Computational Geometry

Tutorial #6 — Algorithm design and Plane partitions

Institut für Betriebssysteme
und Rechnerverbund
Algorithmik

Peter Kramer | *January 23rd, 2025*

Algorithm Design

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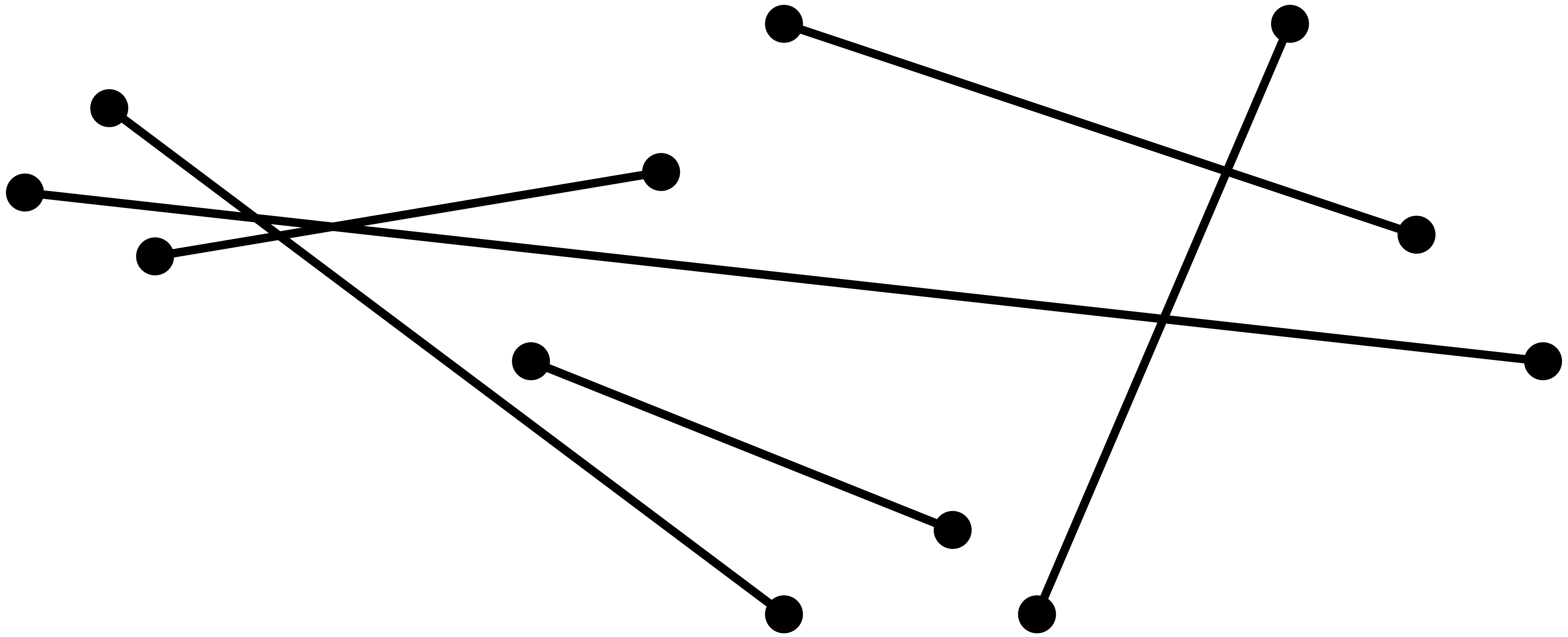
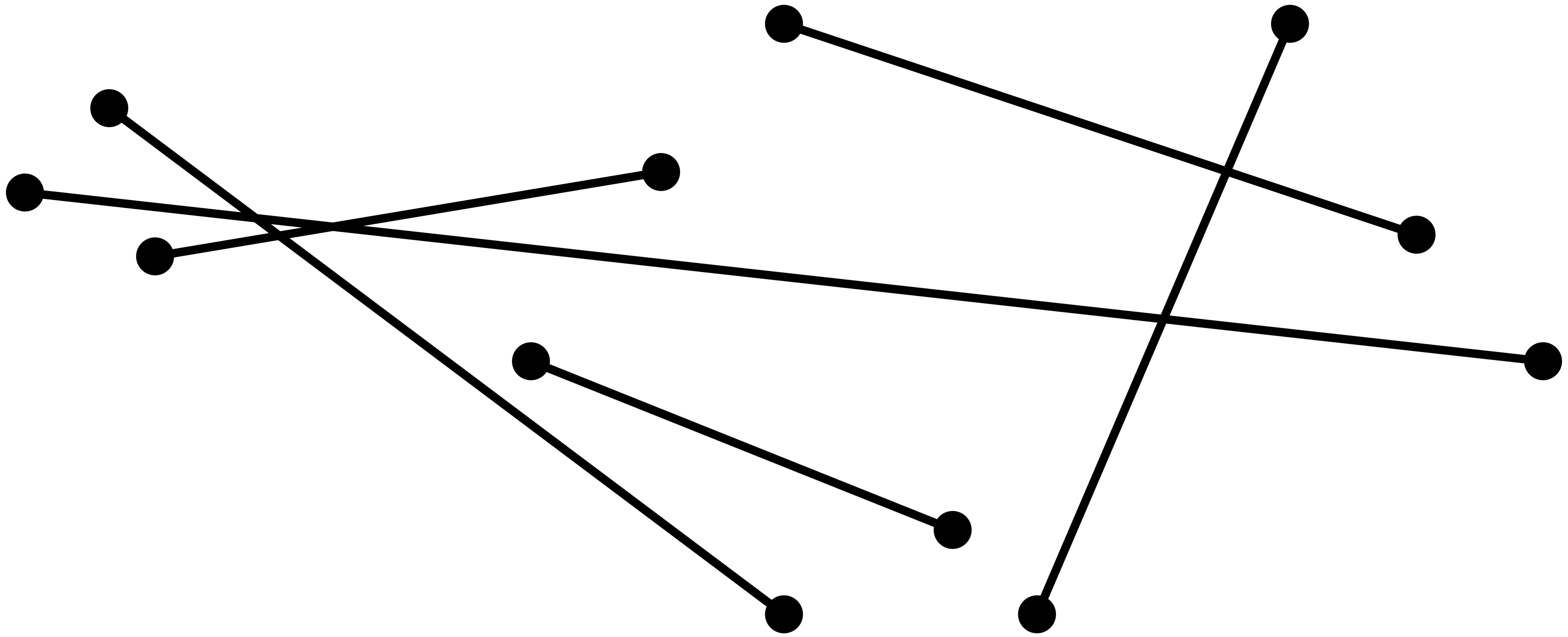
Intersections of line segments
Algorithm design



- **Given:** Endpoints $(p_1, q_1), \ldots, (p_n, q_n)$ of n line segments $\overline{p_i q_i}$ in the plane.
- **Wanted:** Intersections of segments, so …
    - … the number of intersections k, and
    - … the involved edges of each crossing.

Intersections of line segments

Algorithm design



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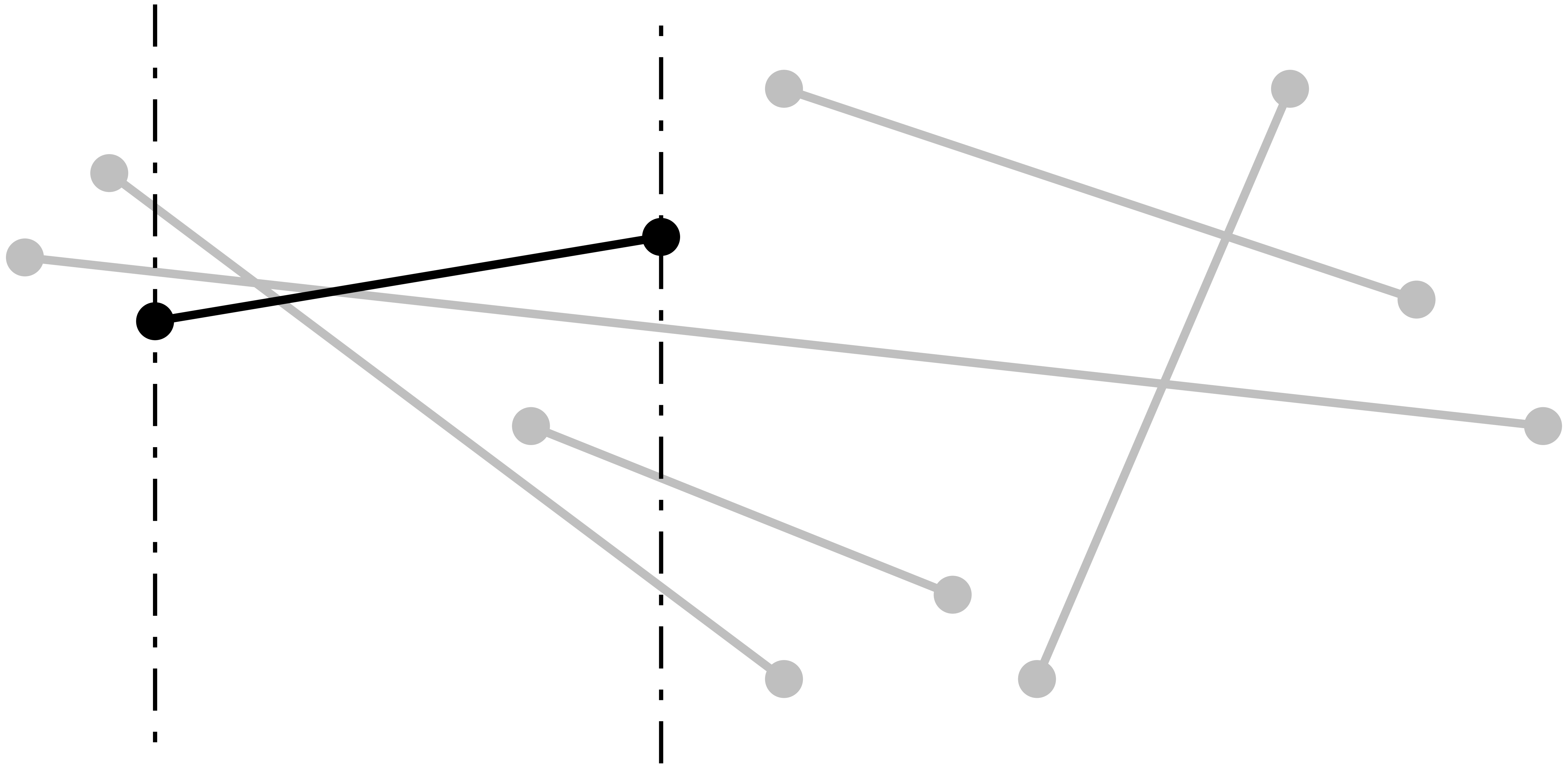
- *When do two lines intersect (criteria)?*
  How many crossings can there be?
  Is there structure to this problem?

Intersections of line segments

Algorithm design



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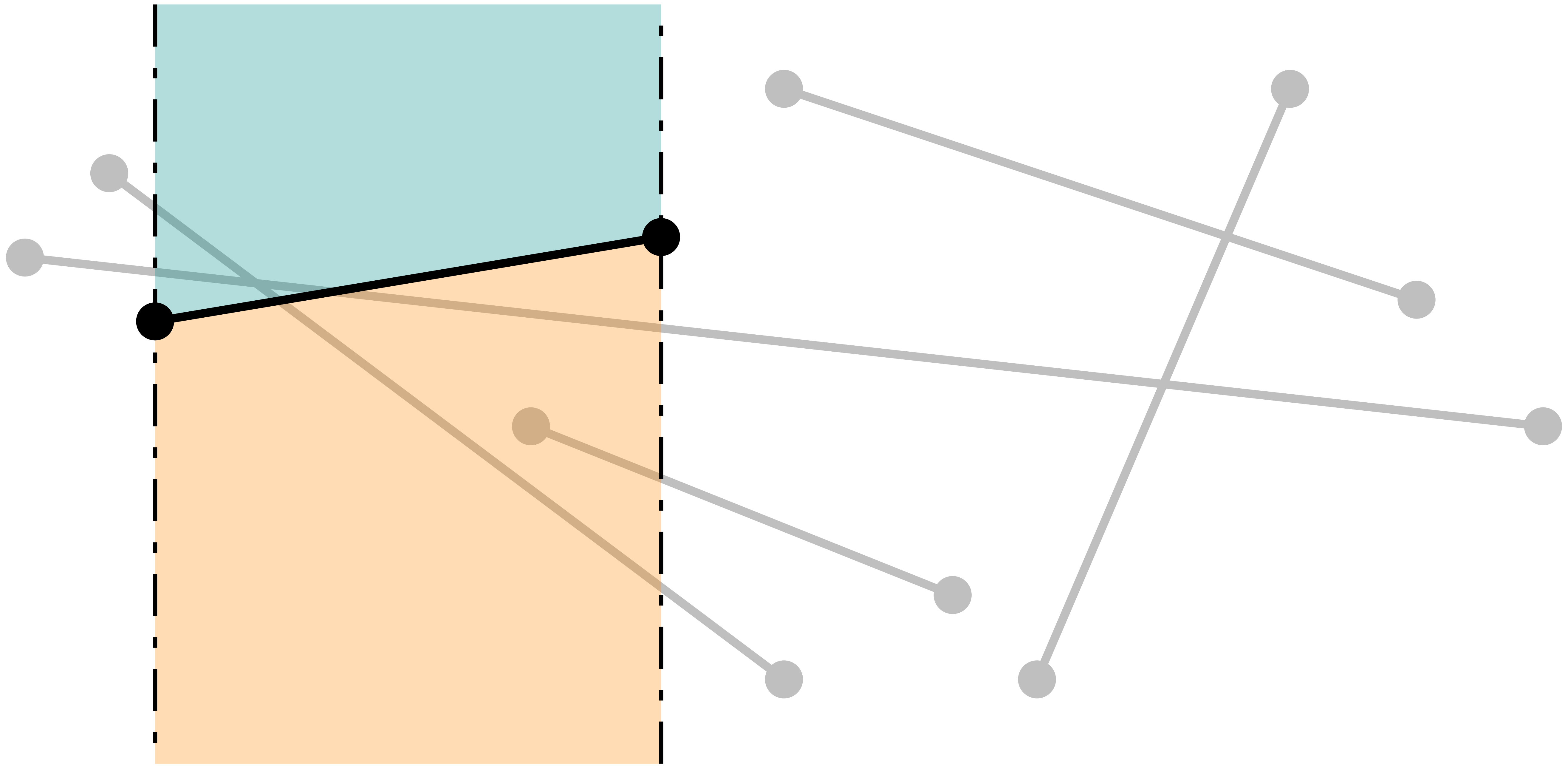

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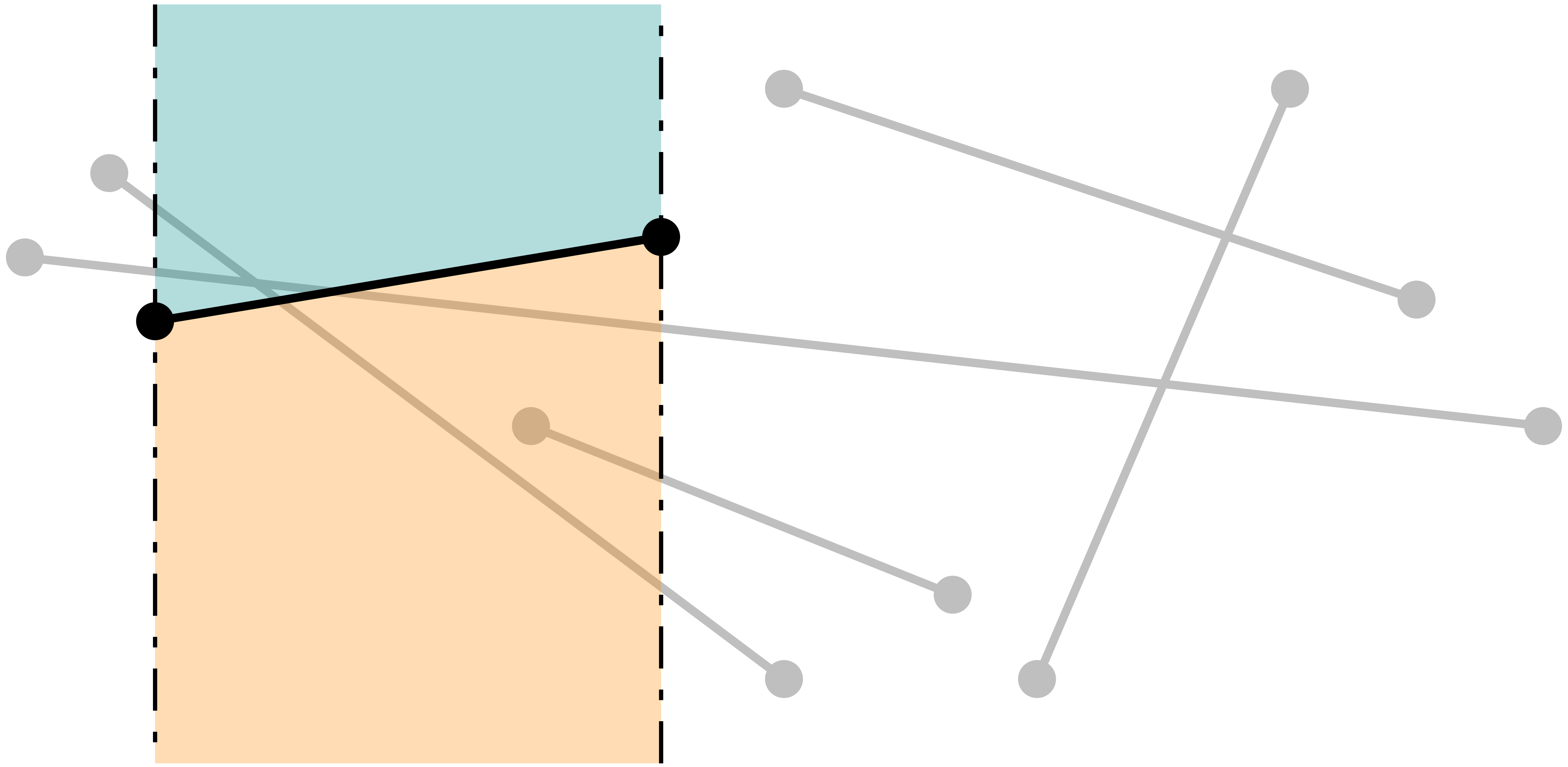

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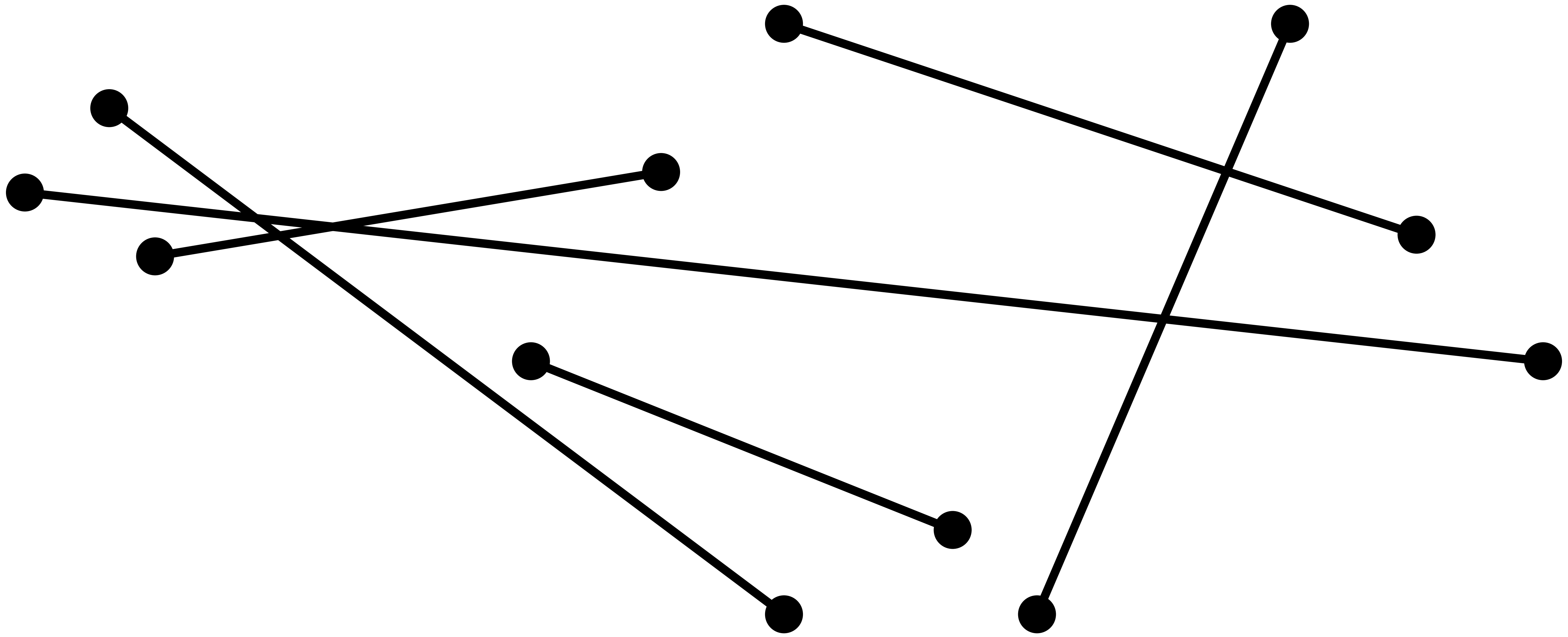
- *When do two lines intersect (criteria)?*
  How many crossings can there be?
  Is there structure to this problem?
- **Goal:** Construct a sweep-line algorithm that computes this in $\mathcal{O}((n+k)\log n)$ time.

Intersections of line segments

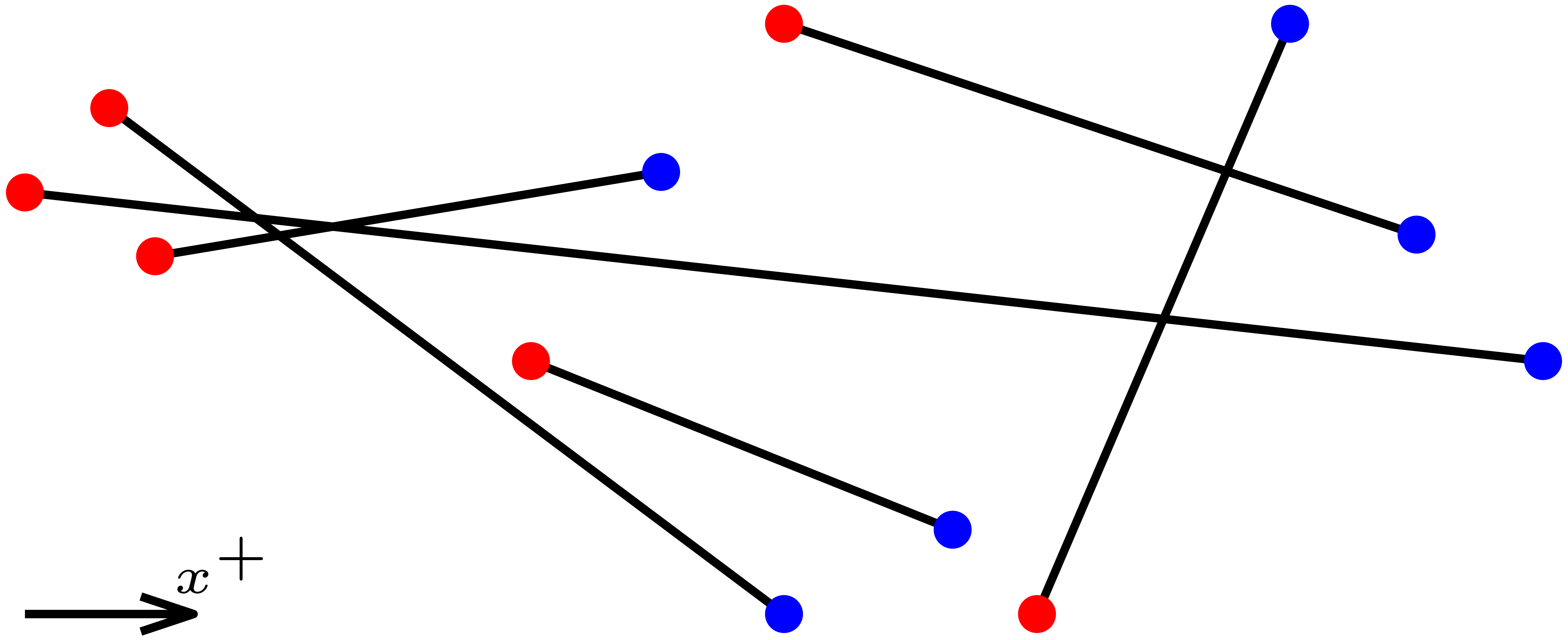
Sweep-Line Algorithm



Intersections of line segments

Sweep-Line Algorithm

(i) ‘Sort’ each segment such that $x(p_i) \leq x(q_i)$.



Intersections of line segments

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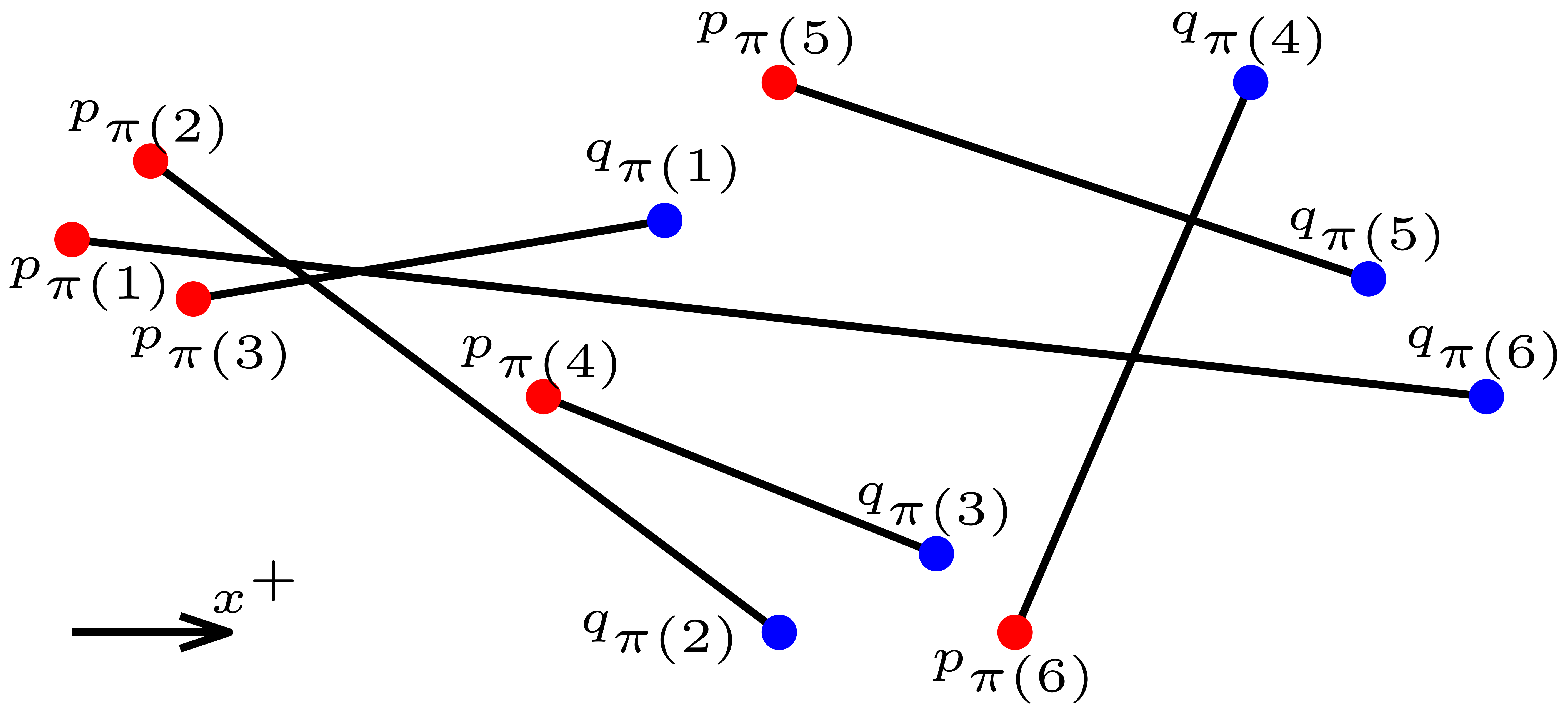

(i) ‘Sort’ each segment such that $x(p_i) \leq x(q_i)$.

(ii) Sort ps and qs by their x-coordinates.
Let $p_{\pi(i)}$ be the element rank i (ith lowest).

Intersections of line segments

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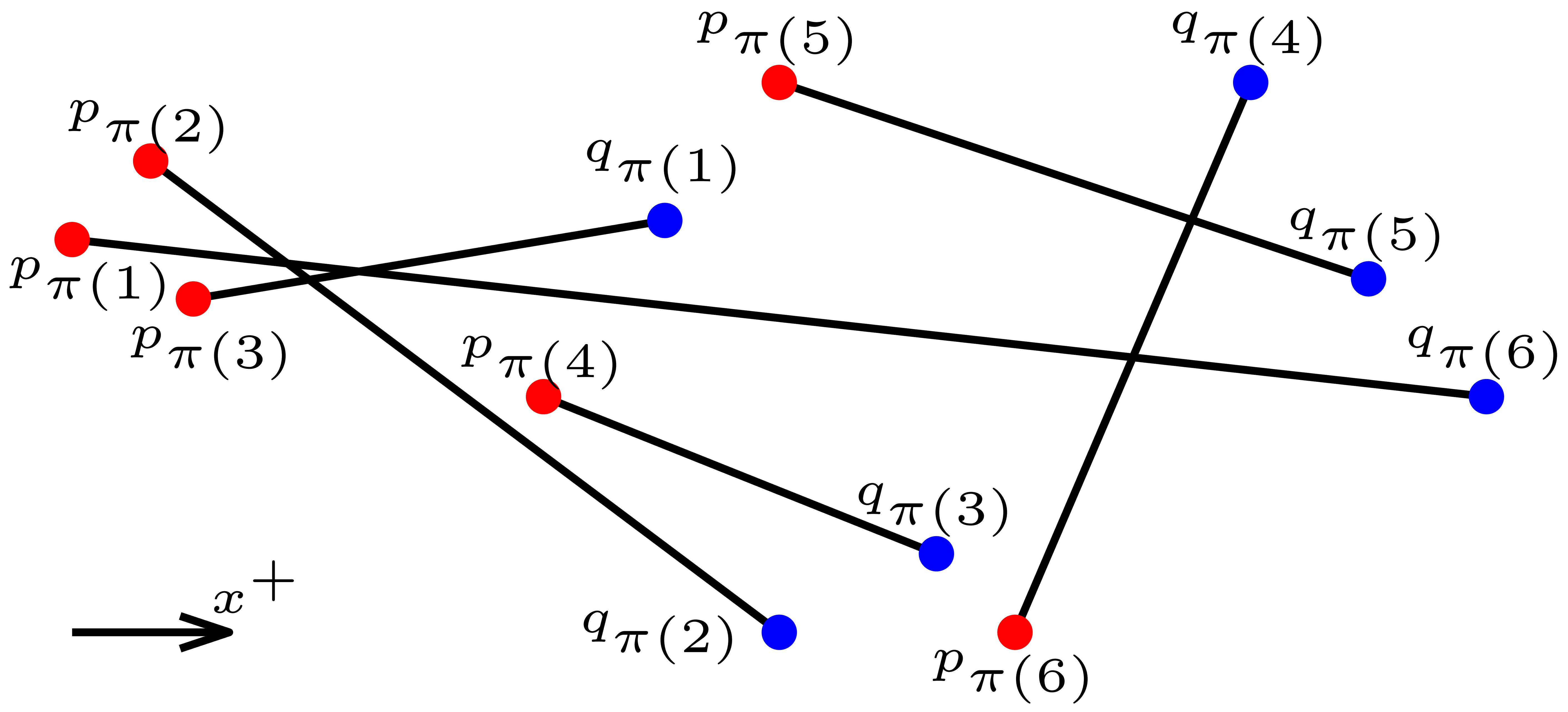

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Intersections of line segments

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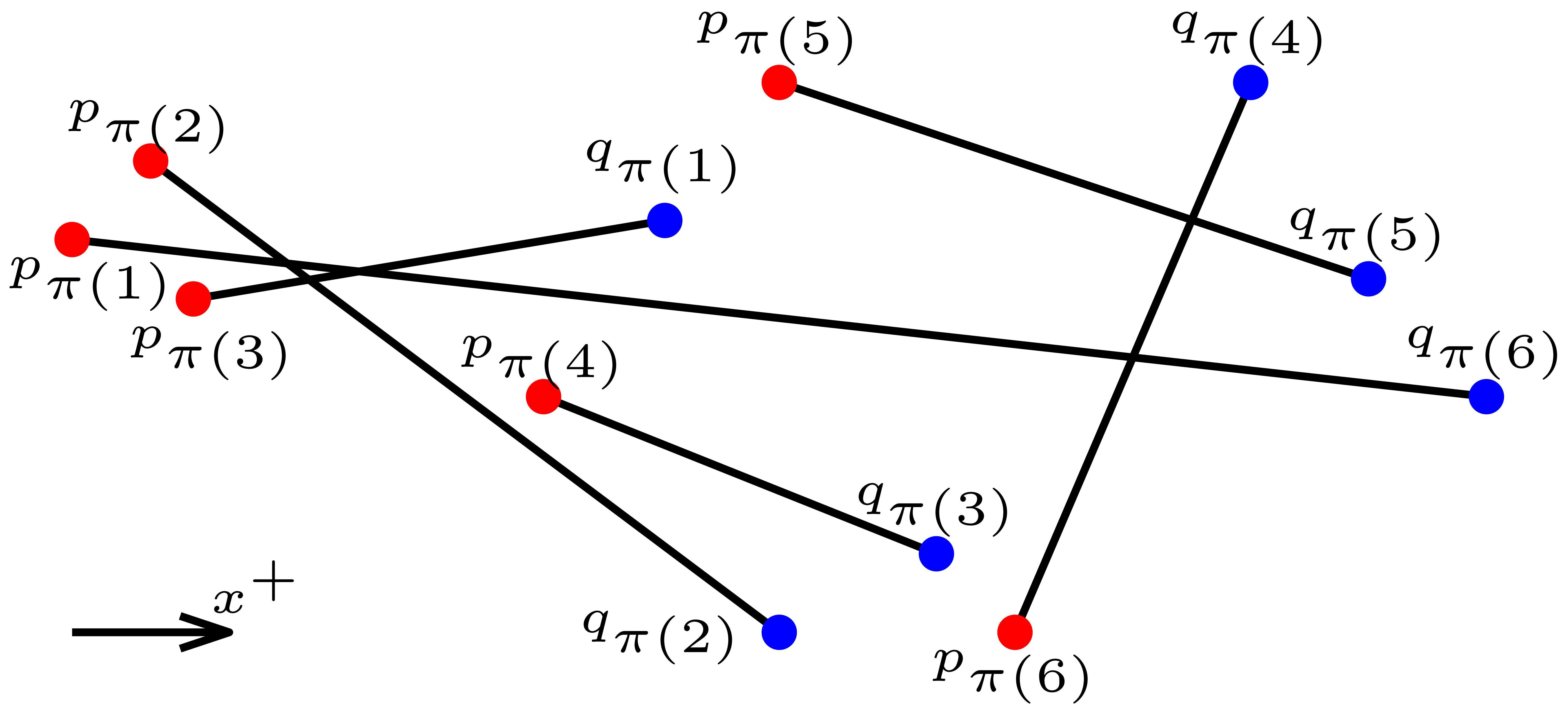

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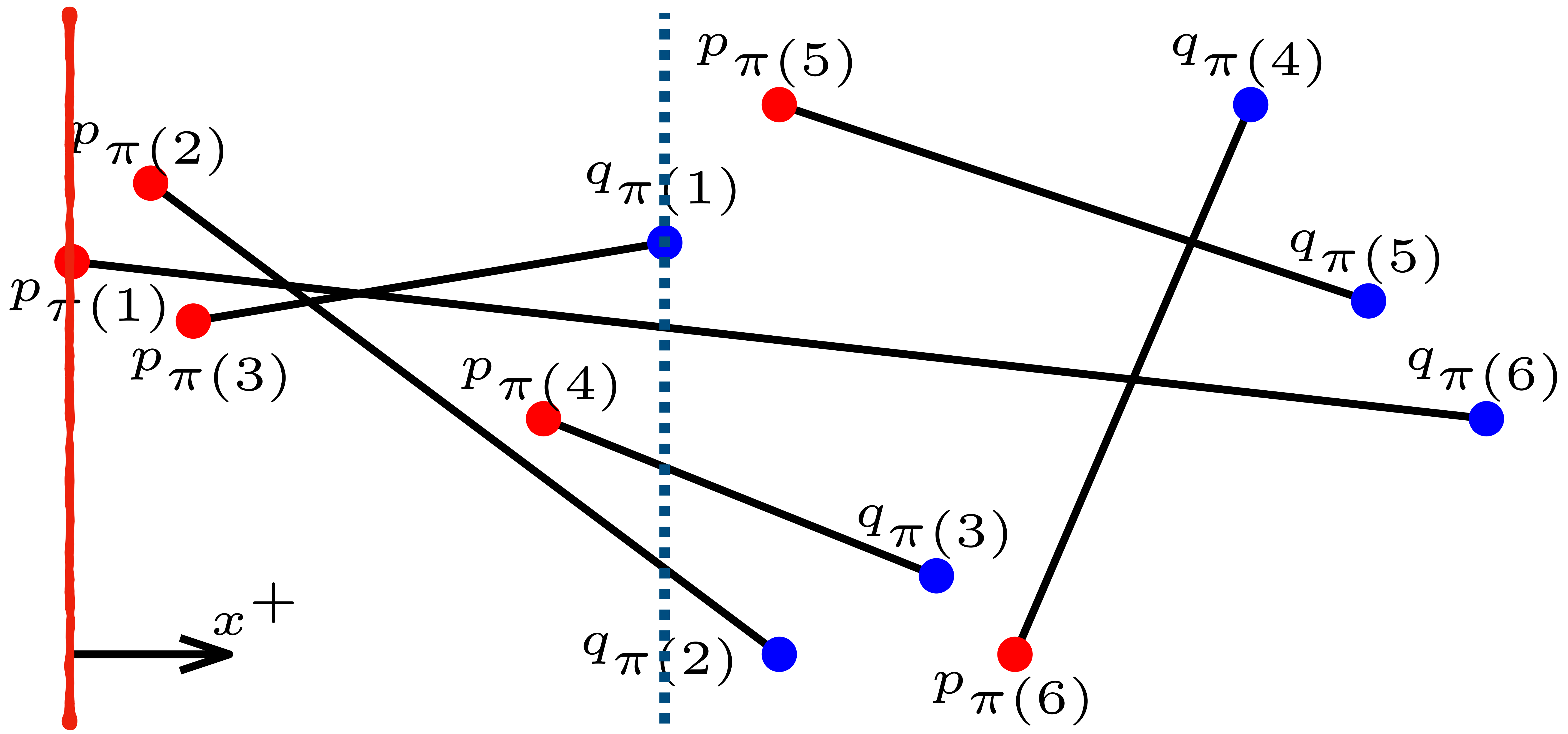
(iii) Let $i, j = 1$, initialise a BST T (sweep state) and a priority queue C (crossings).

(iv) Sweep the vertical bisector at $\pi(i)$ along the x-axis, tracking intersecting segments in y-monotone order using the BST.
This order changes exactly at a crossings.

Intersections of line segments

Sweep-Line Algorithm



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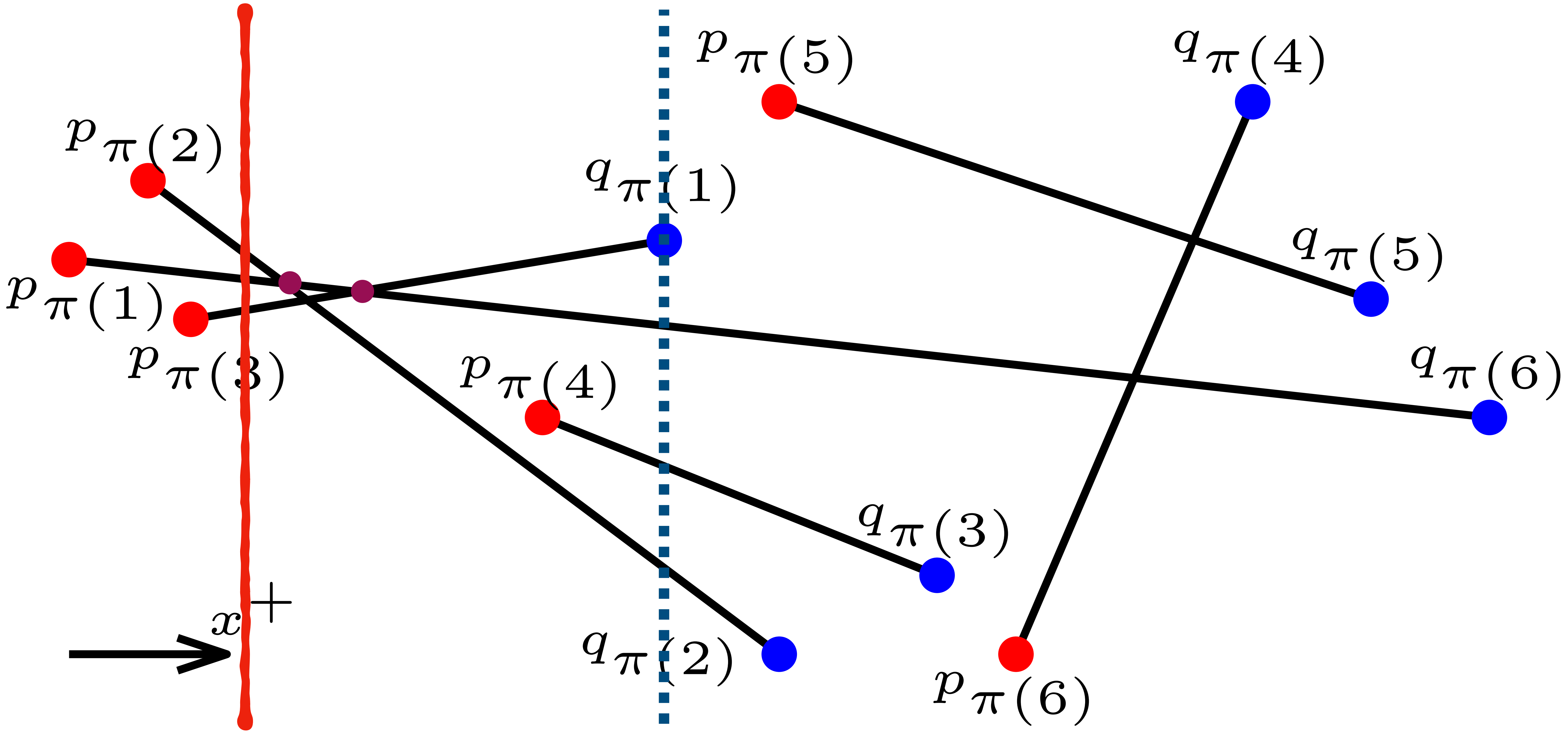
Intersections of line segments

Sweep-Line Algorithm



Intersections of line segments

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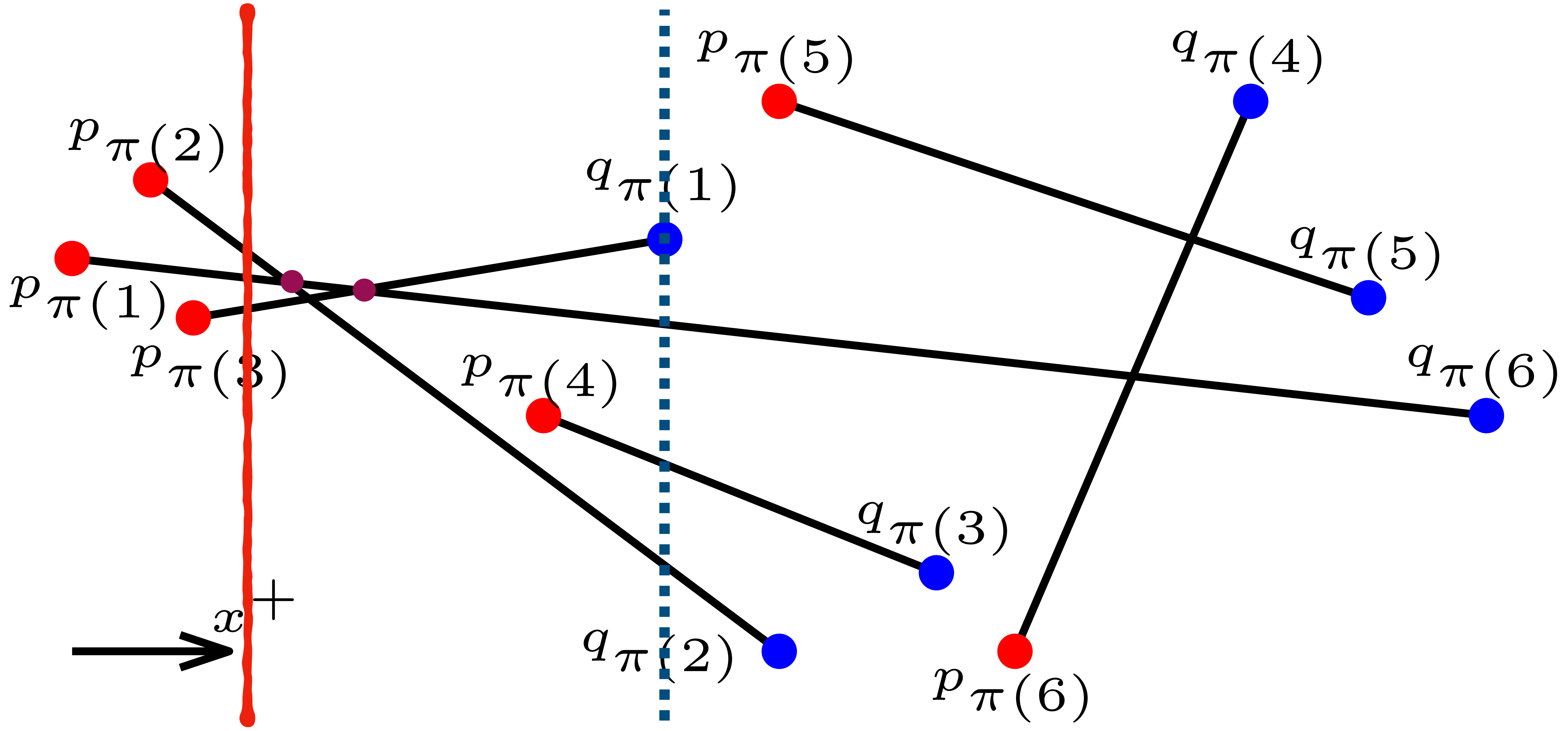
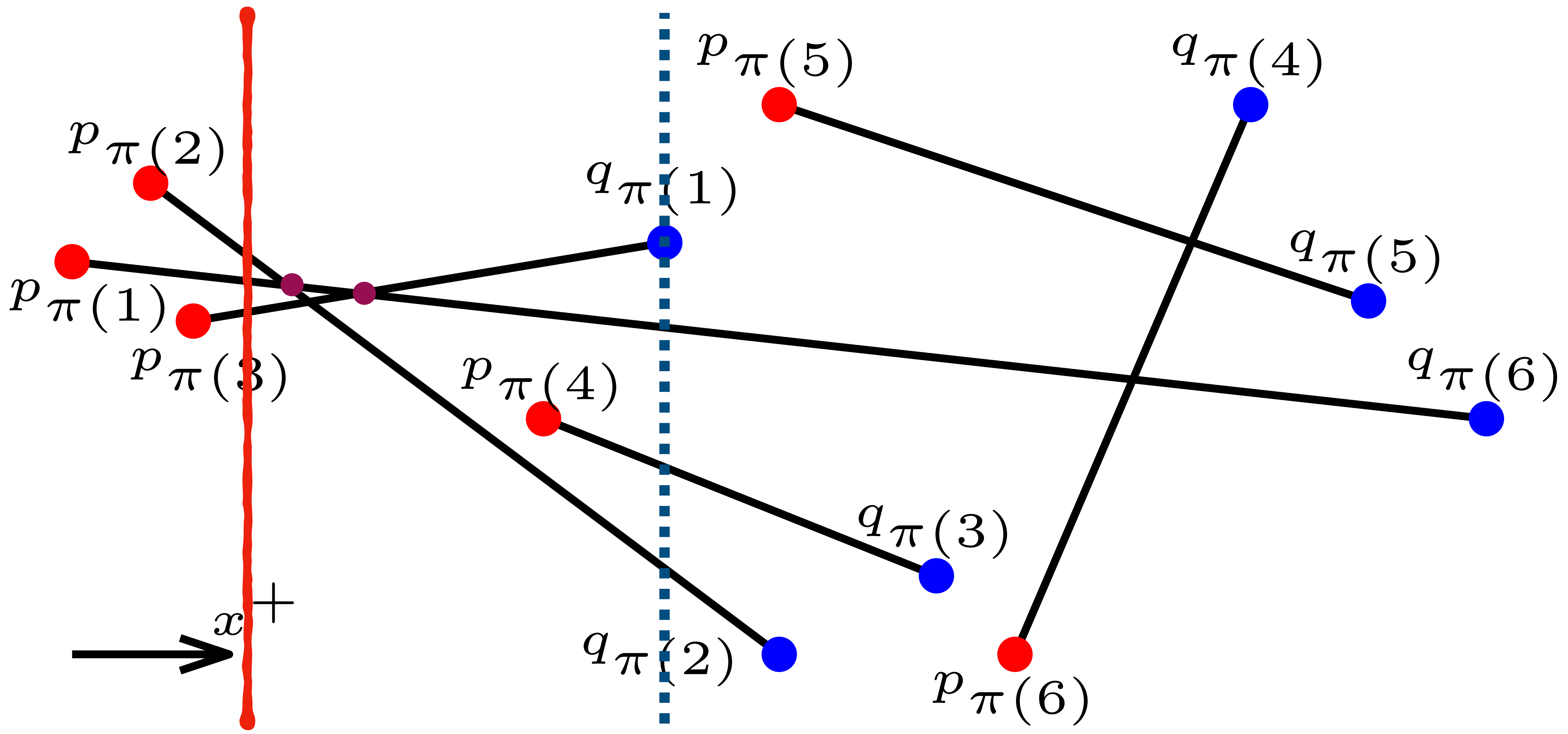


Intersections of line segments

Sweep-Line Algorithm



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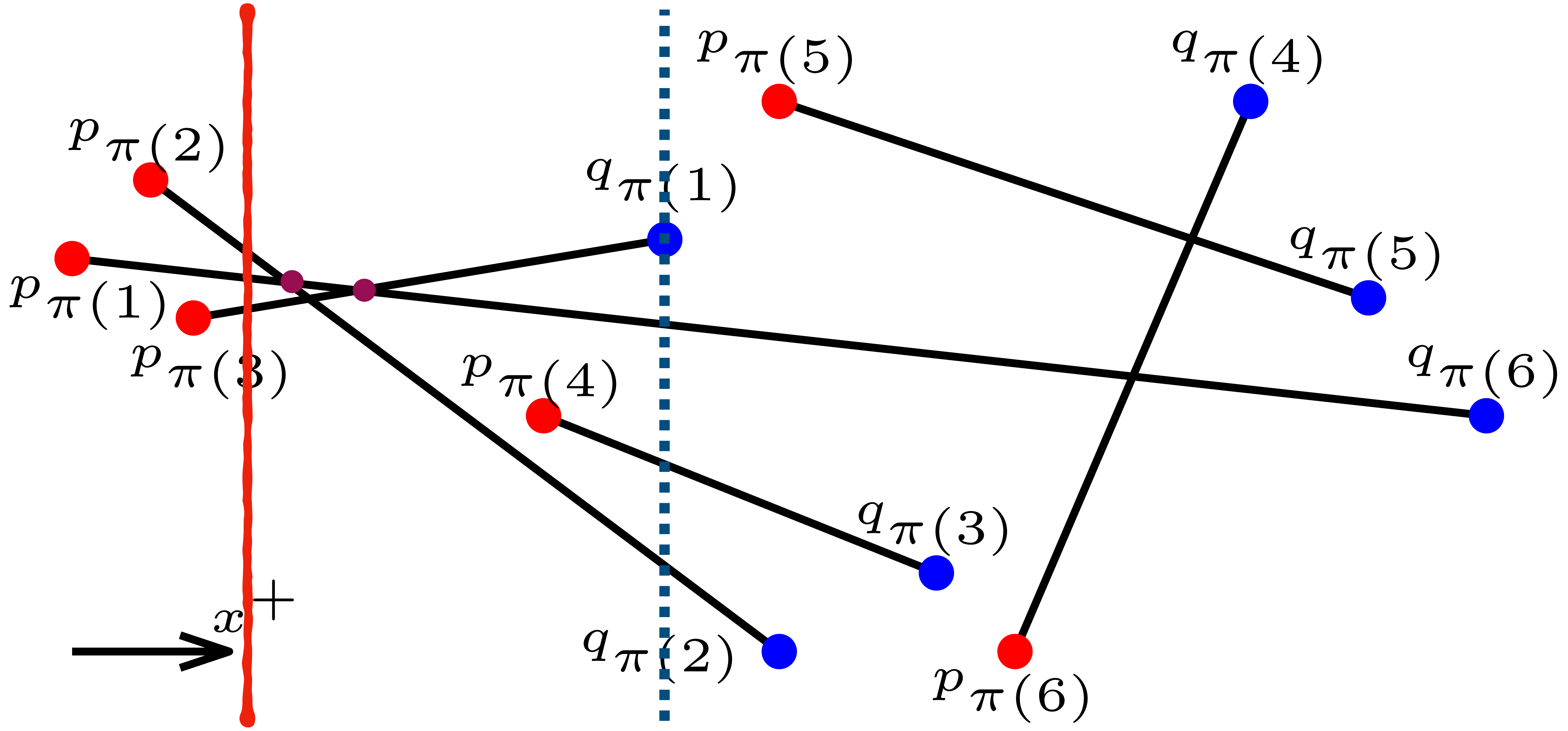

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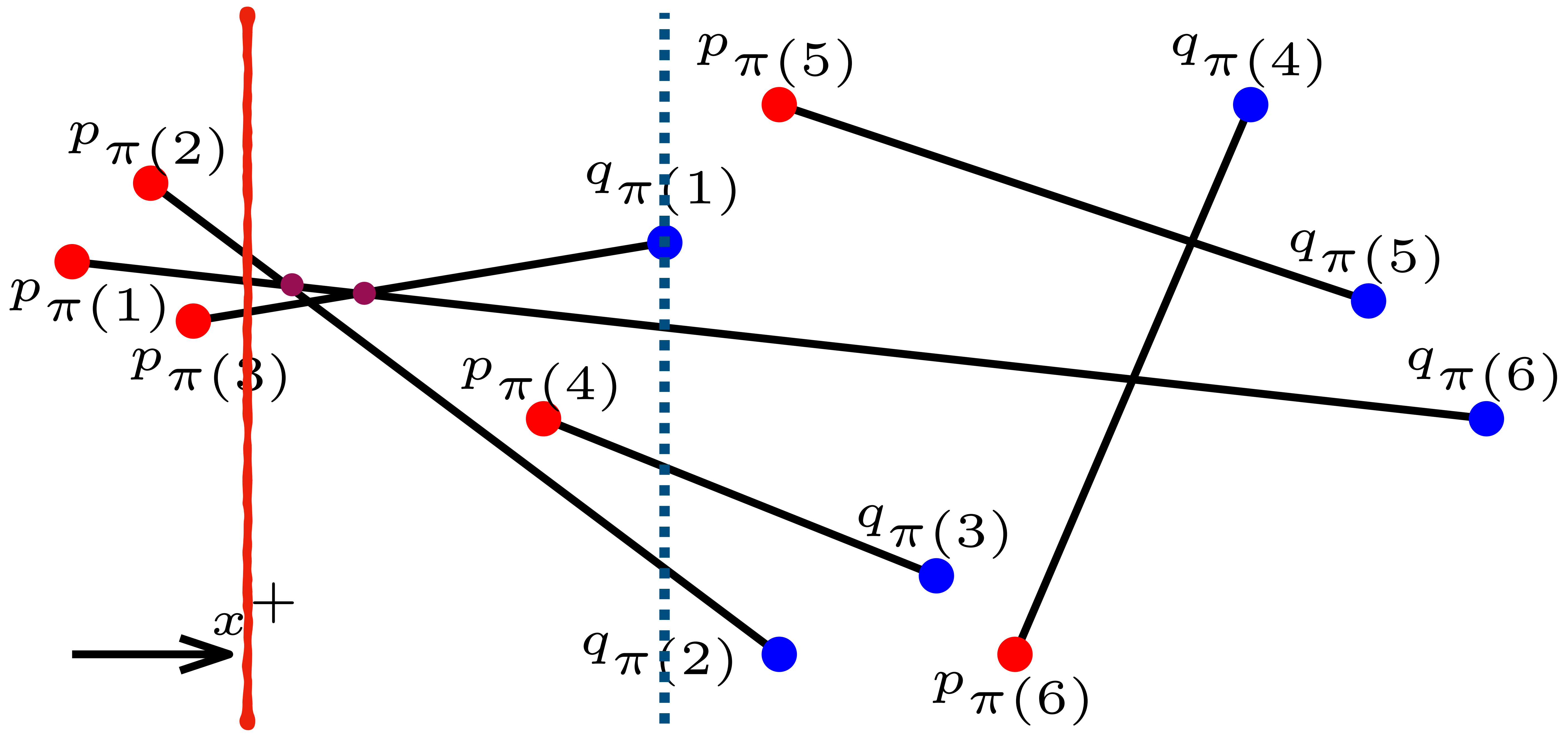

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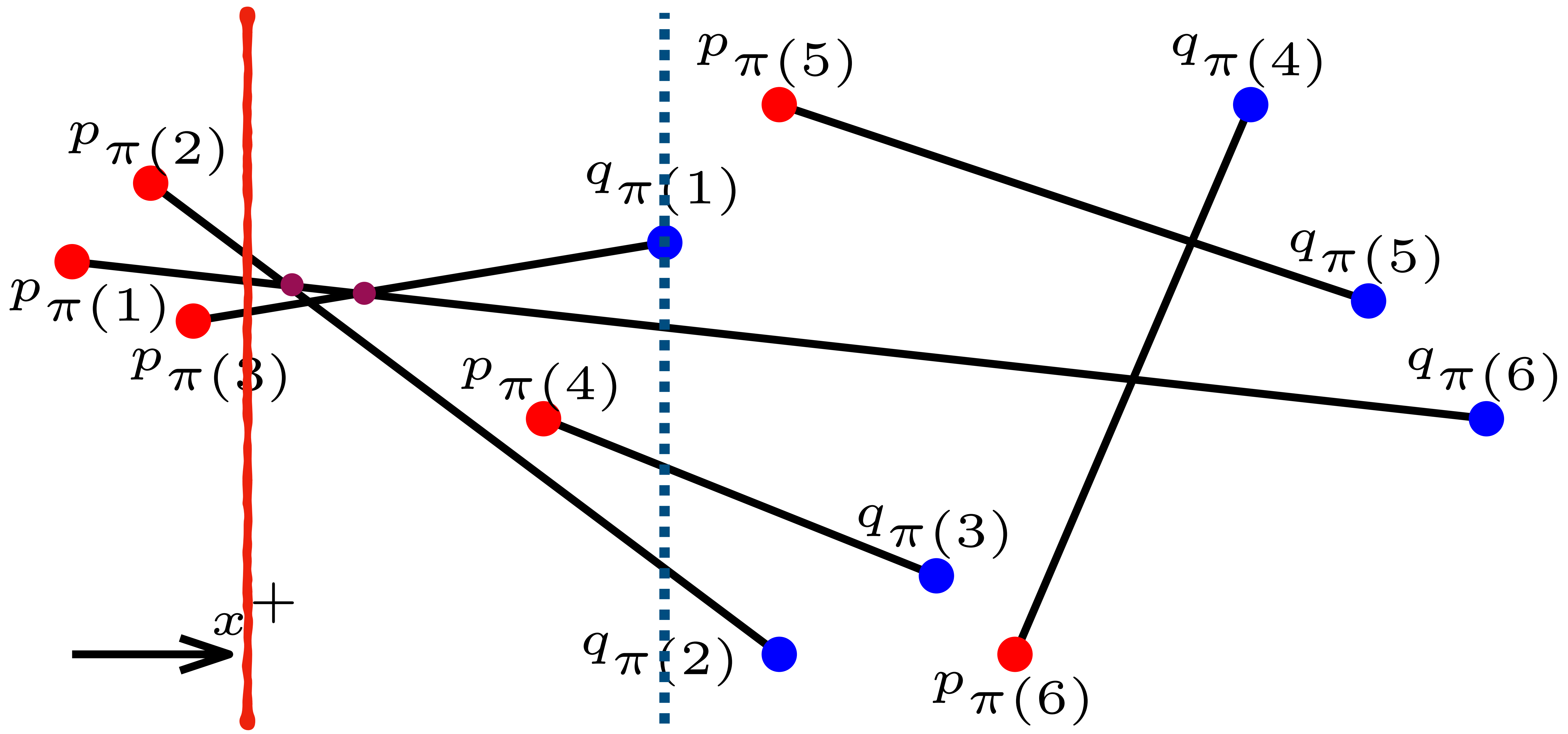
Crossings:

Sweep state:

Intersections of line segments

Sweep-Line Algorithm



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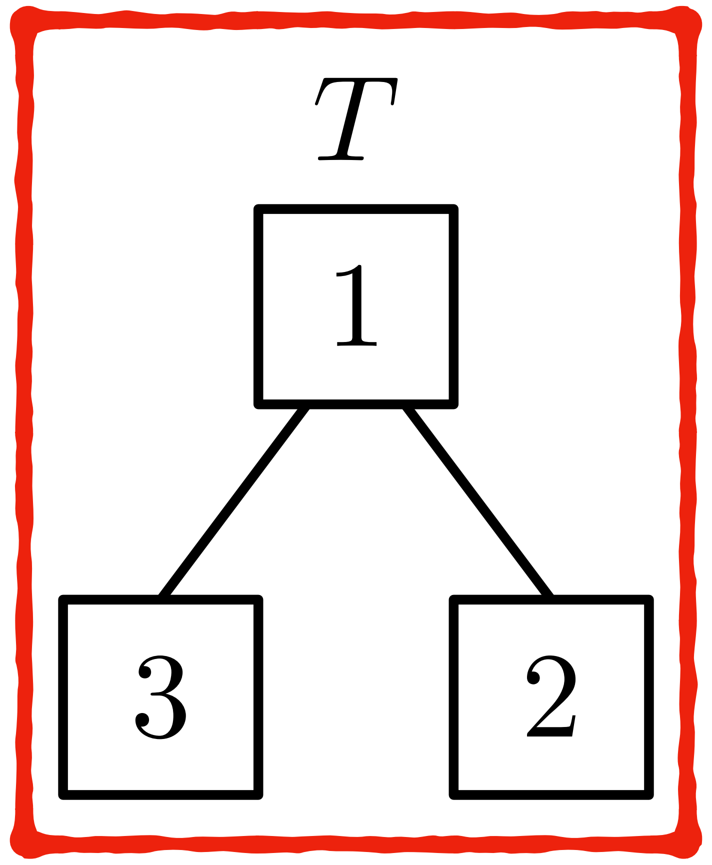
Crossings:

Each $(c, a, b) \in C$ with

- $c \in \mathbb{R}^2$ position
- $a \in [1,n]$ segment A
- $b \in [1,n]$ segment B
- **Priority $x(c)$.**

Sweep state:



Geometric path planning

Peter Kramer | January 23rd, 2025

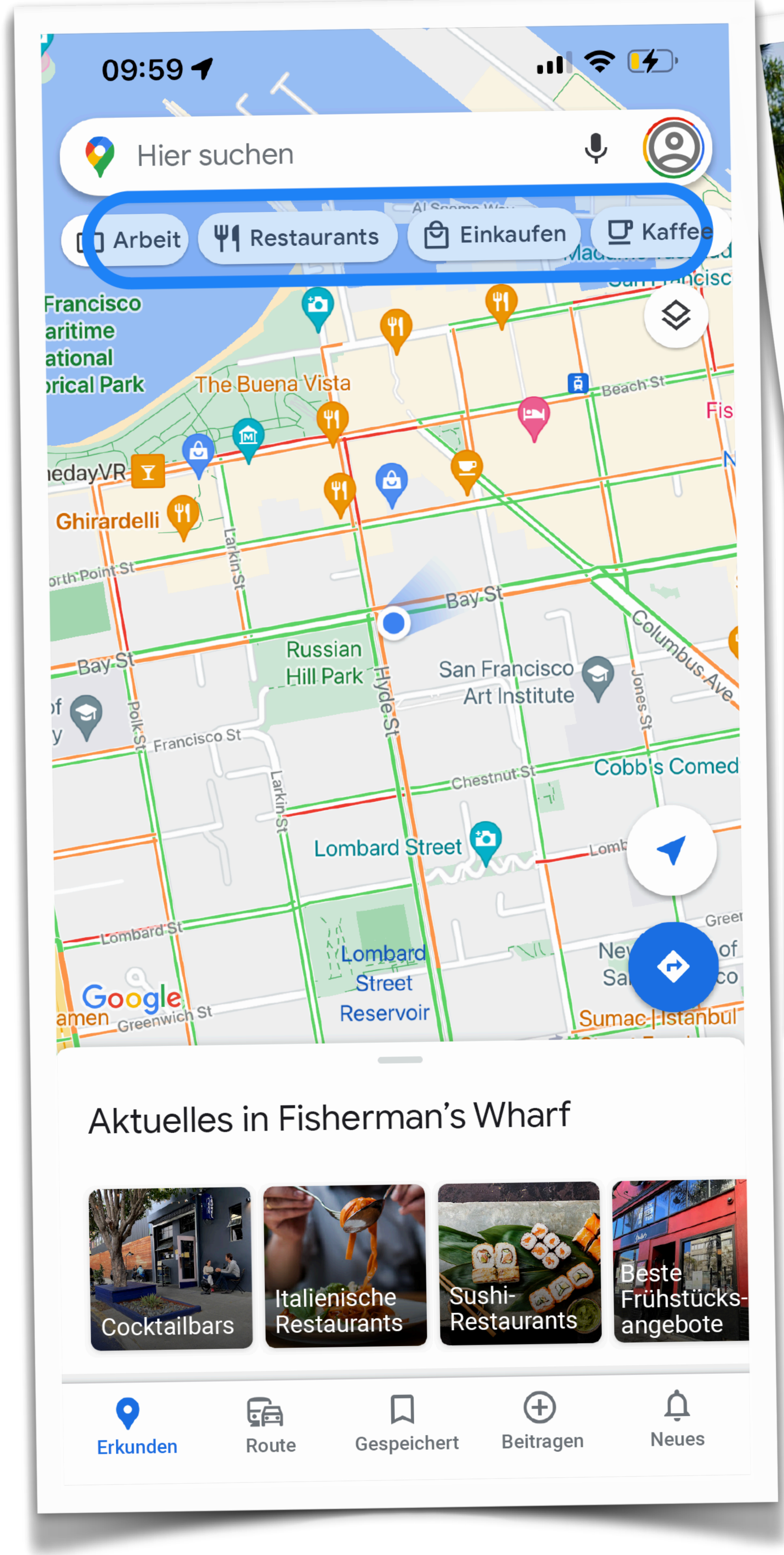
Point Location Problems

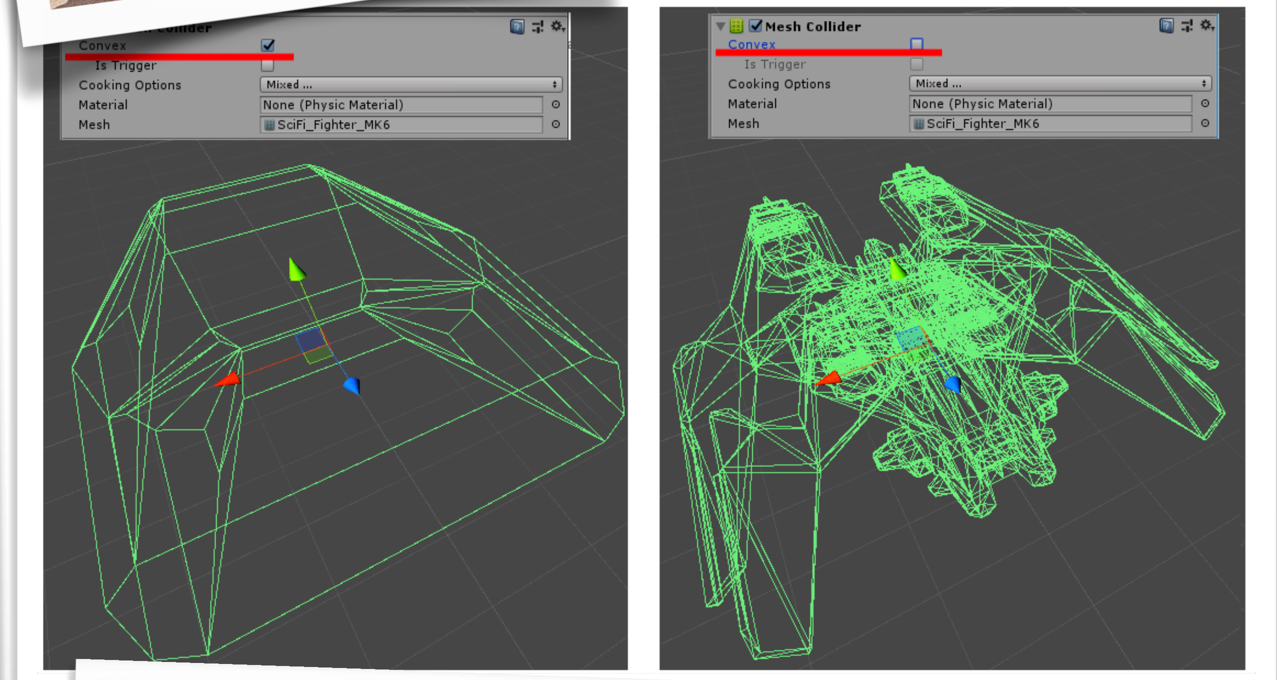
“Where am I?”

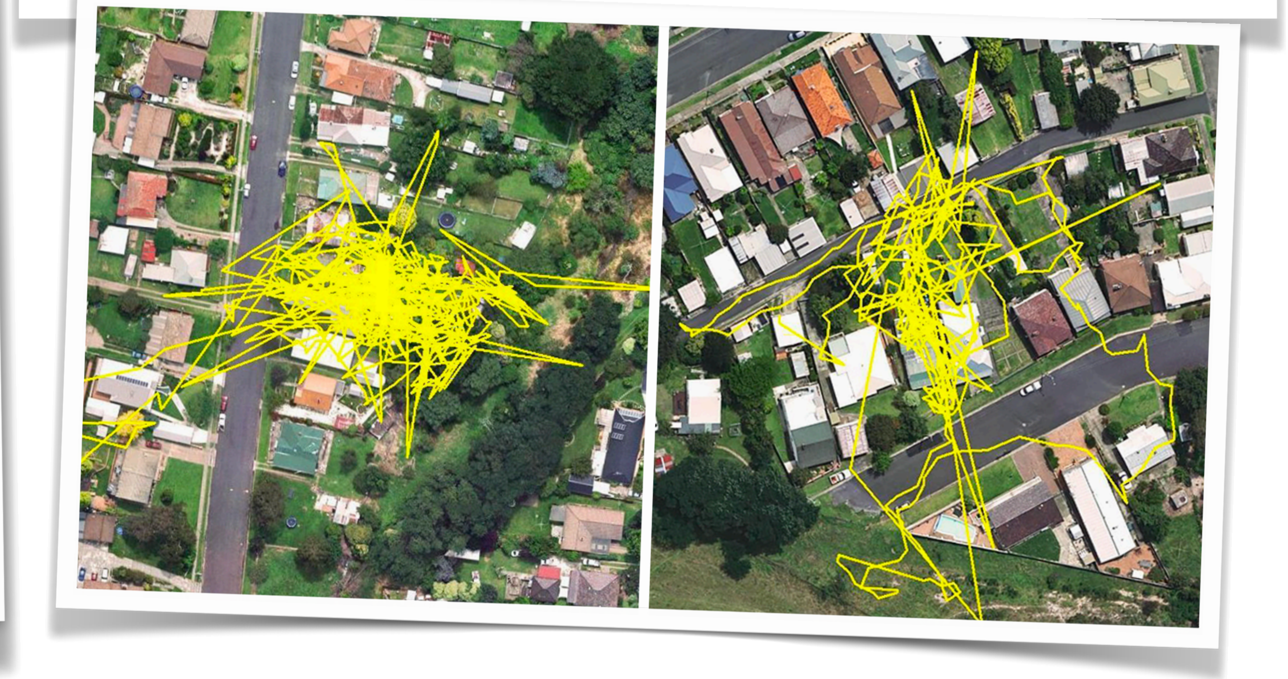
- Given geometric information such as a map in the plane, how can we decide **where** we are?

- *“In which country am I right now?”*
  “Can I leave this scooter here?”
  “Do these virtual objects collide?”

- ***“Is point p inside of region P?”***







Applications: Geofencing, Navigation, Simulation Software, Outlier Detection, …

Geometric path planning

Trapezoidal maps

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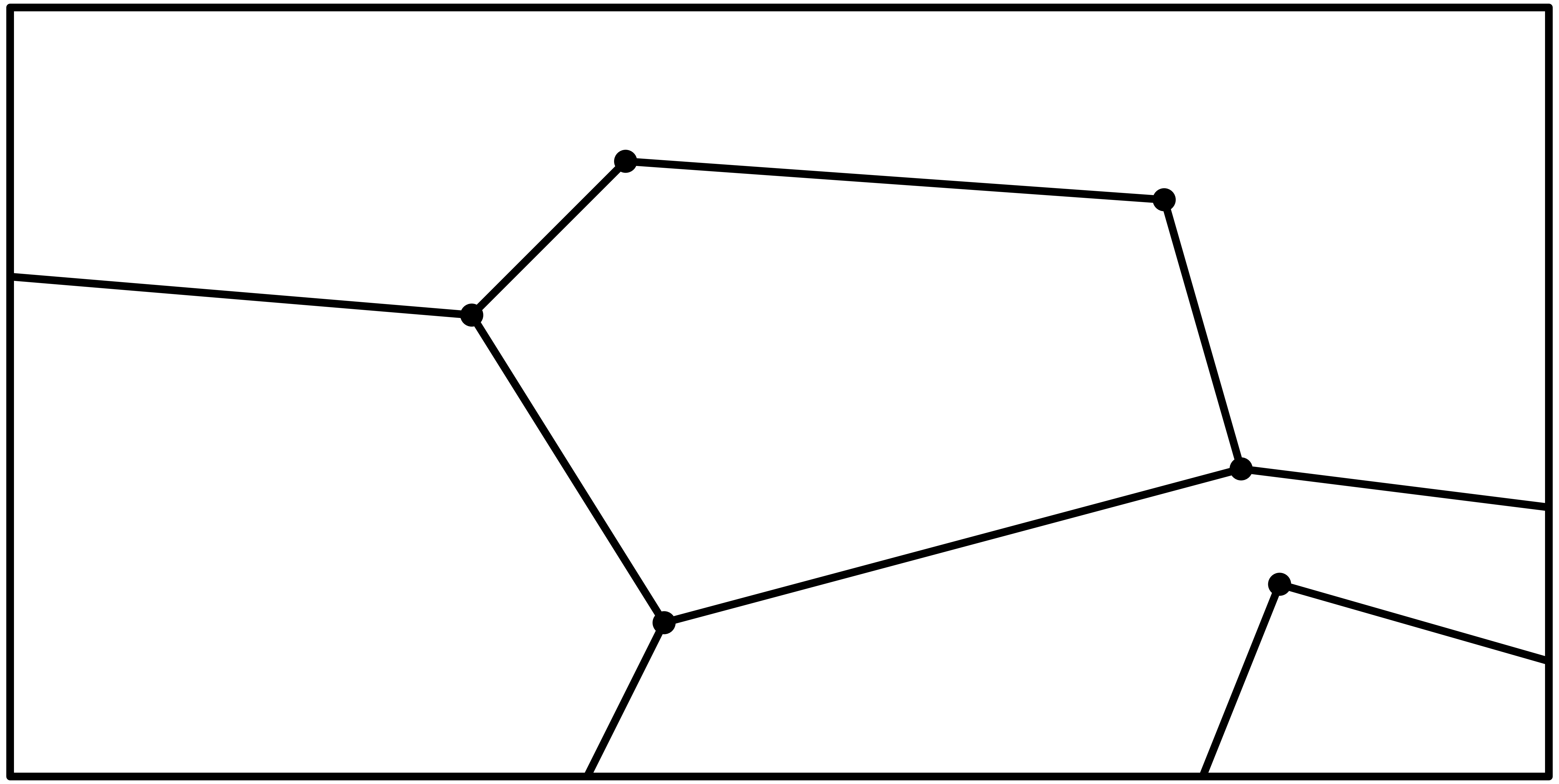
Trapezoidal maps



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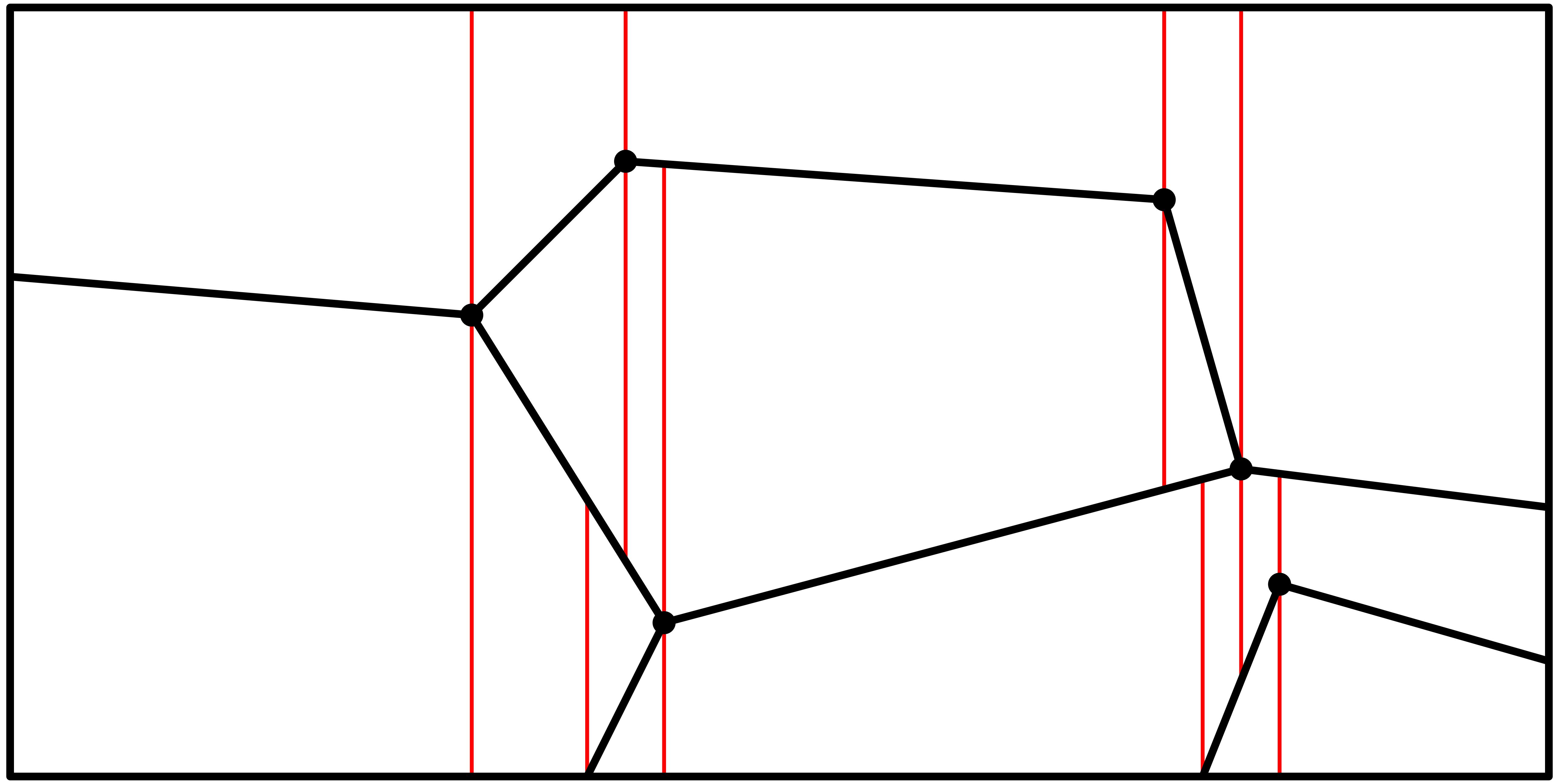

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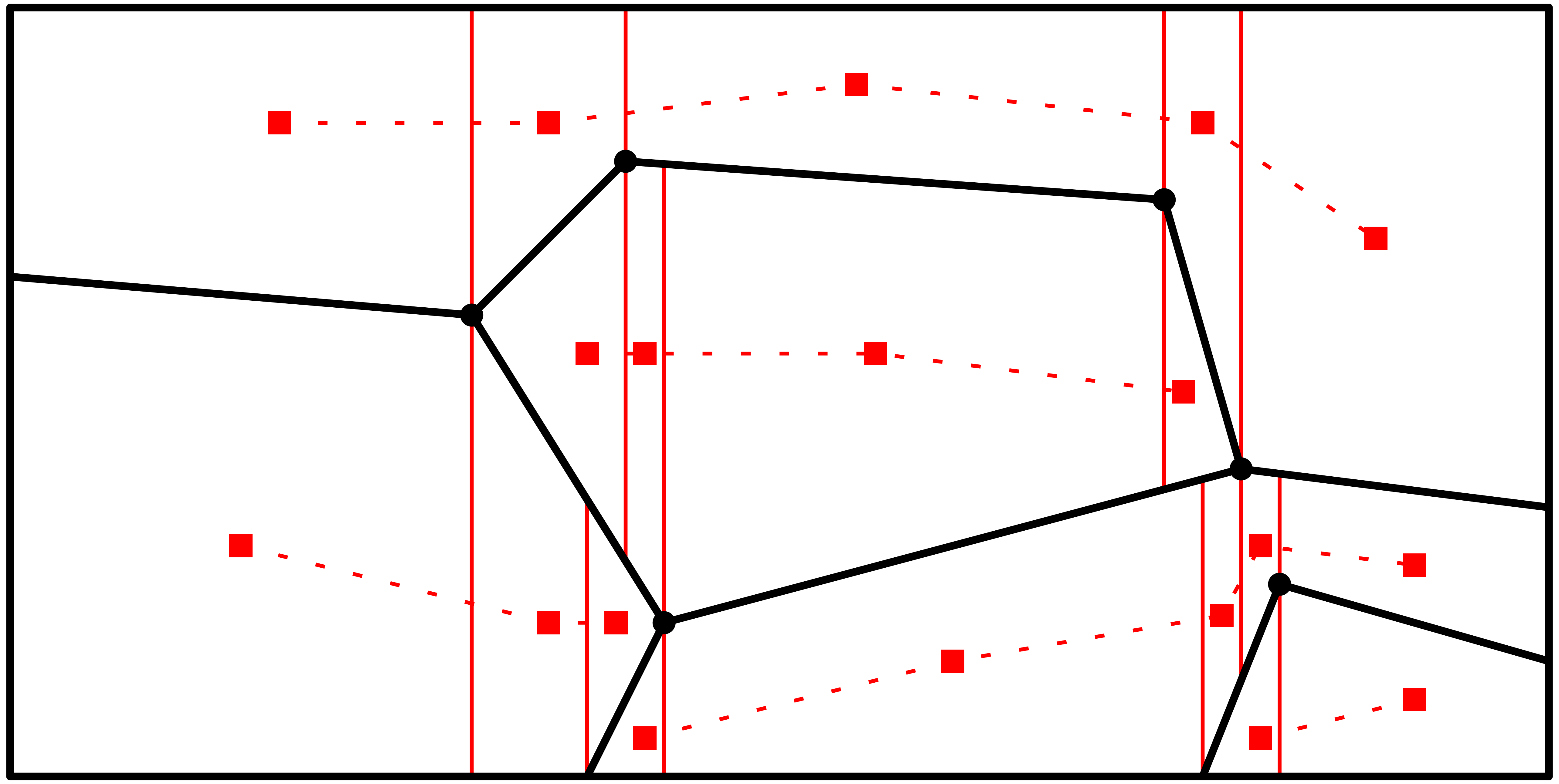

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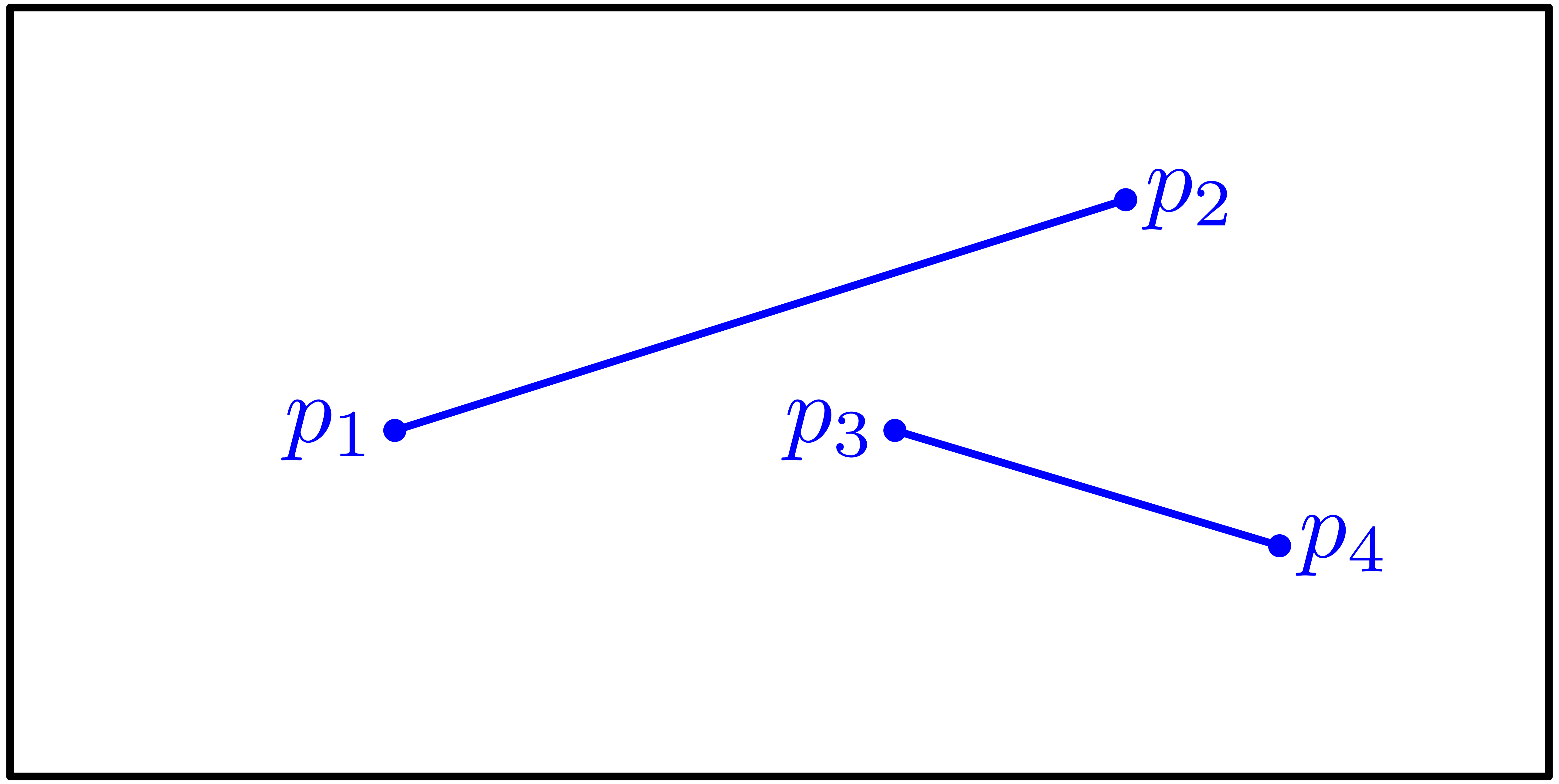

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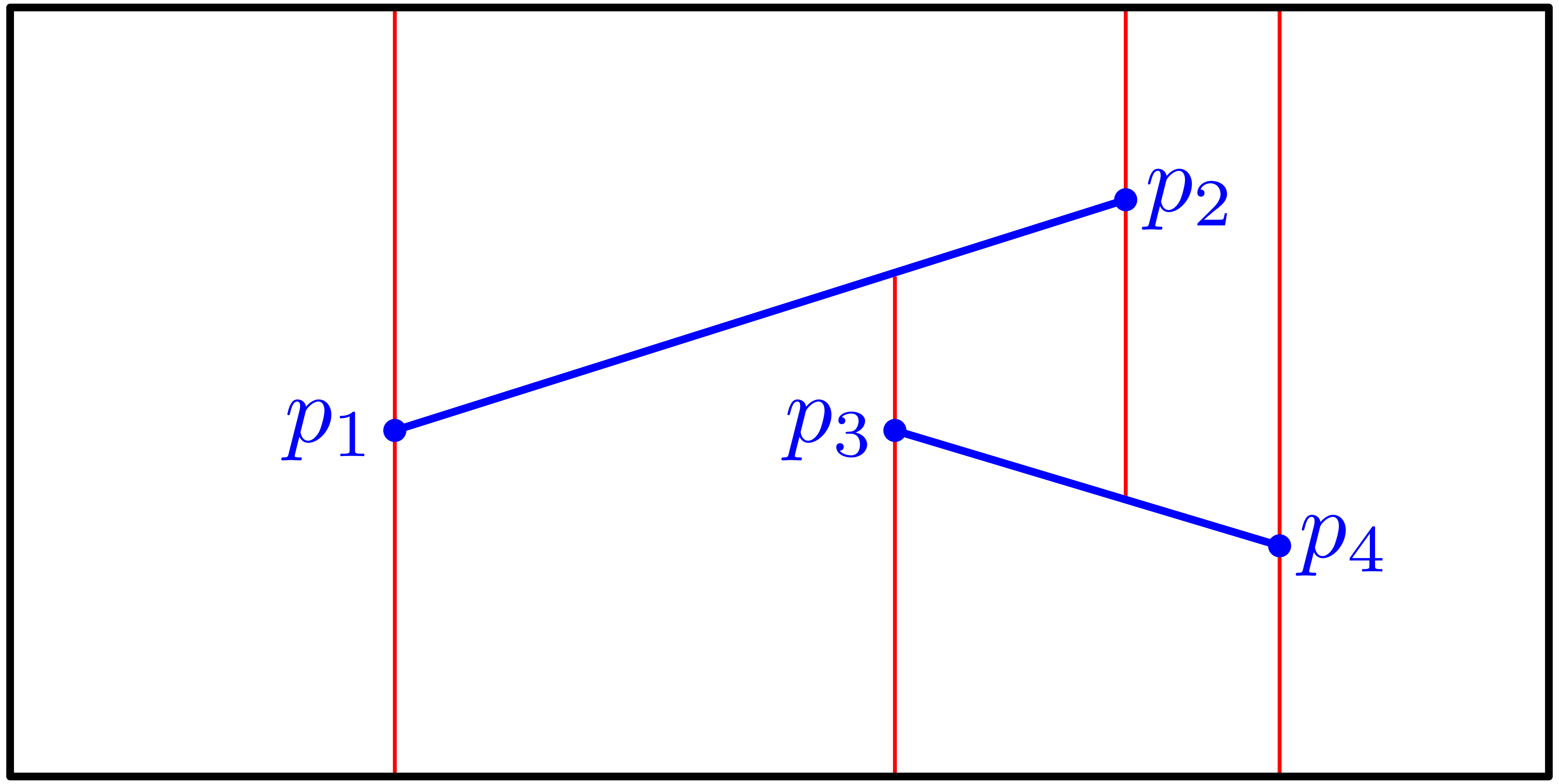

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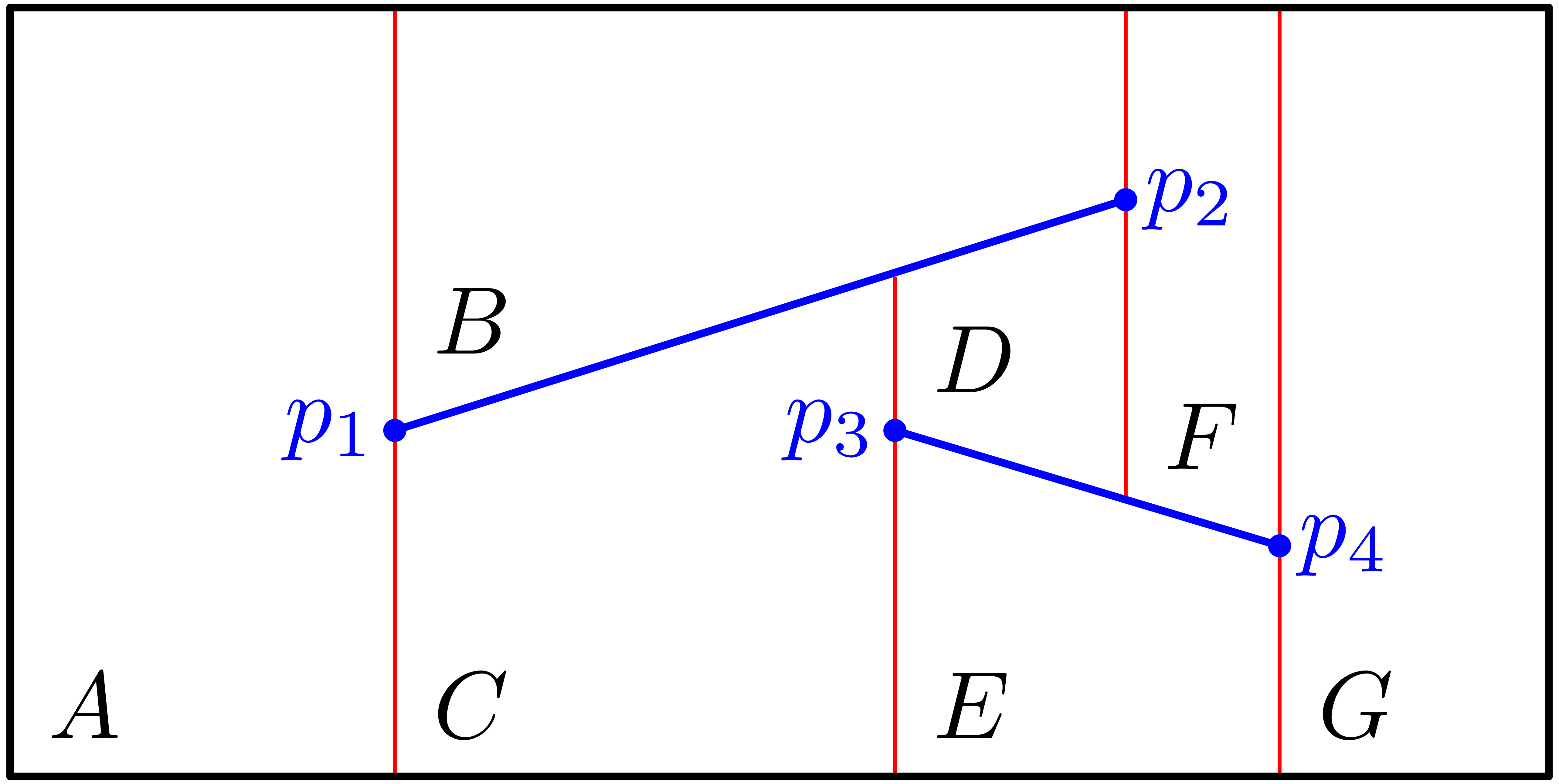
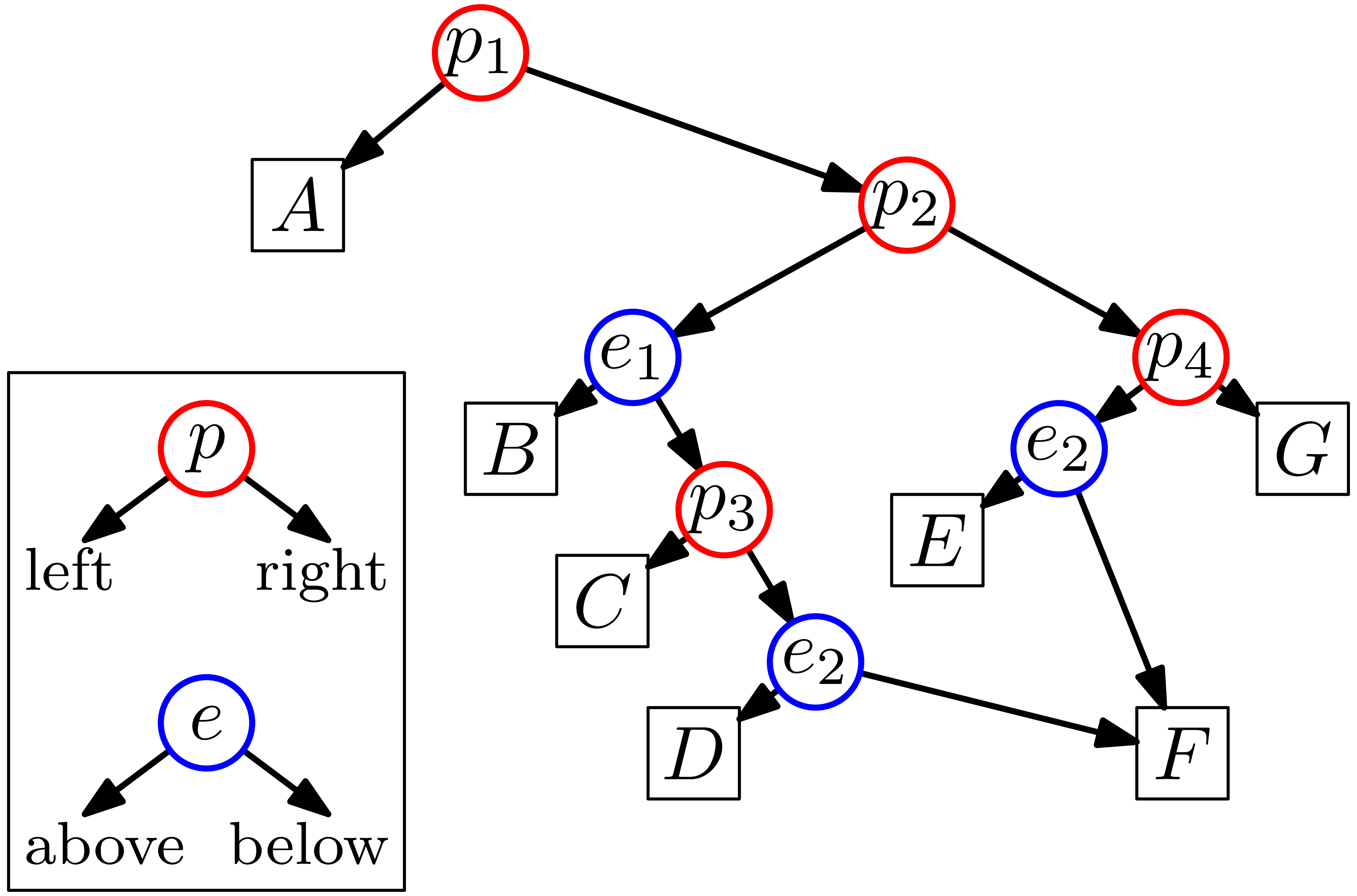
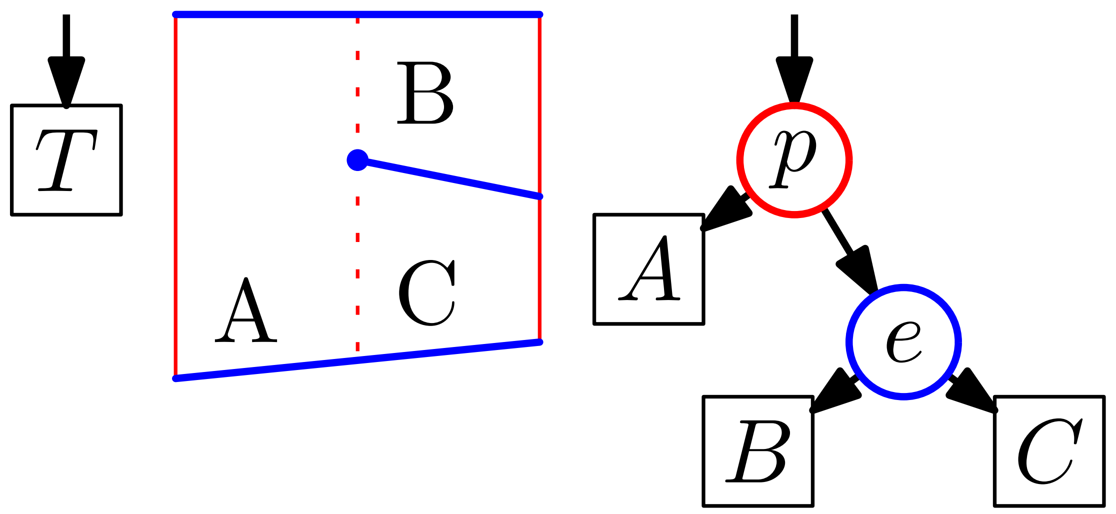
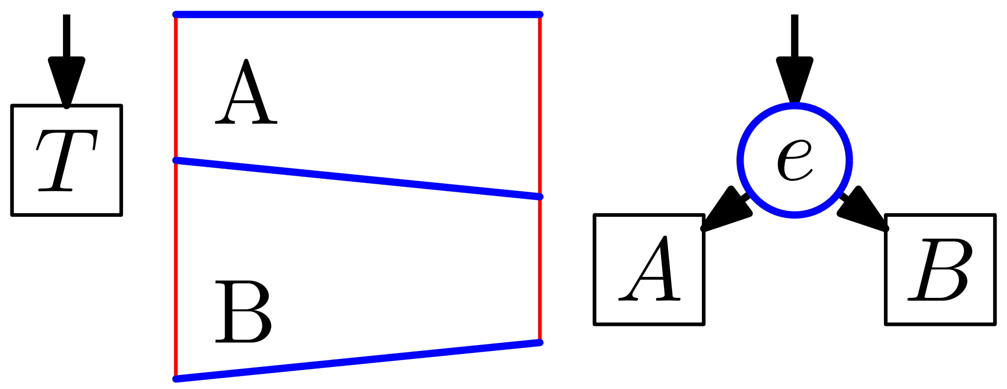


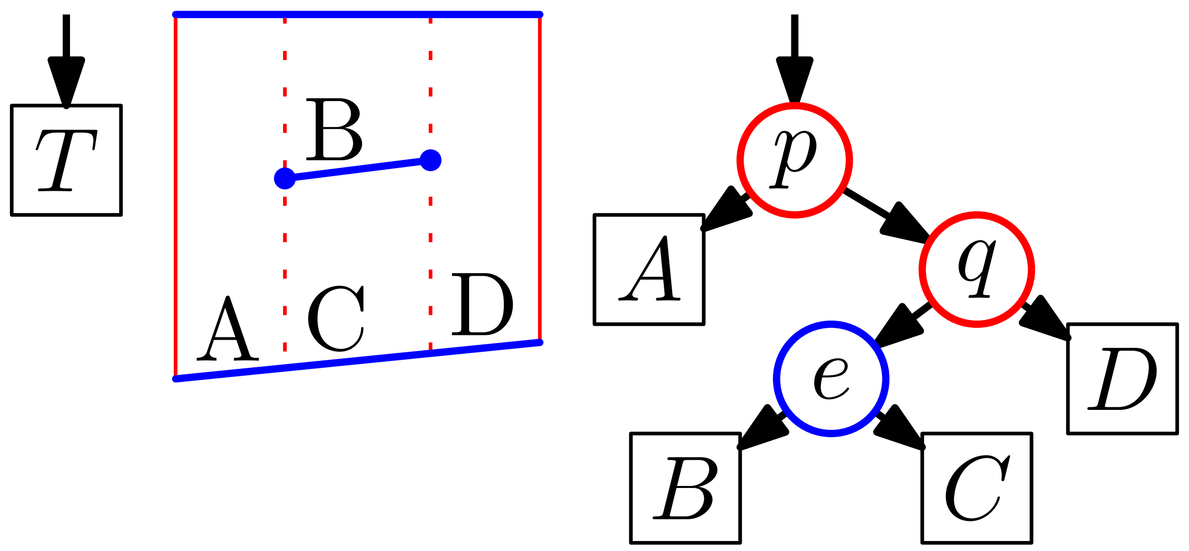


Trapezoidal maps

Incremental construction







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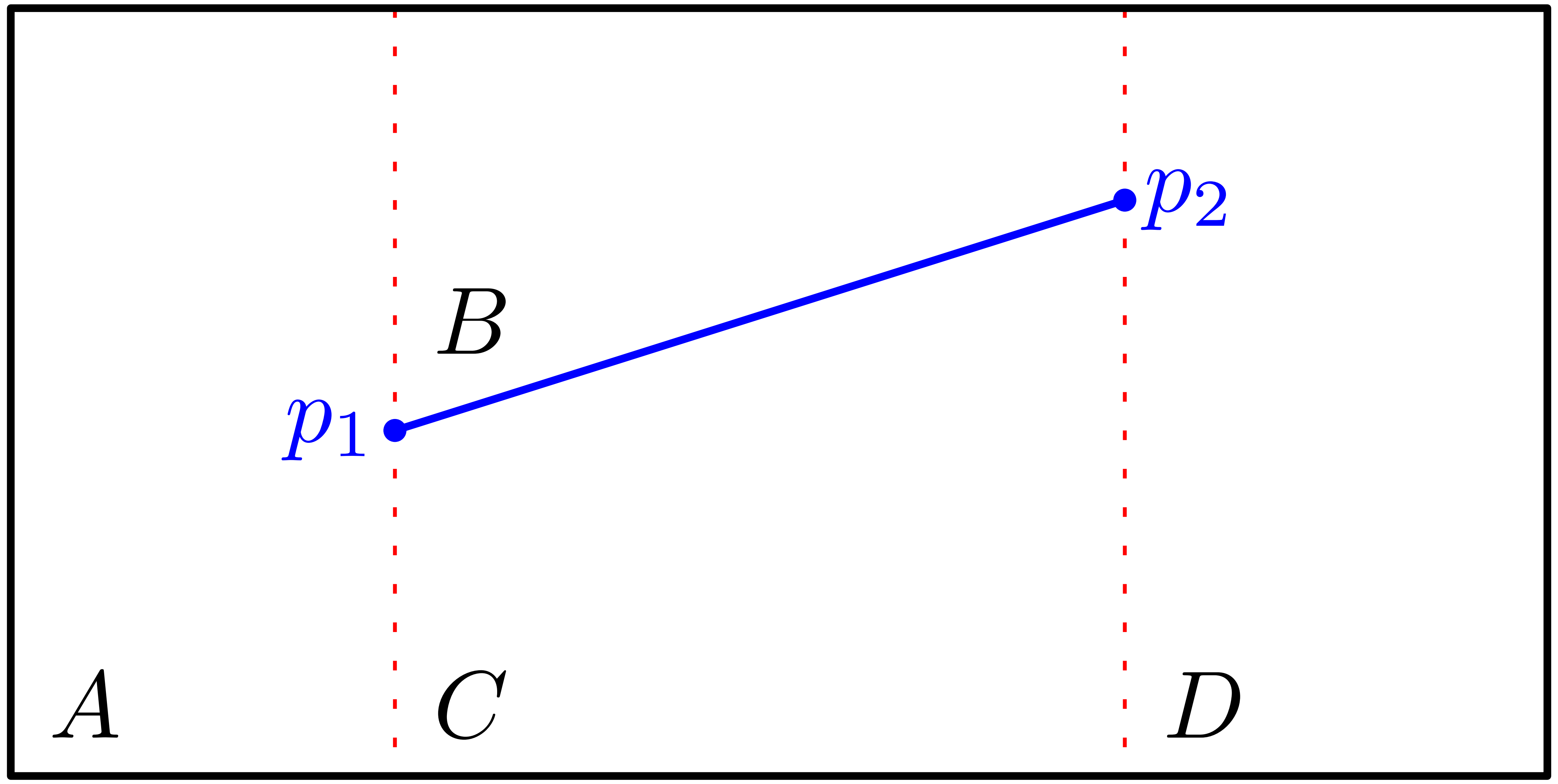

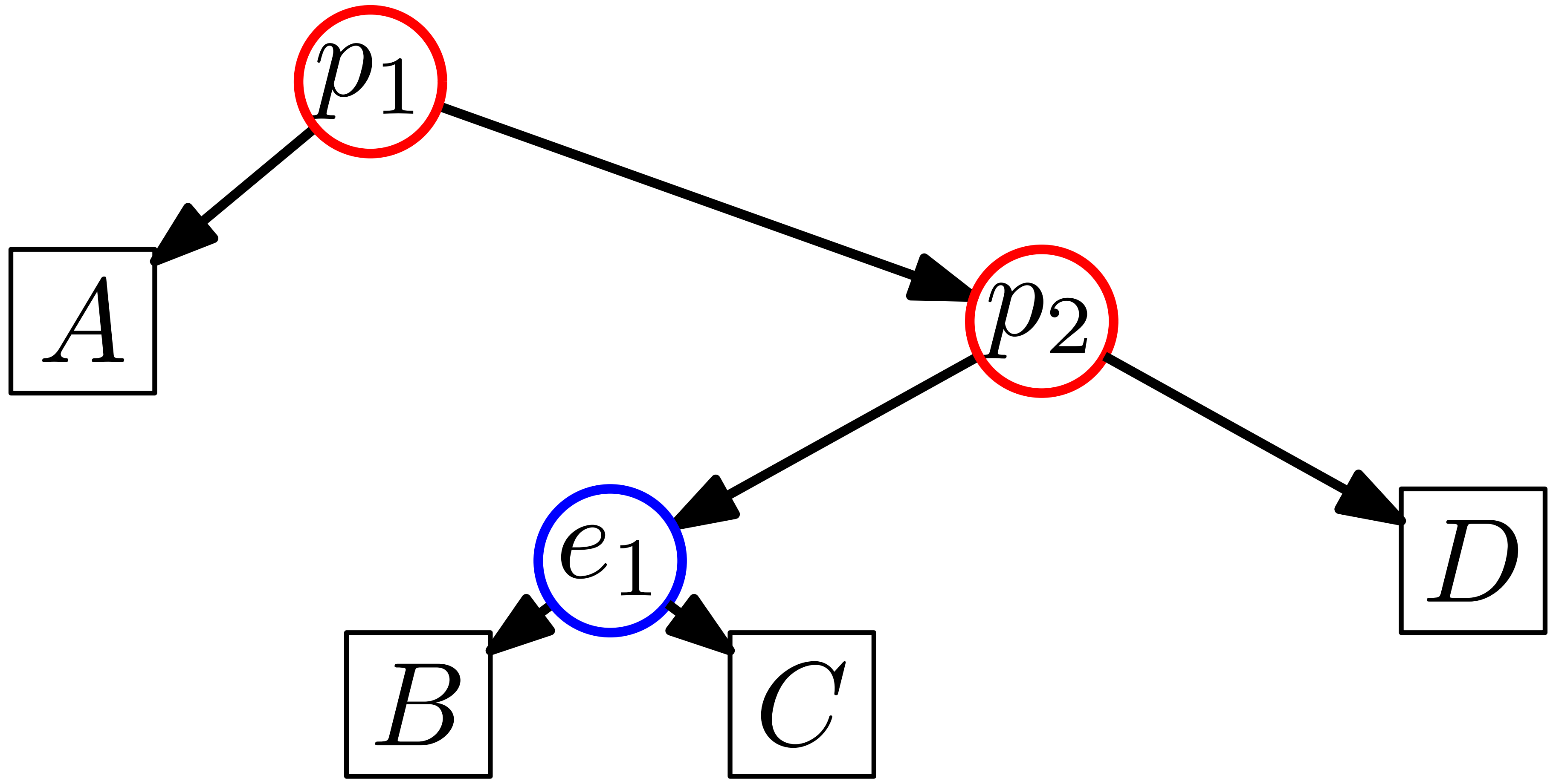

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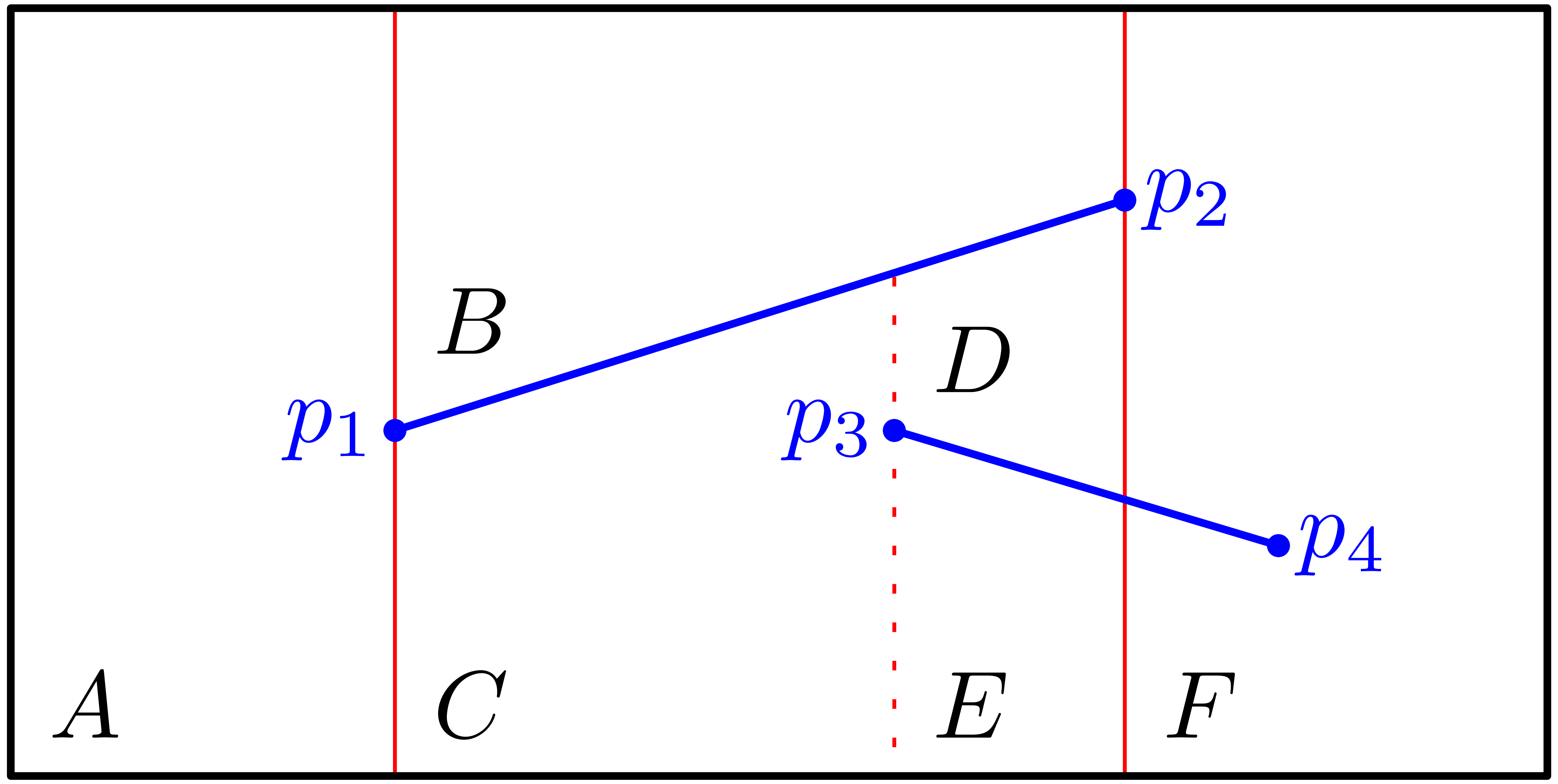

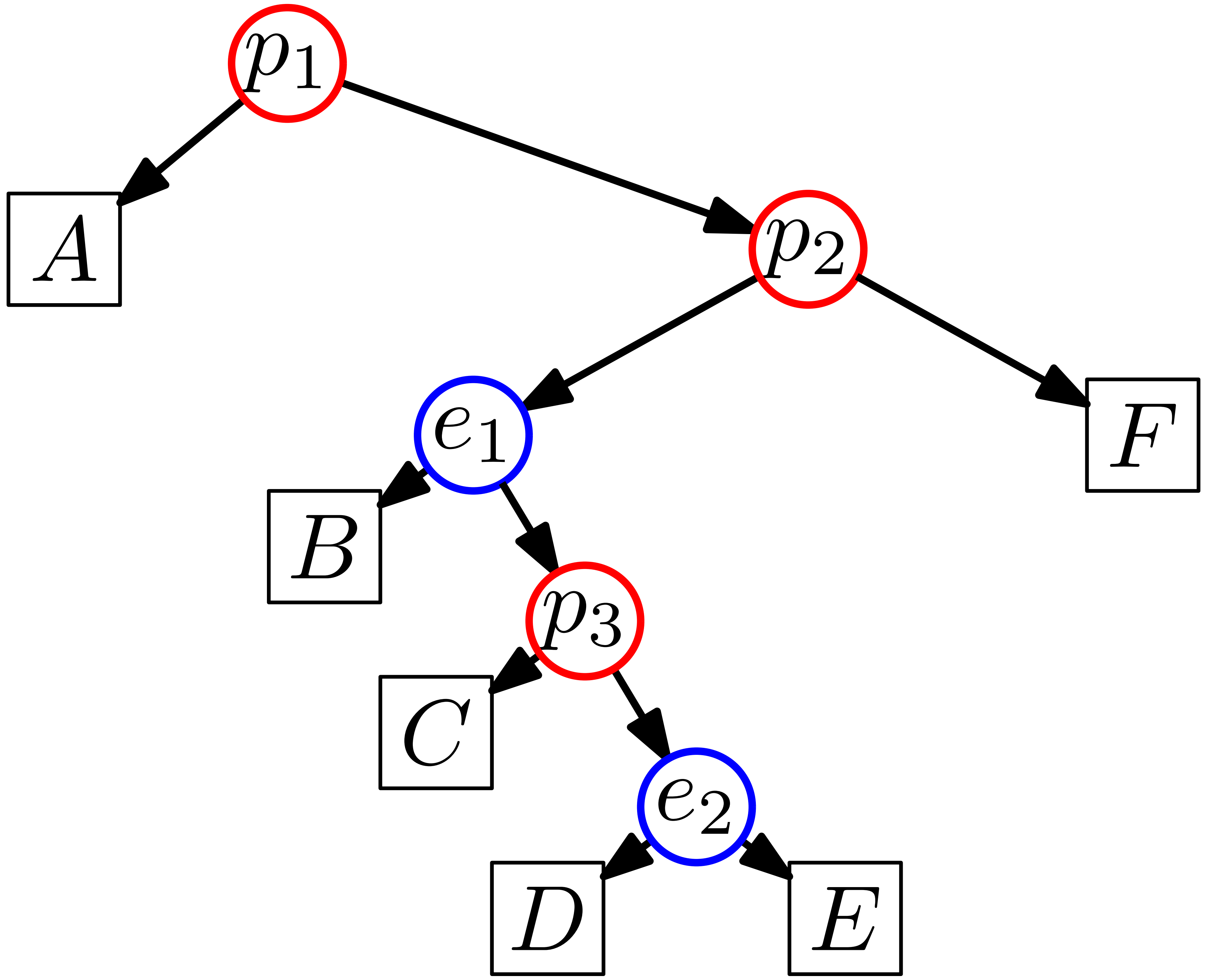

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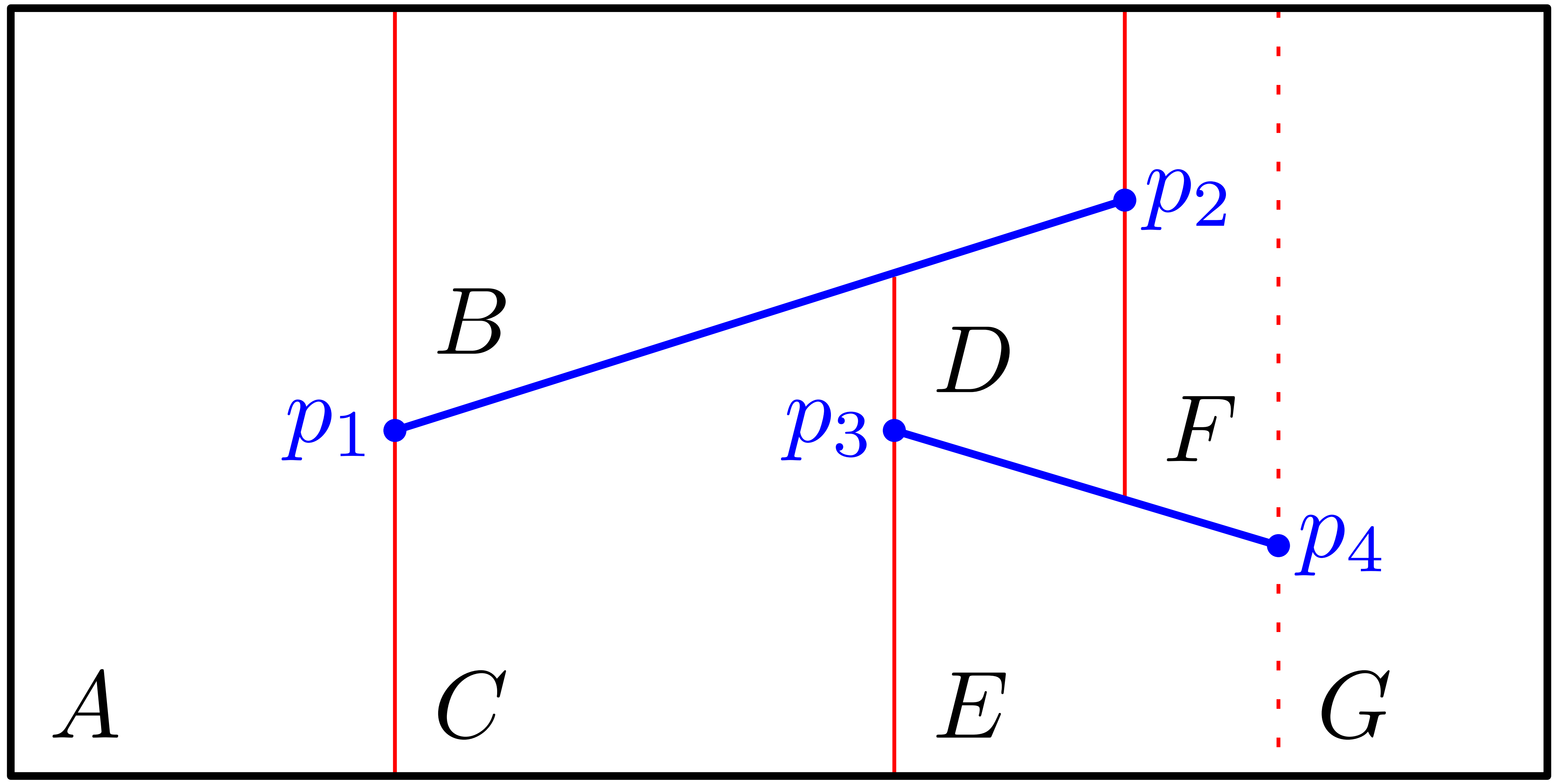

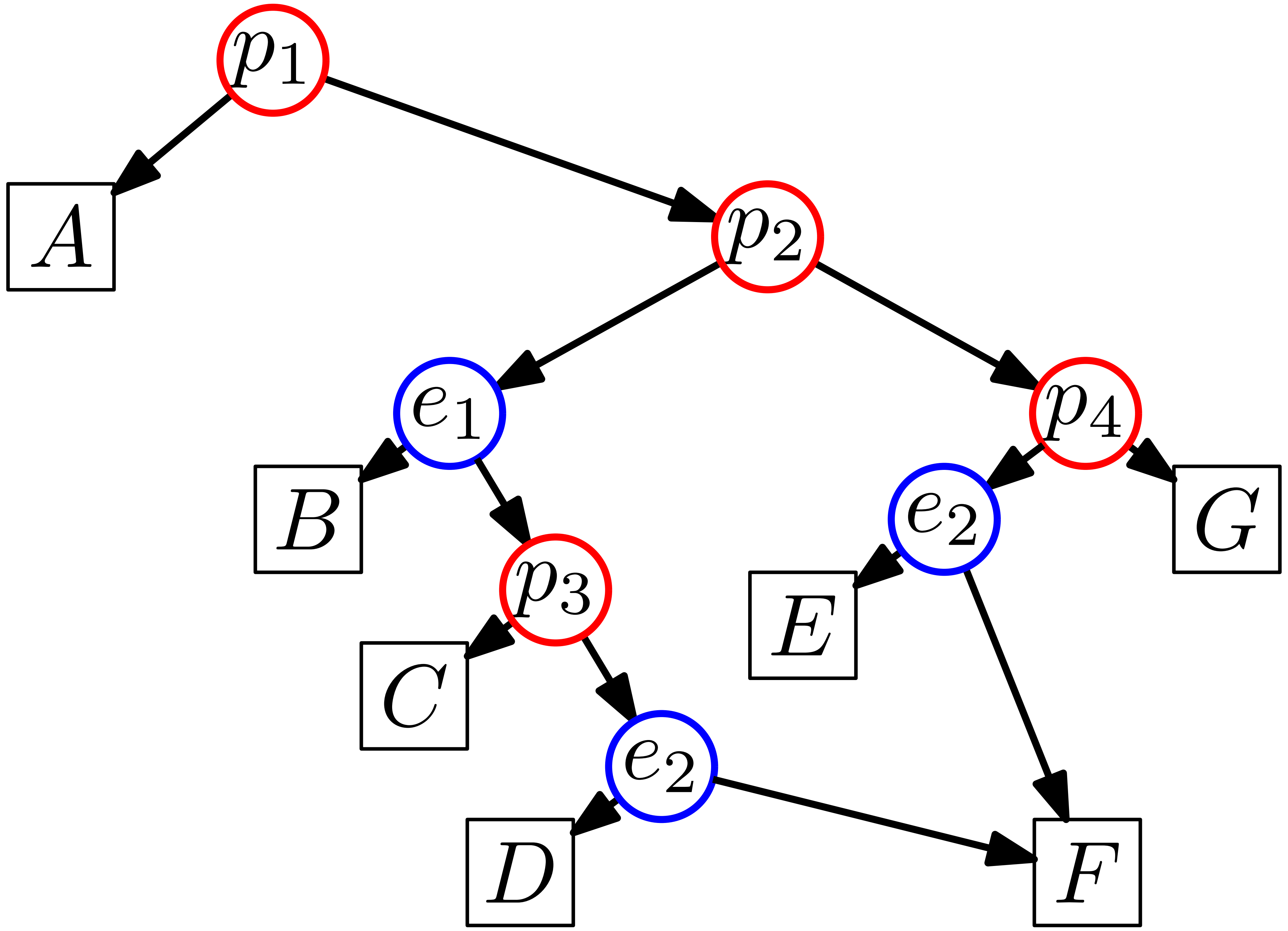

Thank you