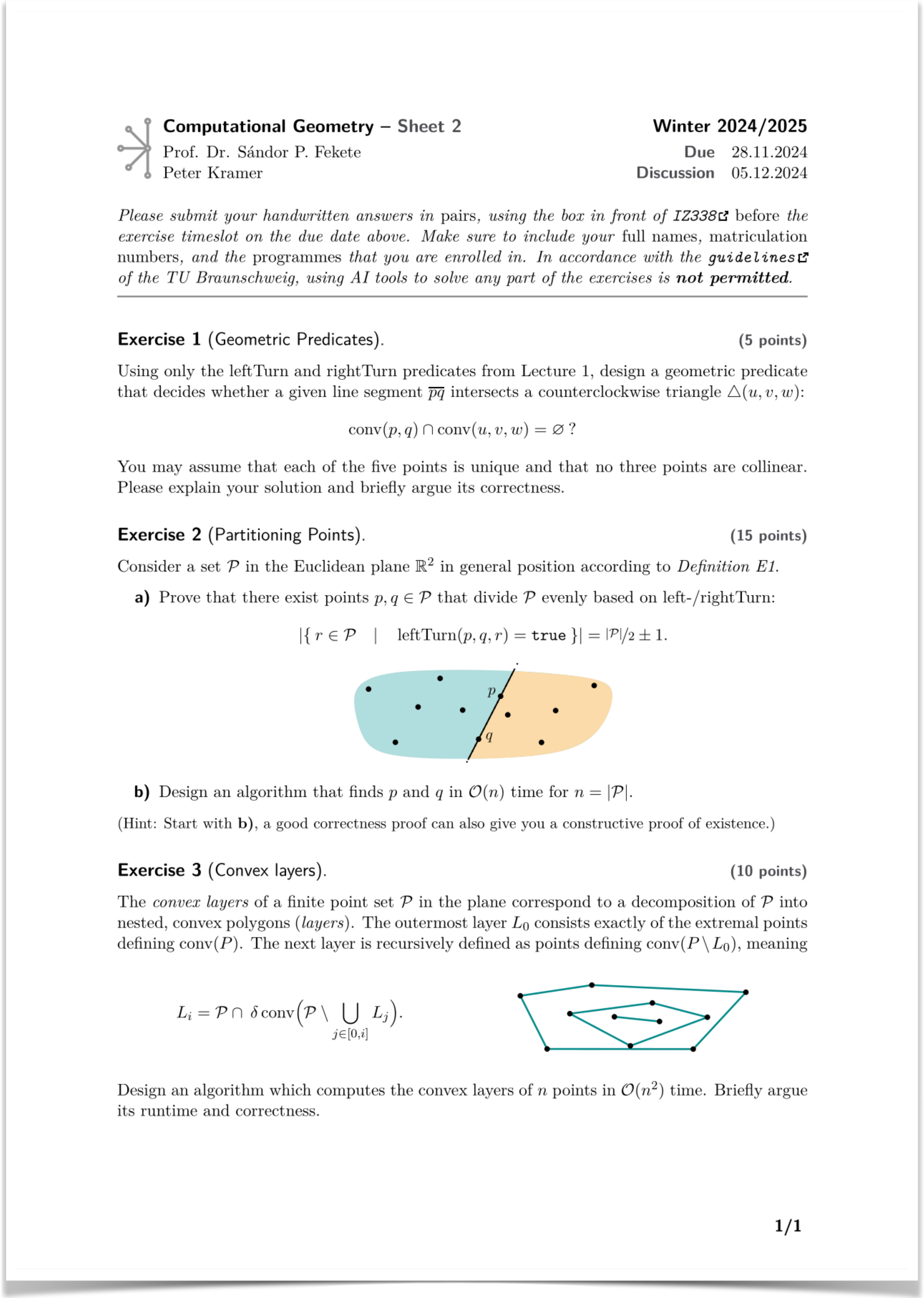
1/1



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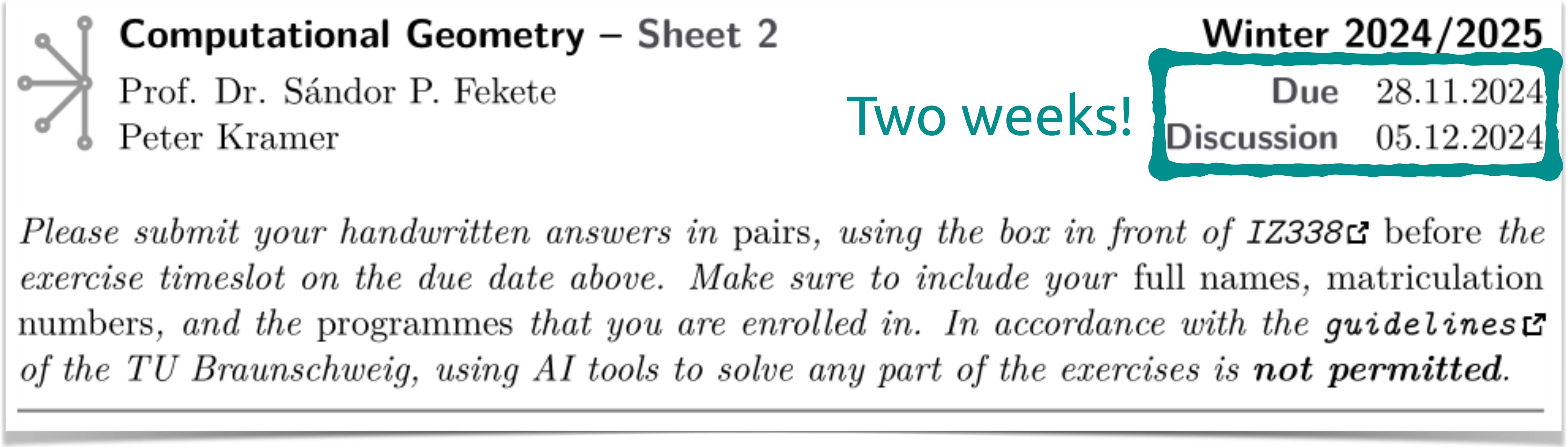

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- You may change homework partners at any time. Grading is tracked individually, not by group.
- A total of **70 points across all sheets** is sufficient for the coursework / Studienleistung.
- **So far:** 35 points, this sheet: 30 points

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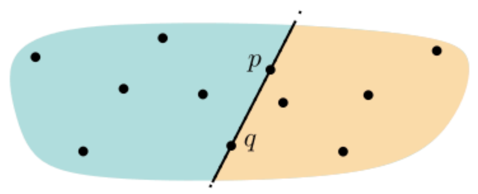

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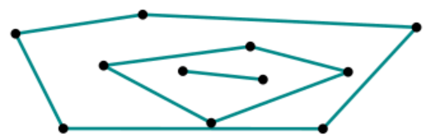

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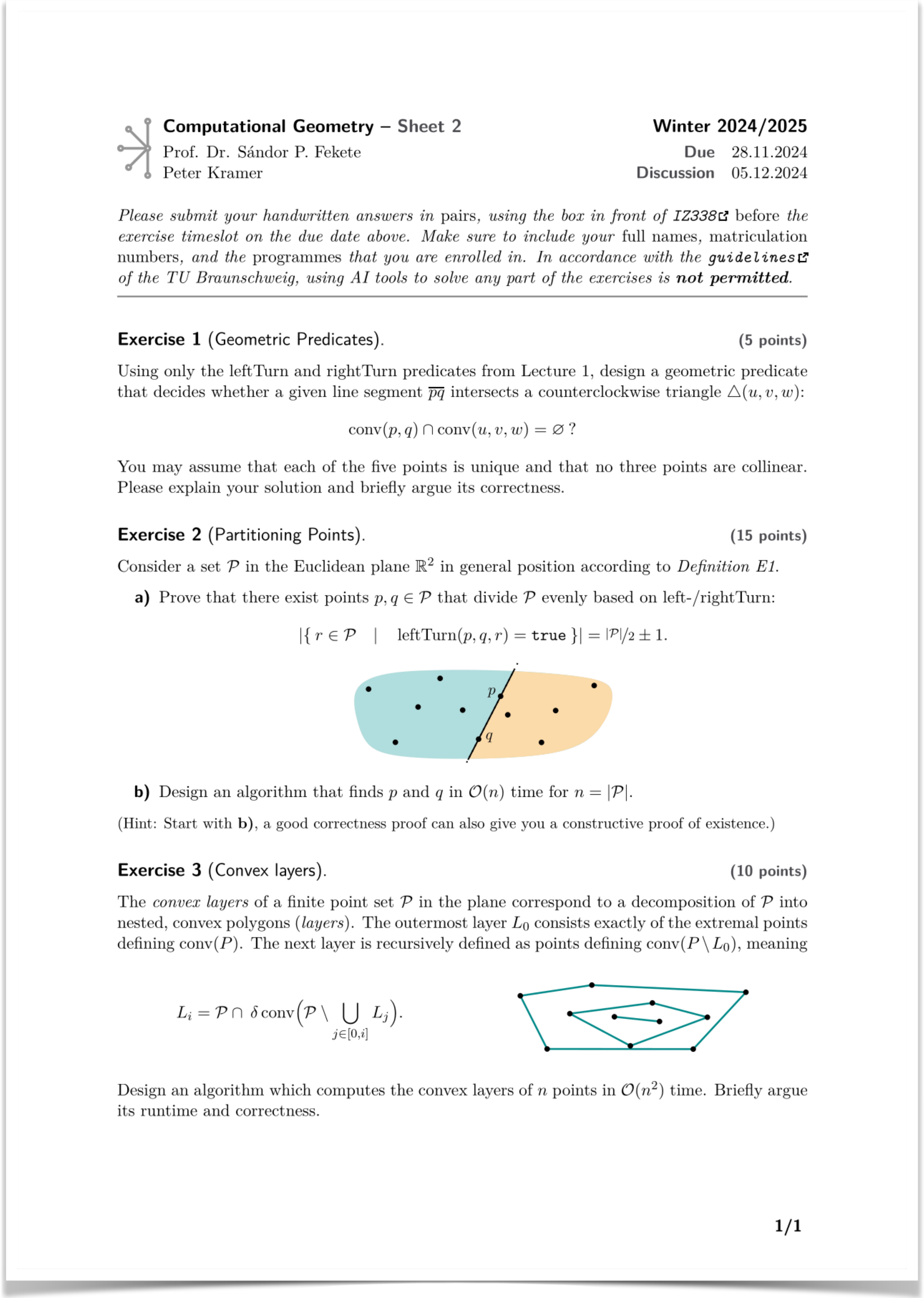

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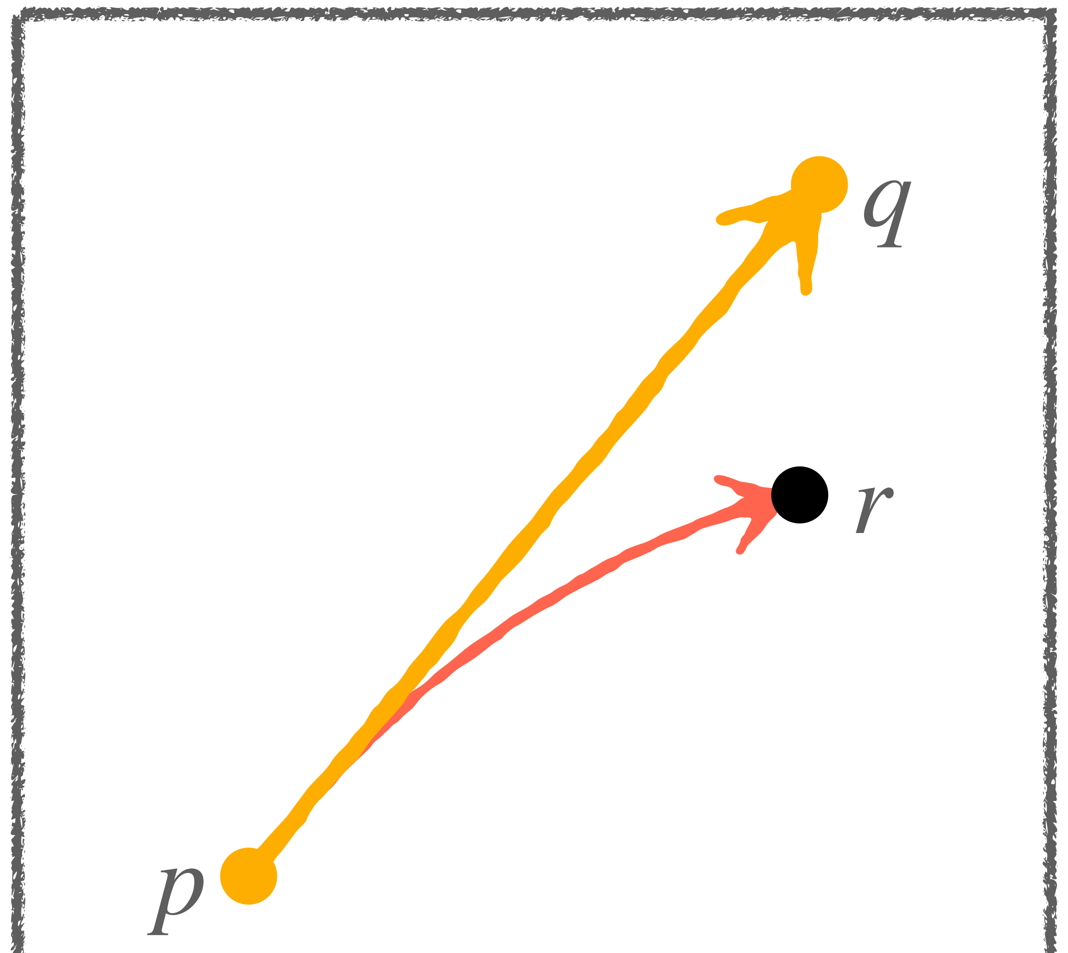
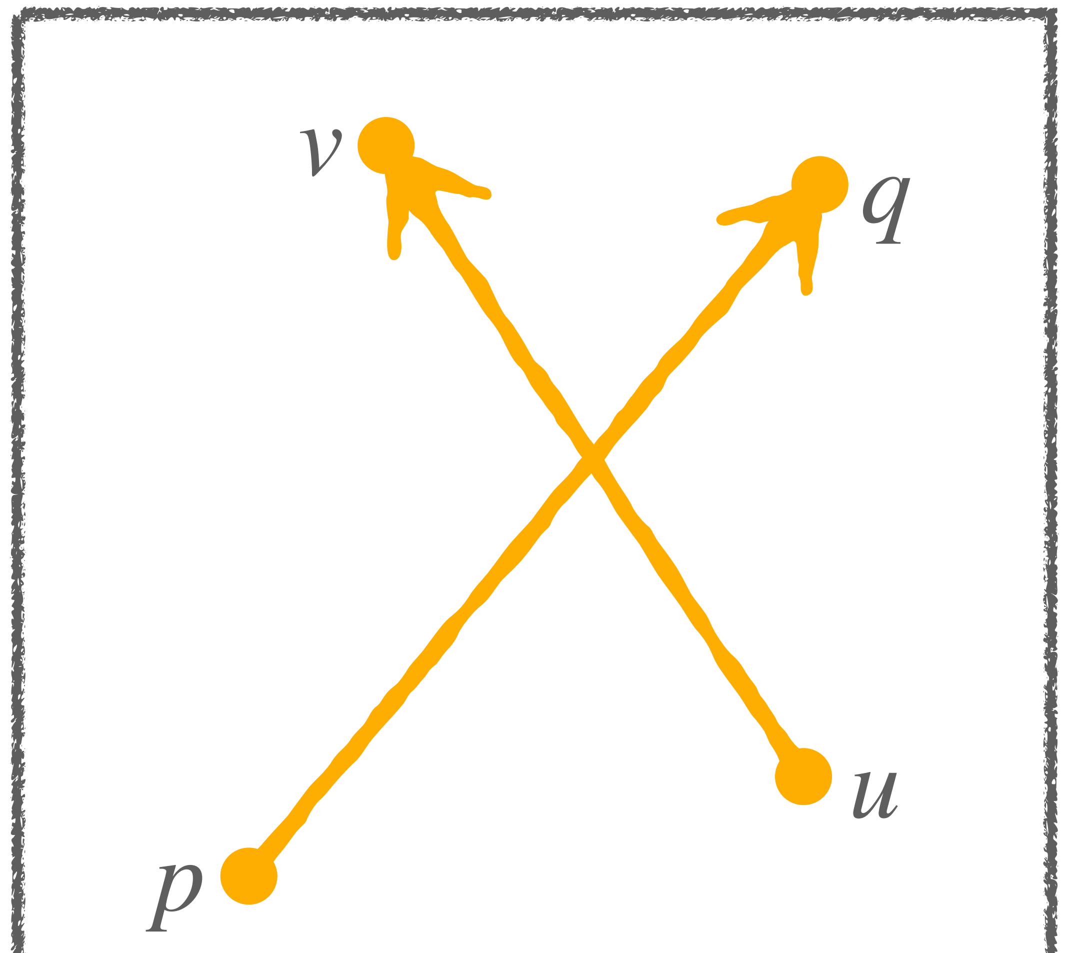
Point-Line Test



Is r collinear/left/right of $\overline{pq}$?

Intersection Test



(Where) do $\overline{pq}$ and $\overline{uv}$ intersect?

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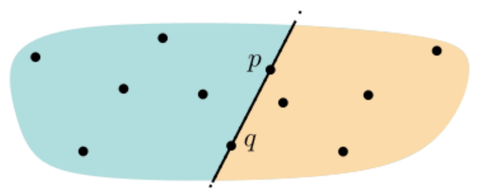

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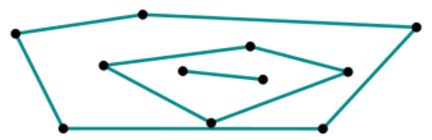

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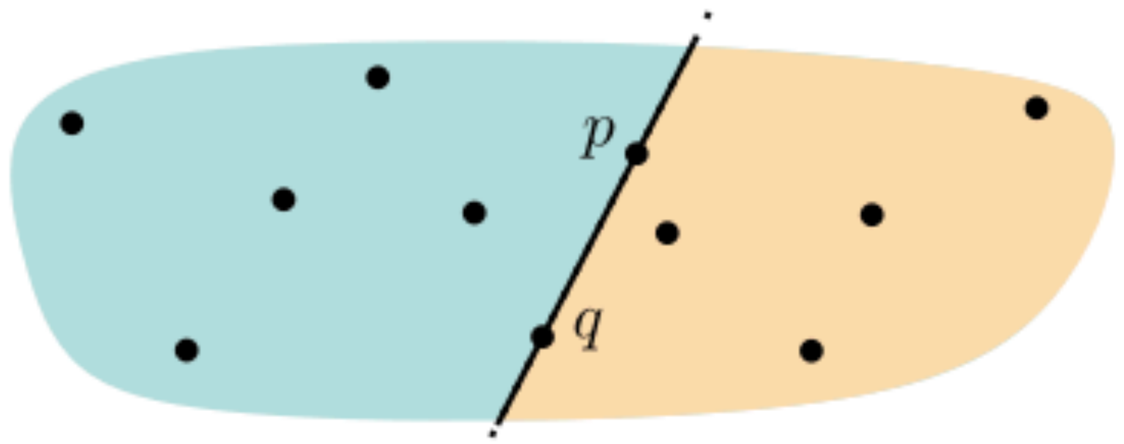

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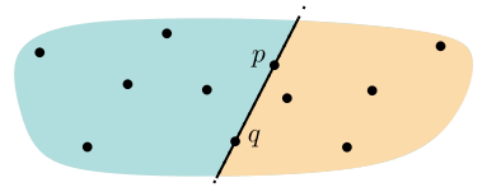

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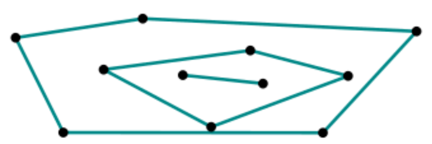
The *convex layers* of a finite point set $\mathcal{P}$ in the plane correspond to a decomposition of $\mathcal{P}$ into nested, convex polygons (*layers*). The outermost layer L_0 consists exactly of the extremal points defining $\mathrm{conv}(P)$. The next layer is recursively defined as points defining $\mathrm{conv}(P \setminus L_0)$, meaning

$$L_i = \mathcal{P} \cap \delta\,\mathrm{conv}\Big(\mathcal{P} \setminus \bigcup_{j \in [0,i]} L_j\Big).$$



Design an algorithm which computes the convex layers of n points in $\mathcal{O}(n^2)$ time. Briefly argue its runtime and correctness.

1/1

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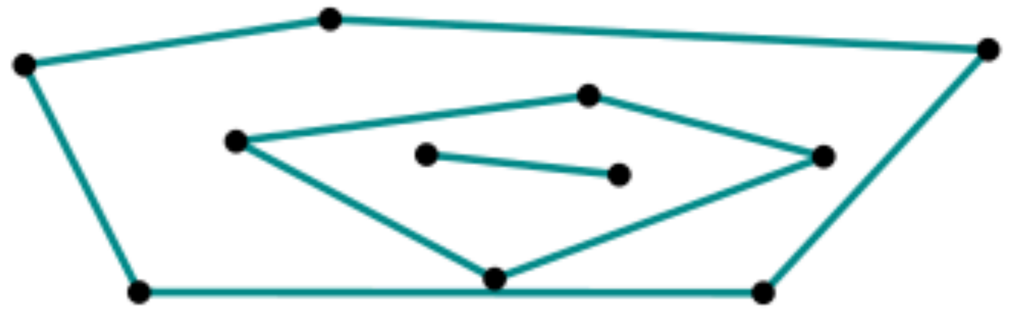

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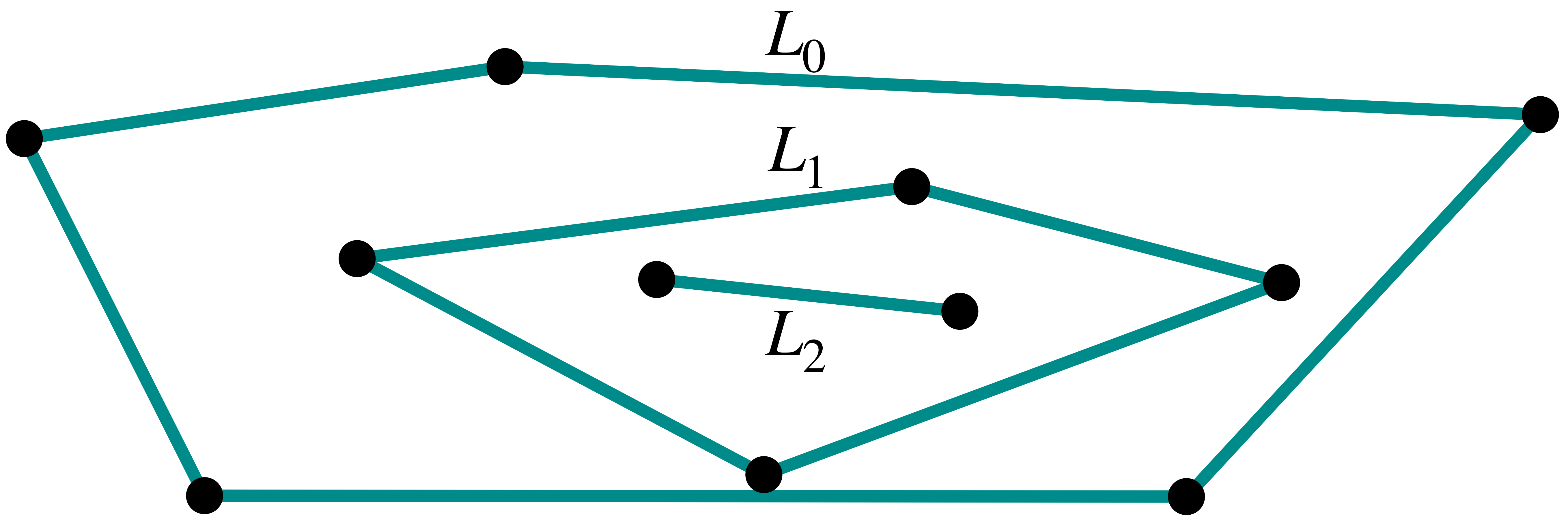

Thank you.