Computational Geometry

Lecture #4 — Closest Pairs II

Peter Kramer | November 12th, 2024

Recap:

Master Theorem

“Median of Medians” Algorithm

Closest pair — Bounds

Closest Pair — Divide and conquer

Master Theorem

Analysis of recursive algorithms

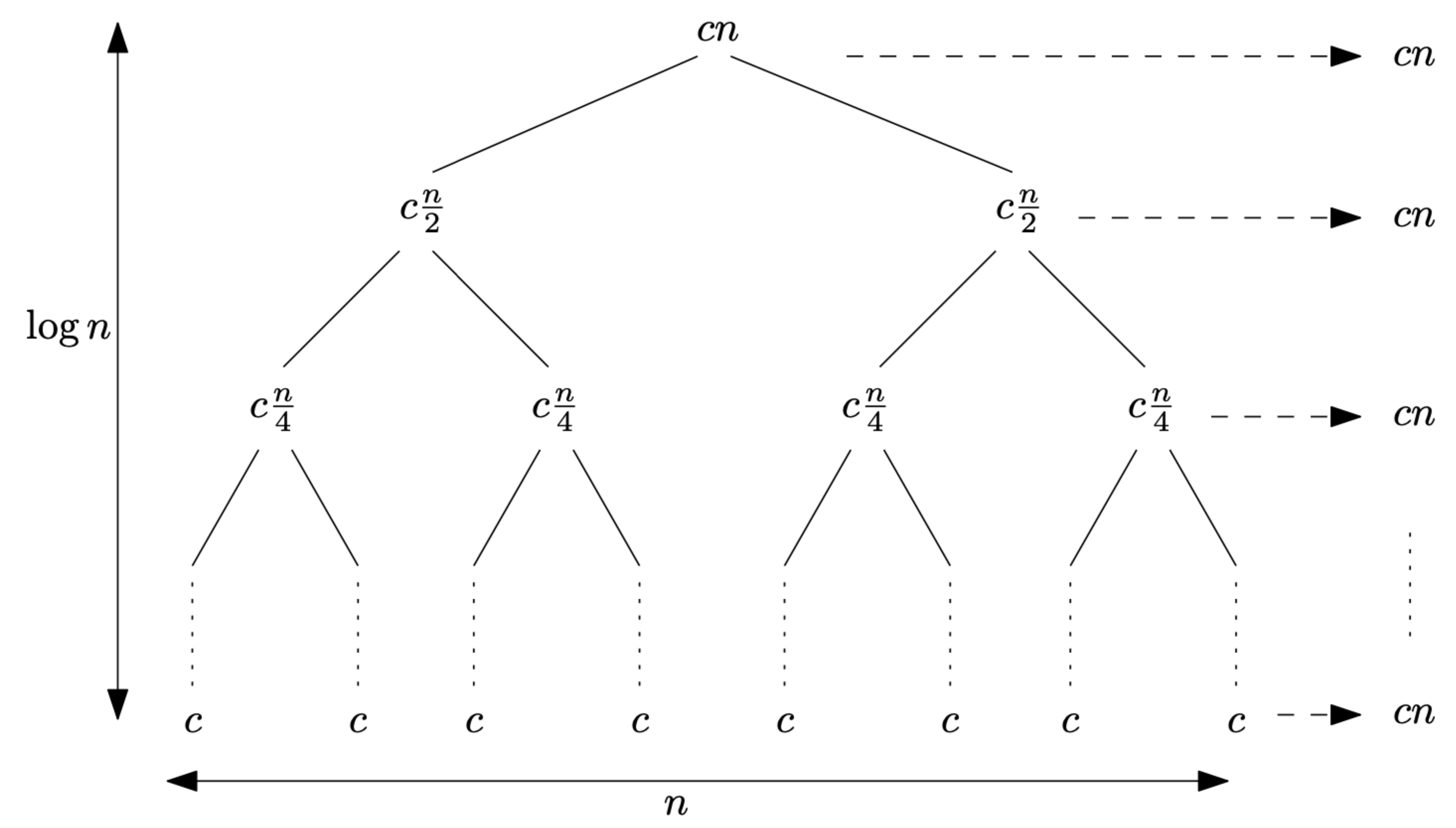
- Runtime based on subproblem size reduction before recursion, in sum:
  - "less than original" $\Rightarrow \Theta(n^k)$
  - "same as original" $\Rightarrow \Theta(n^k \log n)$
  - "more than original" $\Rightarrow \Theta(n^c)$
- k determined by cost of "merge" step
- e.g., Mergesort has $\Theta(n \log n)$

Theorem 3.3 (Master Theorem) *Let* $T : \mathbb{N} \to \mathbb{R}$ *with*

$$T(n) = \sum_{i=1}^{m} T(\alpha_i \cdot n) + \Theta(n^k),$$

where $\alpha_i \in \mathbb{R}$ *with* $0 < \alpha_i < 1,\ m \in \mathbb{N}$ *and* $k \in \mathbb{R}$. *Then*

$$T(n) \in \begin{cases} \Theta(n^k) & \textit{for } \sum_{i=1}^{m} \alpha_i^k < 1 \\ \Theta(n^k \log(n)) & \textit{for } \sum_{i=1}^{m} \alpha_i^k = 1 \\ \Theta(n^c) & \textit{with } \sum_{i=1}^{m} \alpha_i^c = 1 \textit{ for } \sum_{i=1}^{m} \alpha_i^k > 1 \end{cases}$$



Recap:

“Median of Medians” Algorithm

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“Median of Medians” Algorithm
Medians [Blum, Floyd, Pratt, Rivest, Tarjan 1973]

Theorem 3.4: A median of n numbers can be computed in $\mathcal{O}(n)$.

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- Group the numbers into sets of 5.

1	6	11	17	3
22	18	16	5	9
10	21	2	12	15
13	4	20	19	7
24	25	8	14	23

$\mathcal{O}(n)$

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- Compute the median of each group.

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- Use the *median of medians* as a pivot.

≥ n/4 numbers

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3	2	1	5	4
7	8	10	12	6
9	11	13	14	18
15	16	22	17	21
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$T(\frac{3n}{4})$

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Theorem 3.4: A median of n numbers can be computed in $\mathcal{O}(n)$.

$$T(n) = T(\frac{n}{5}) + T(\frac{3n}{4}) + \Theta(n)$$

$$\sum_{i=1}^{m} \alpha_i^k = \frac{1}{5} + \frac{3}{4} = \frac{19}{20} < 1.$$

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Recap:

Closest pair — Bounds

Closest Pair — Divide and conquer

Lower bound for the Closest pair Problem

Via SORTING and ELEMENT UNIQUENESS

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- **We showed:** Closest pair solves the Element Uniqueness Problem: If closest pair has distance zero, the are not unique.

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- Both require at least $\Omega(n \log n)$ comparisons to solve! $\Rightarrow$ Decision trees
- **We showed:** CLOSEST PAIR solves the ELEMENT UNIQUENESS PROBLEM: If closest pair has distance zero, the are not unique.

Relevant Paper: Ben-Or, "Lower Bounds For Algebraic Computation Trees"

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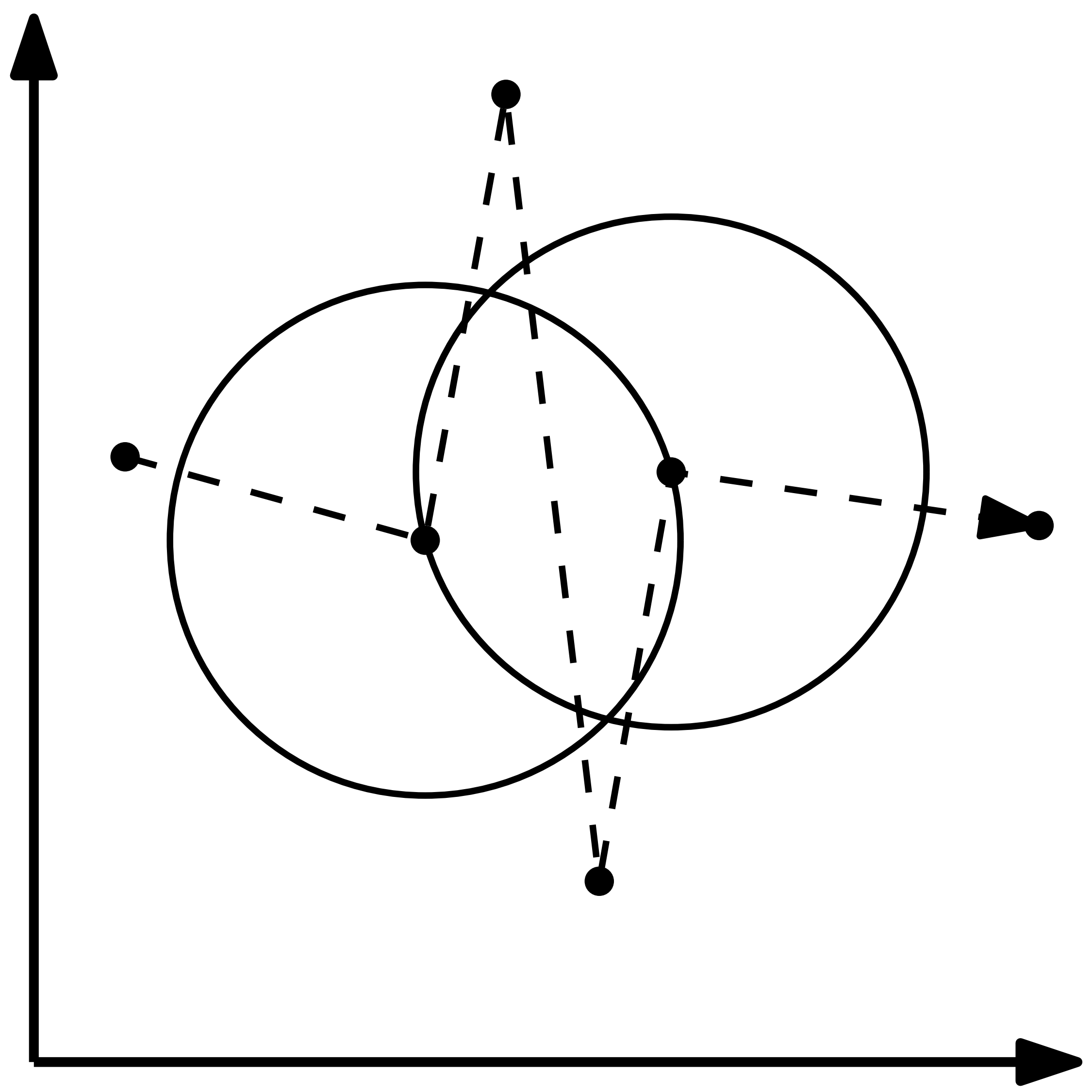

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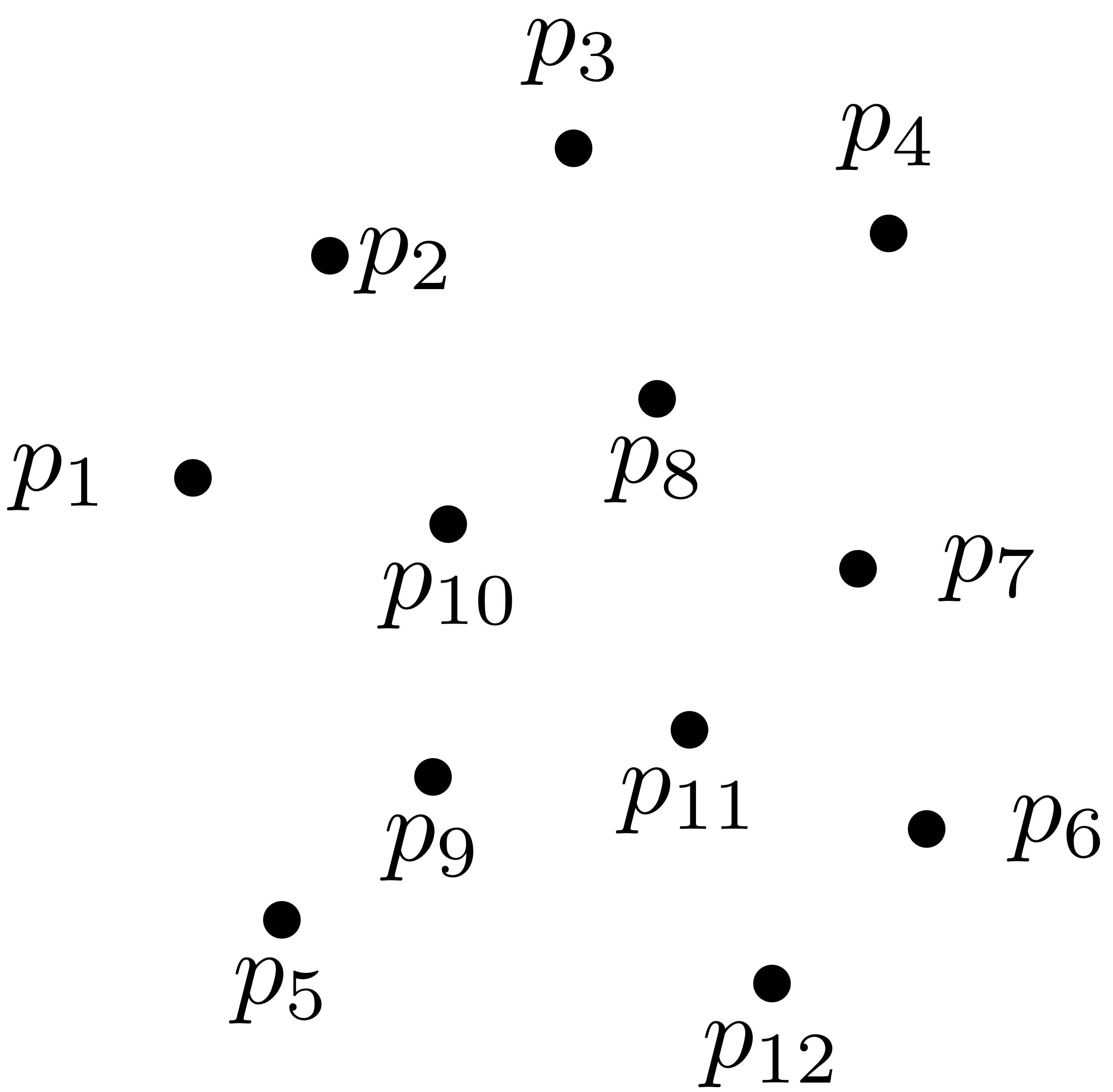
Recap:

Closest Pair — Divide and conquer

Peter Kramer | November 12th, 2024

Closest Pair — Divide and conquer Bentley & Shamos, 1976

- **Theorem 3.8:** A closest pair for n numbers in $\mathbb{R}^2$ can be found in $\mathcal{O}(n \log n)$.



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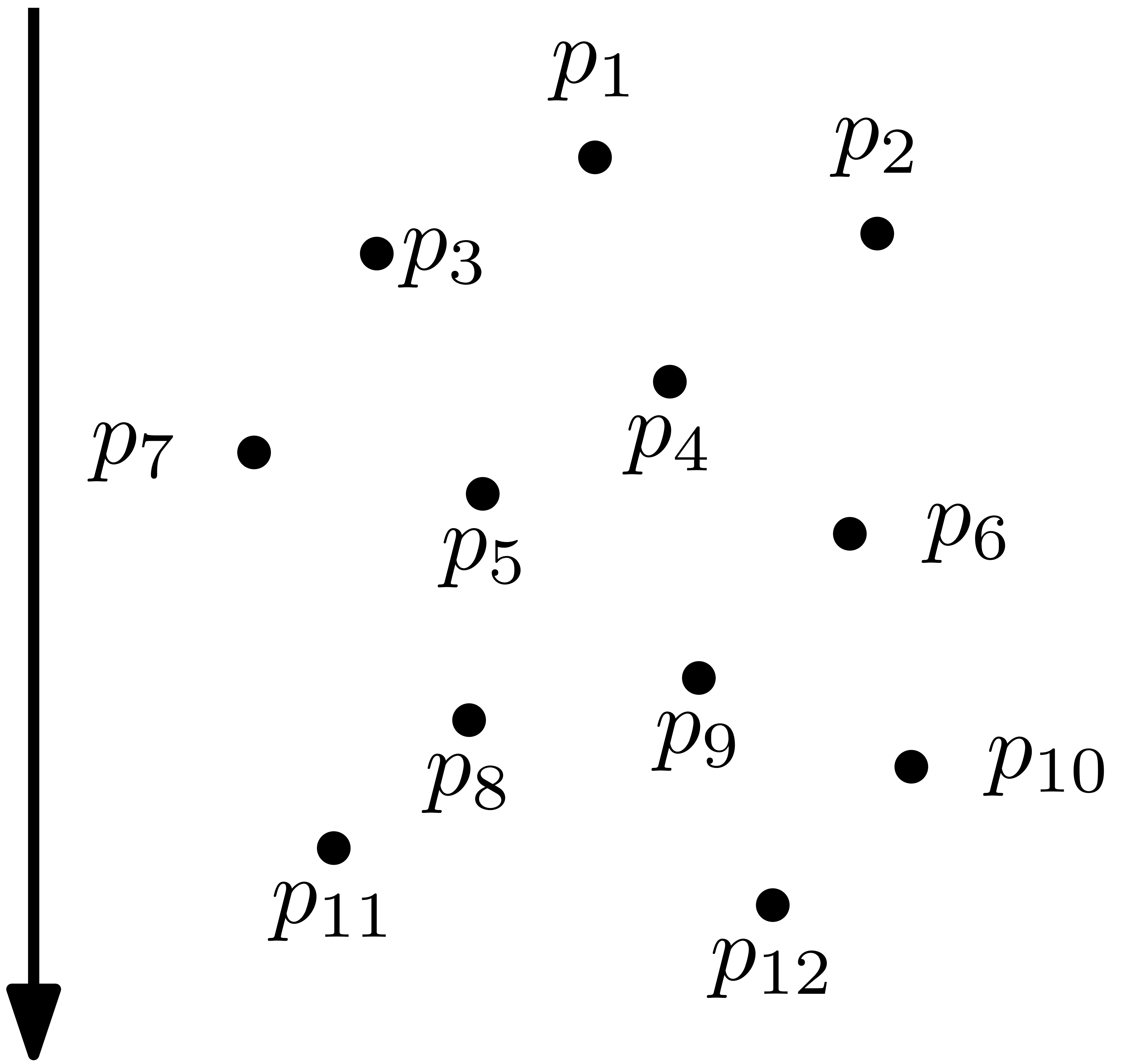

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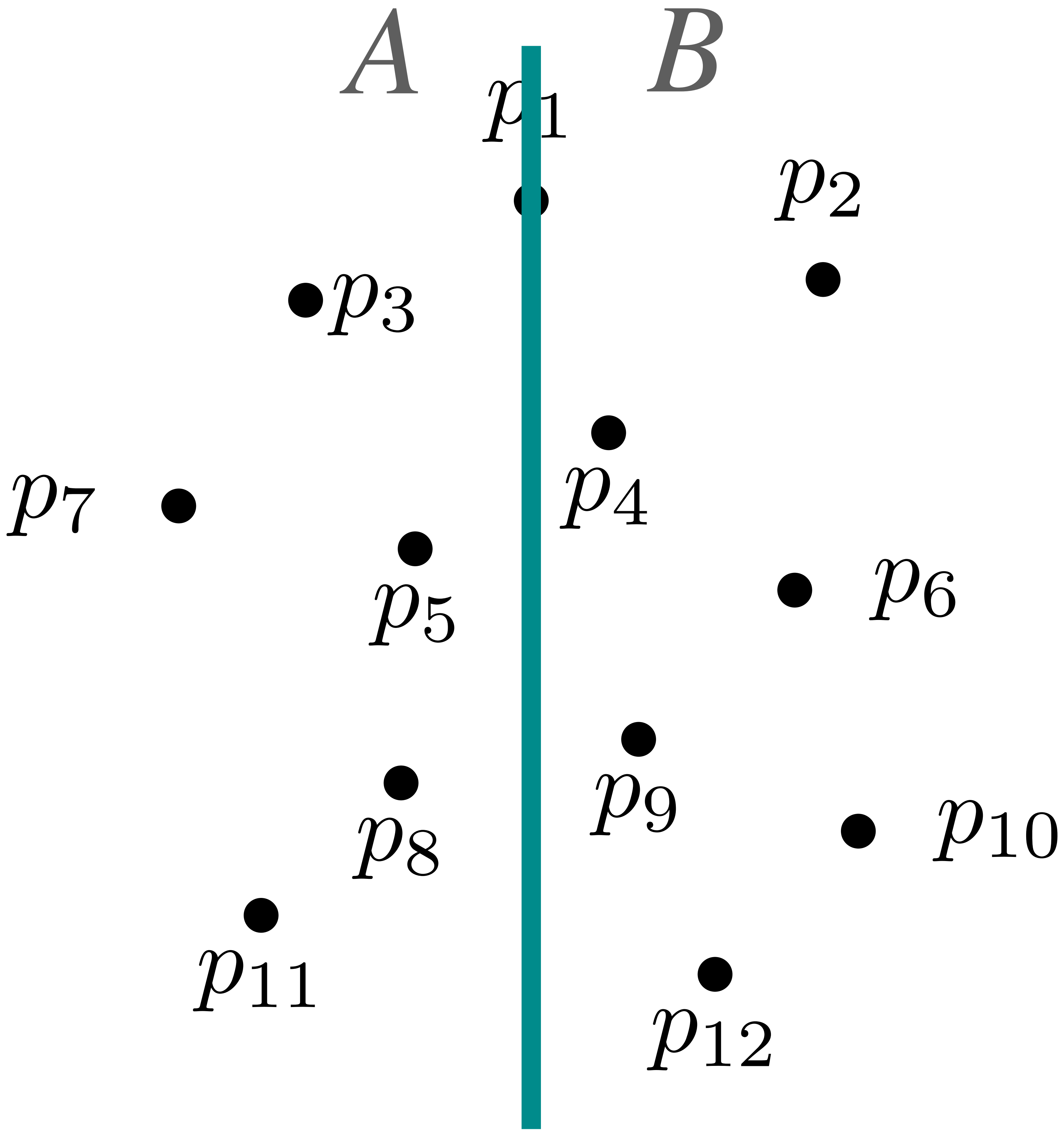

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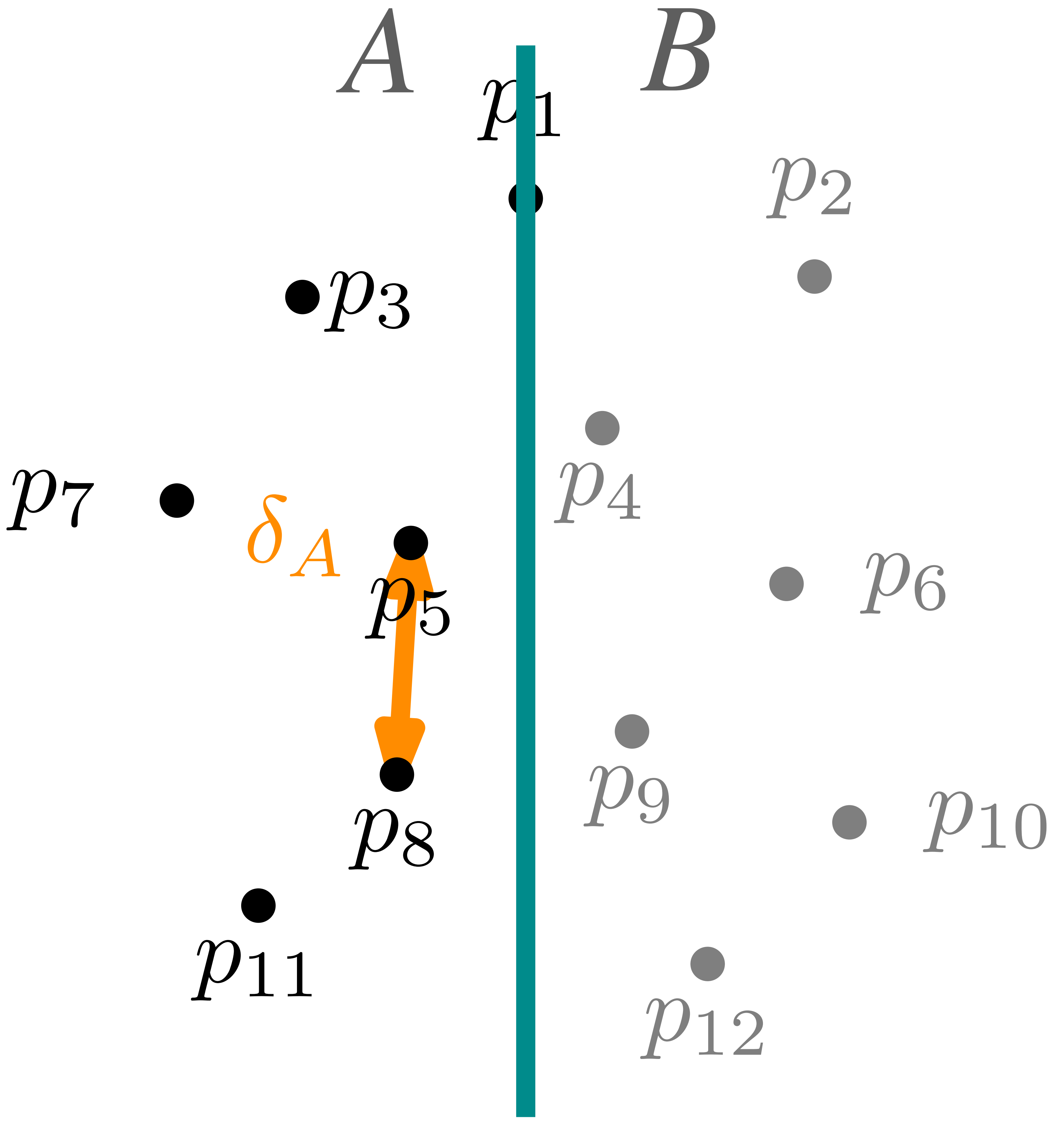

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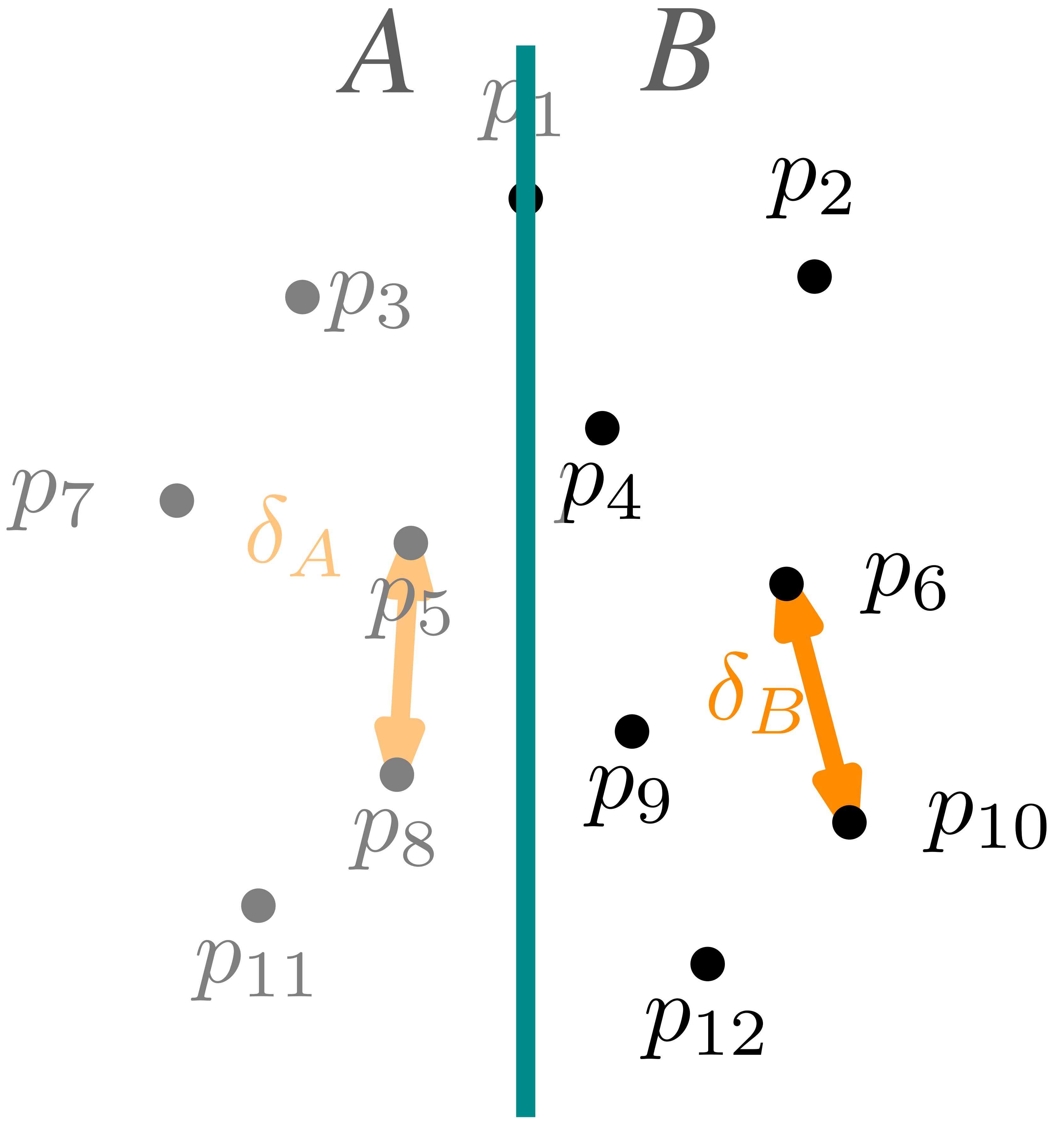

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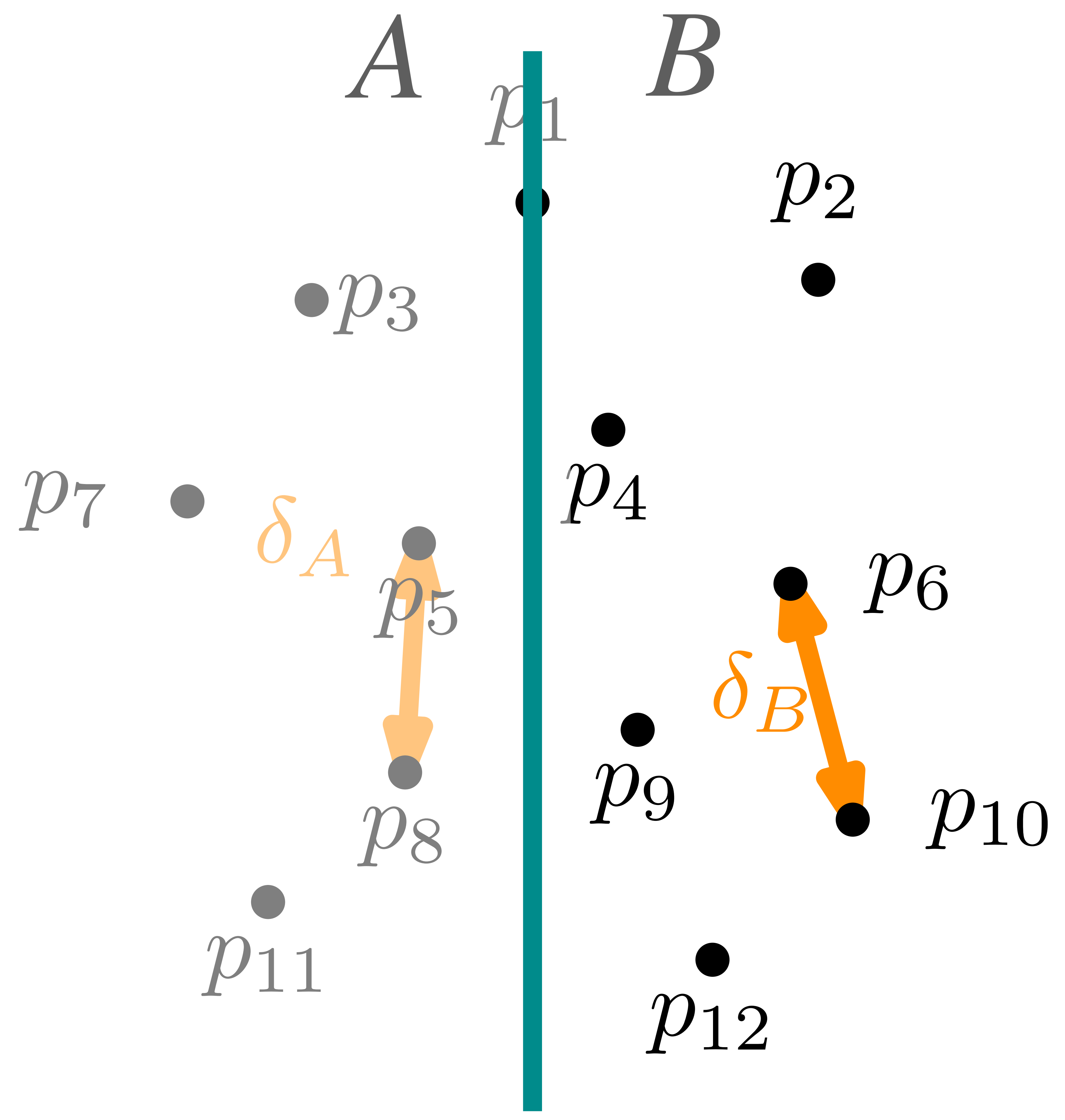
4. … merge by checking pairs in $A \times B$.



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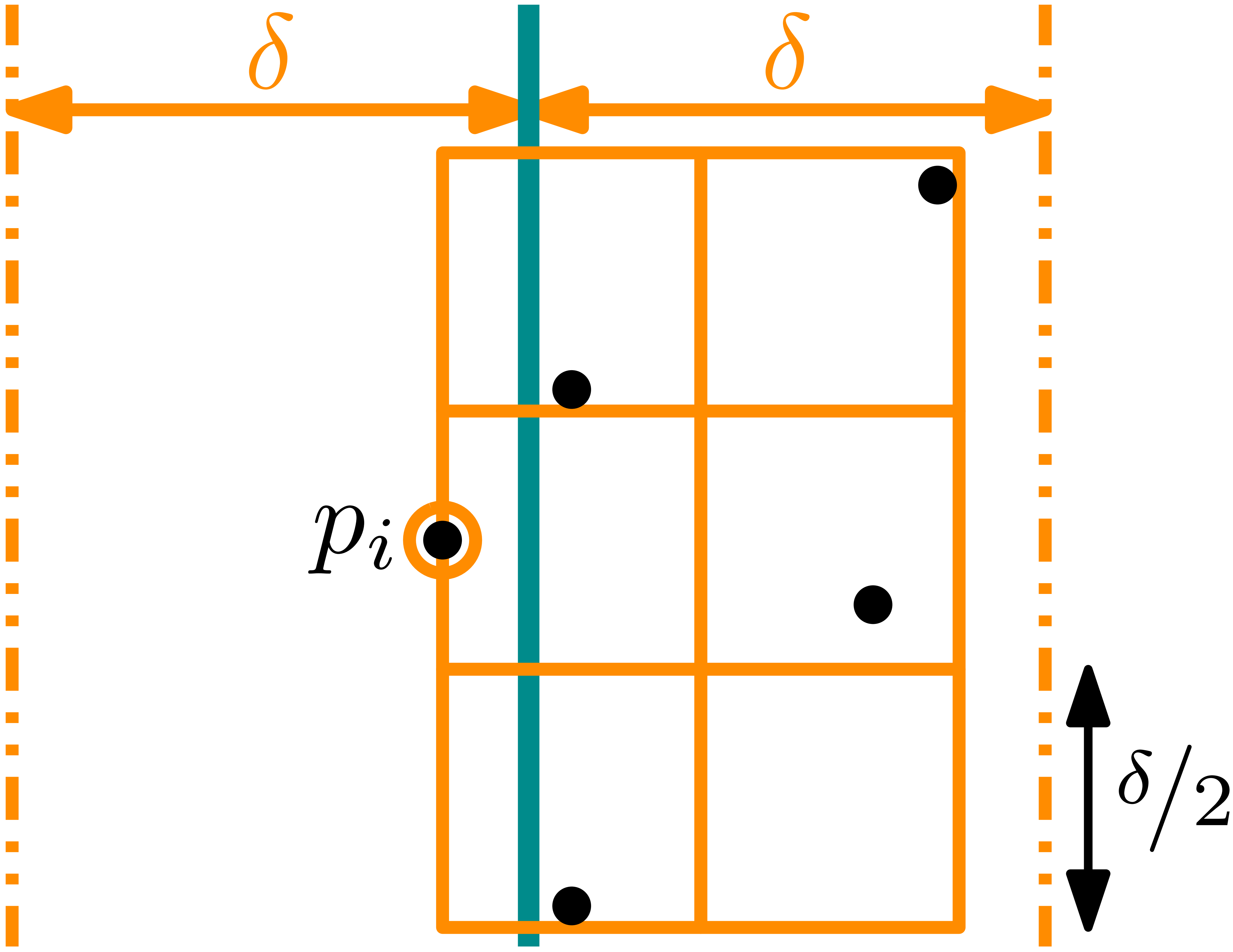
4. … merge by checking pairs in $A \times B$?

- Candidate pairs in $A \times B$ must have distance at most δ to the median line.
- Point pairs in A (B) have distance $\geq \delta$.
- Therefore: For each point in $p_i \in A$, there can be only constantly many points in B that are closer than δ!



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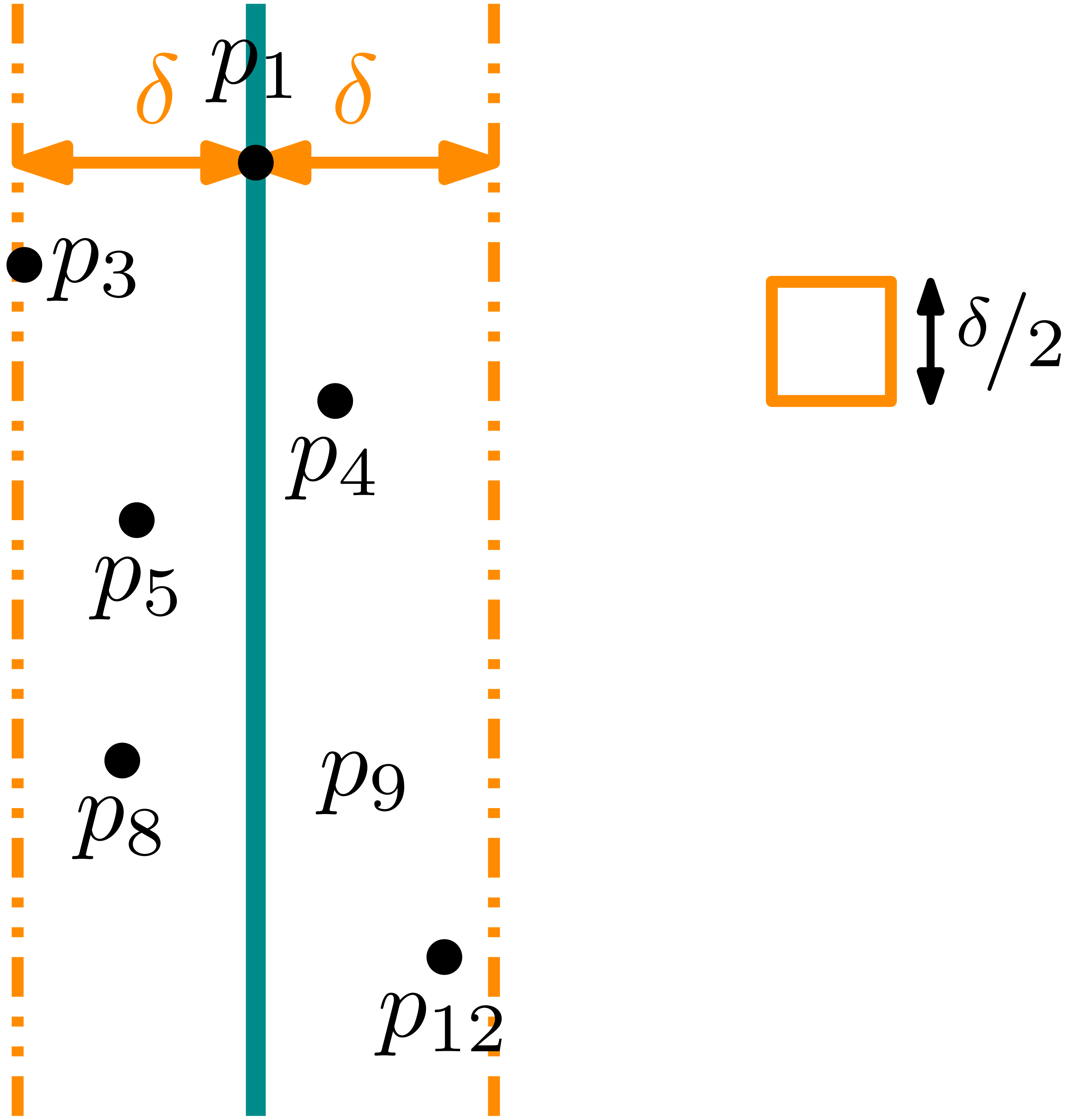
4. … merge by checking pairs in $A \times B$:

- Filter based on distance to median line, remove if farther than $\delta = \min(\delta_A, \delta_B)$.
- Linear scan along one side, y-maximal to y-minimal coordinates.
- Keep track of candidates on the other side of the median line.
- This idea generalizes to higher dimensions!



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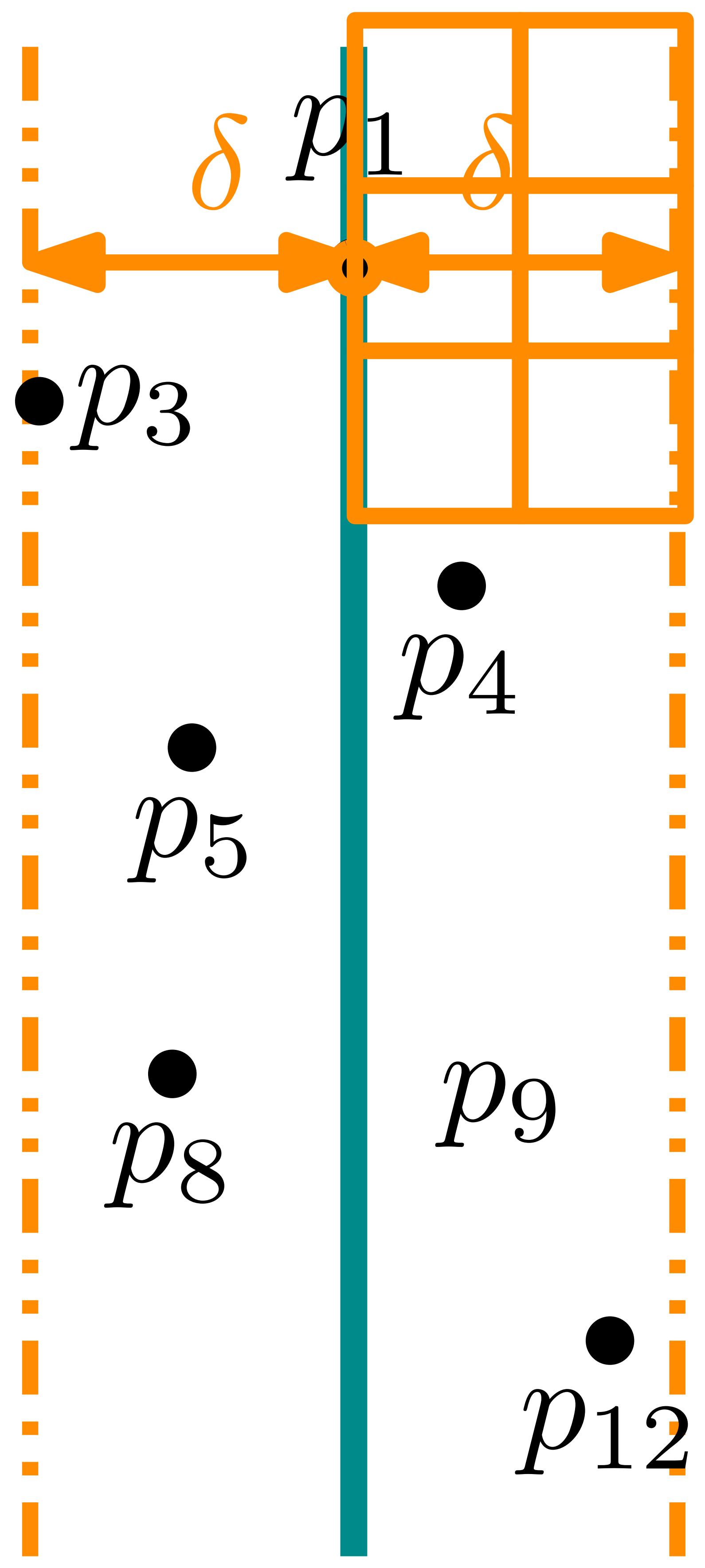

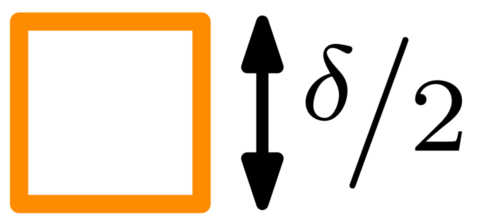

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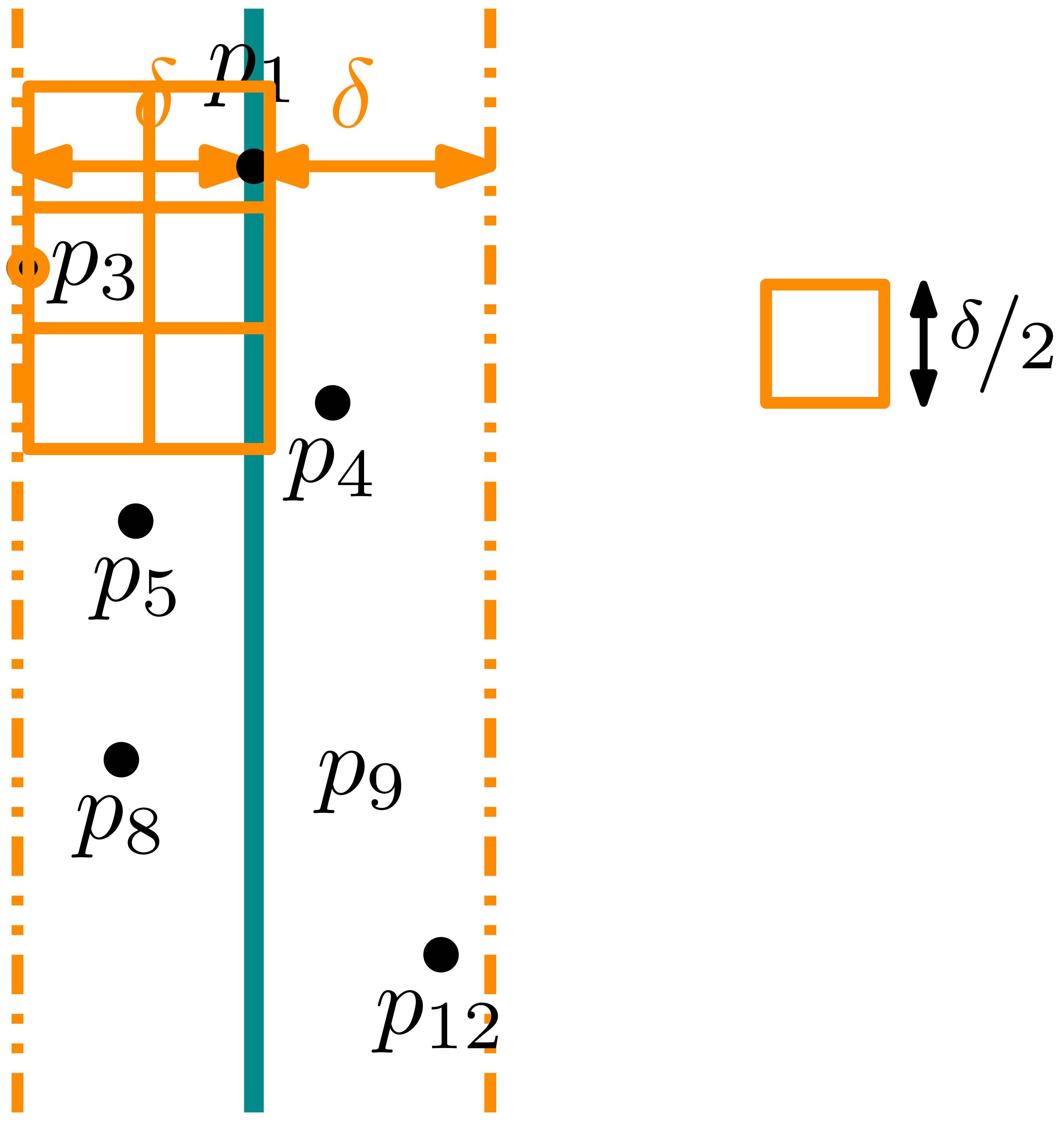

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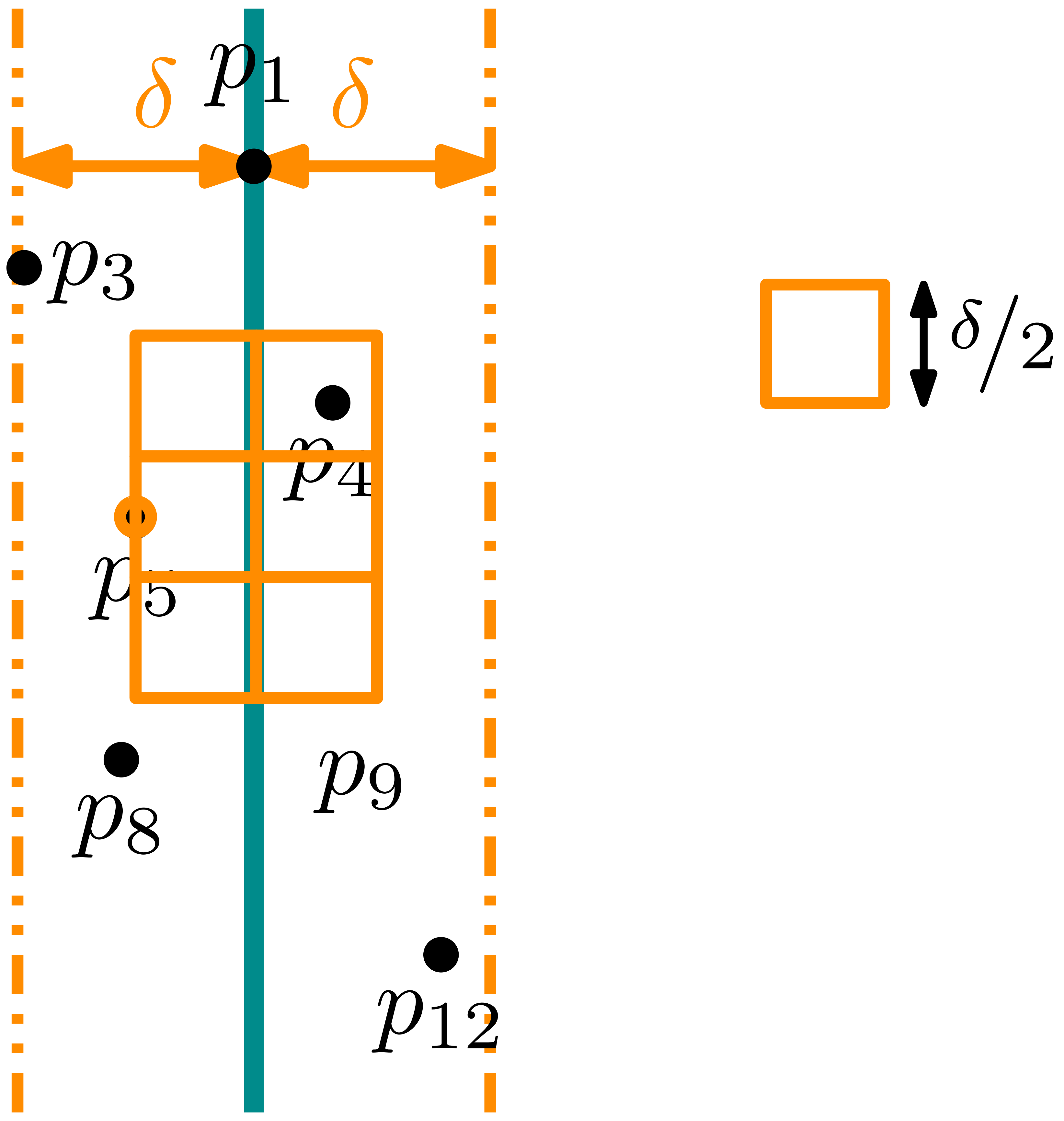

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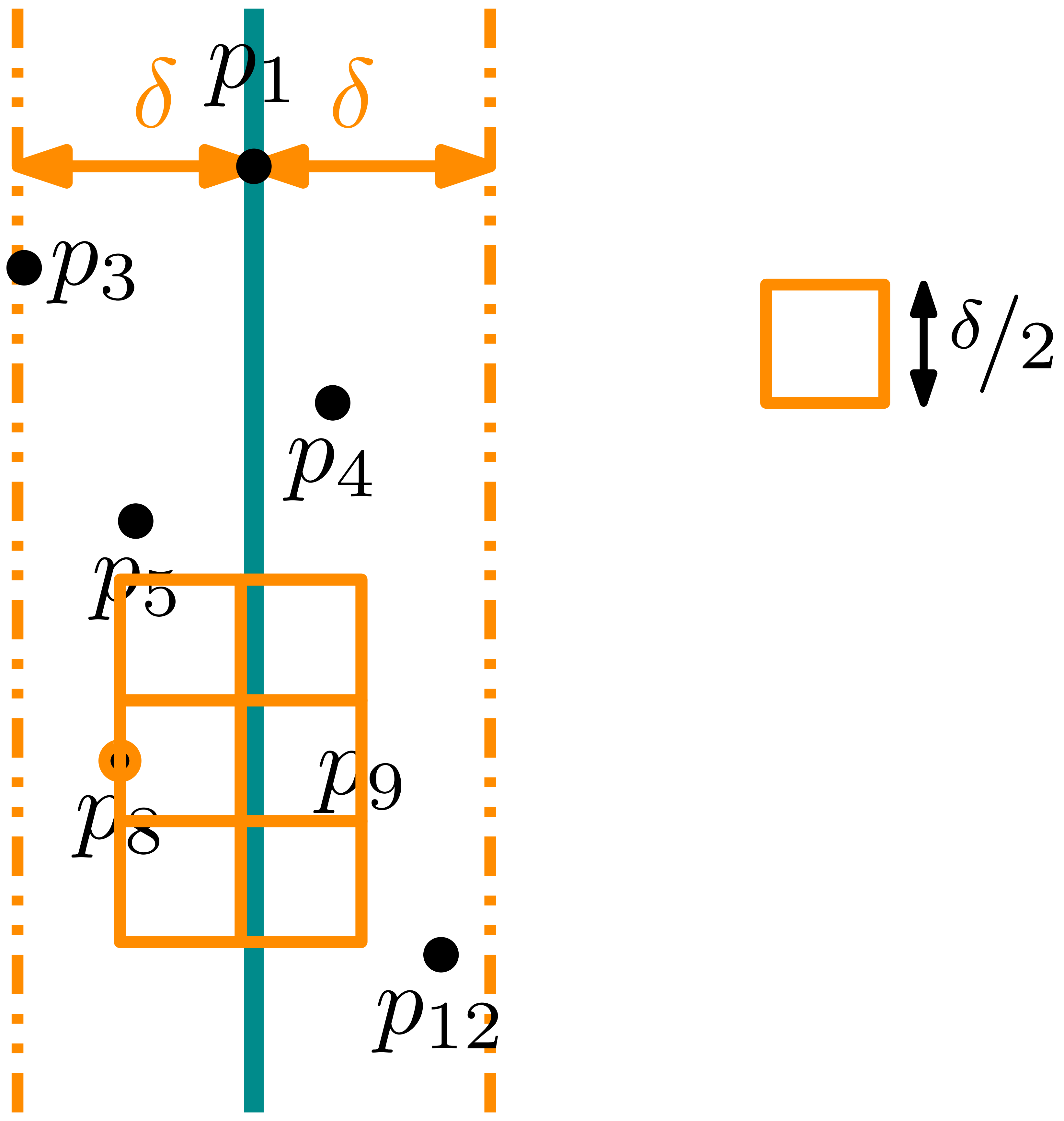

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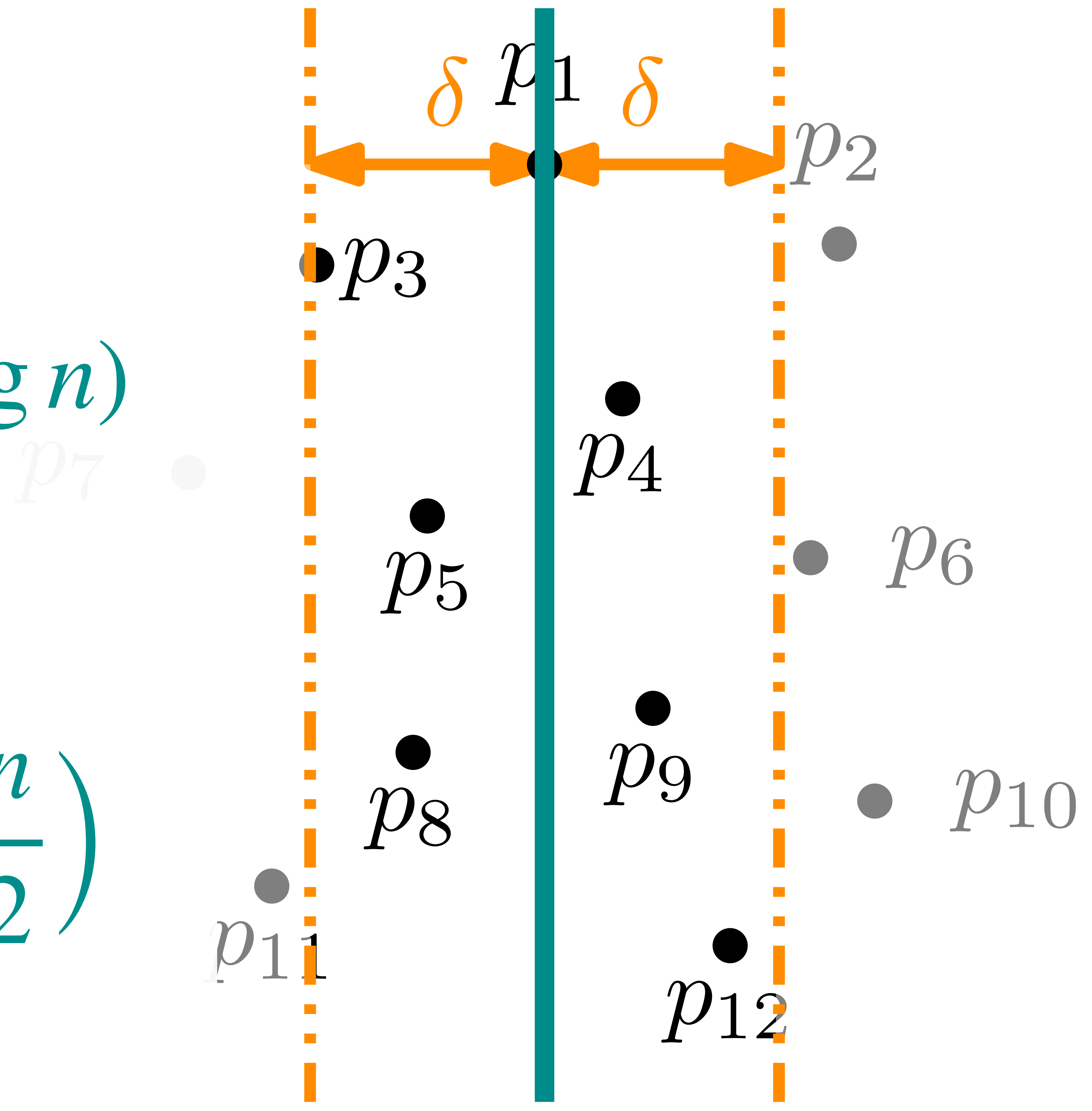

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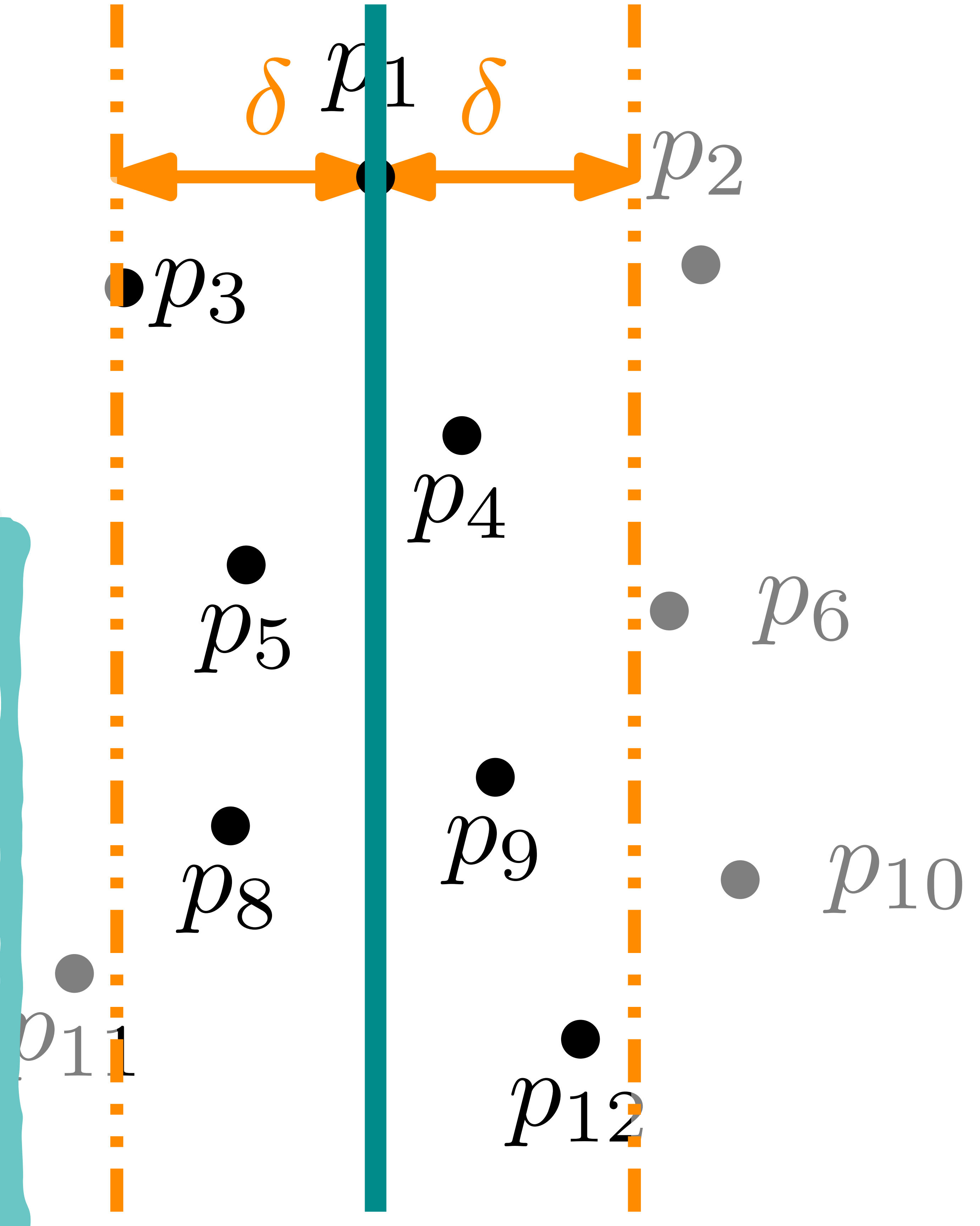

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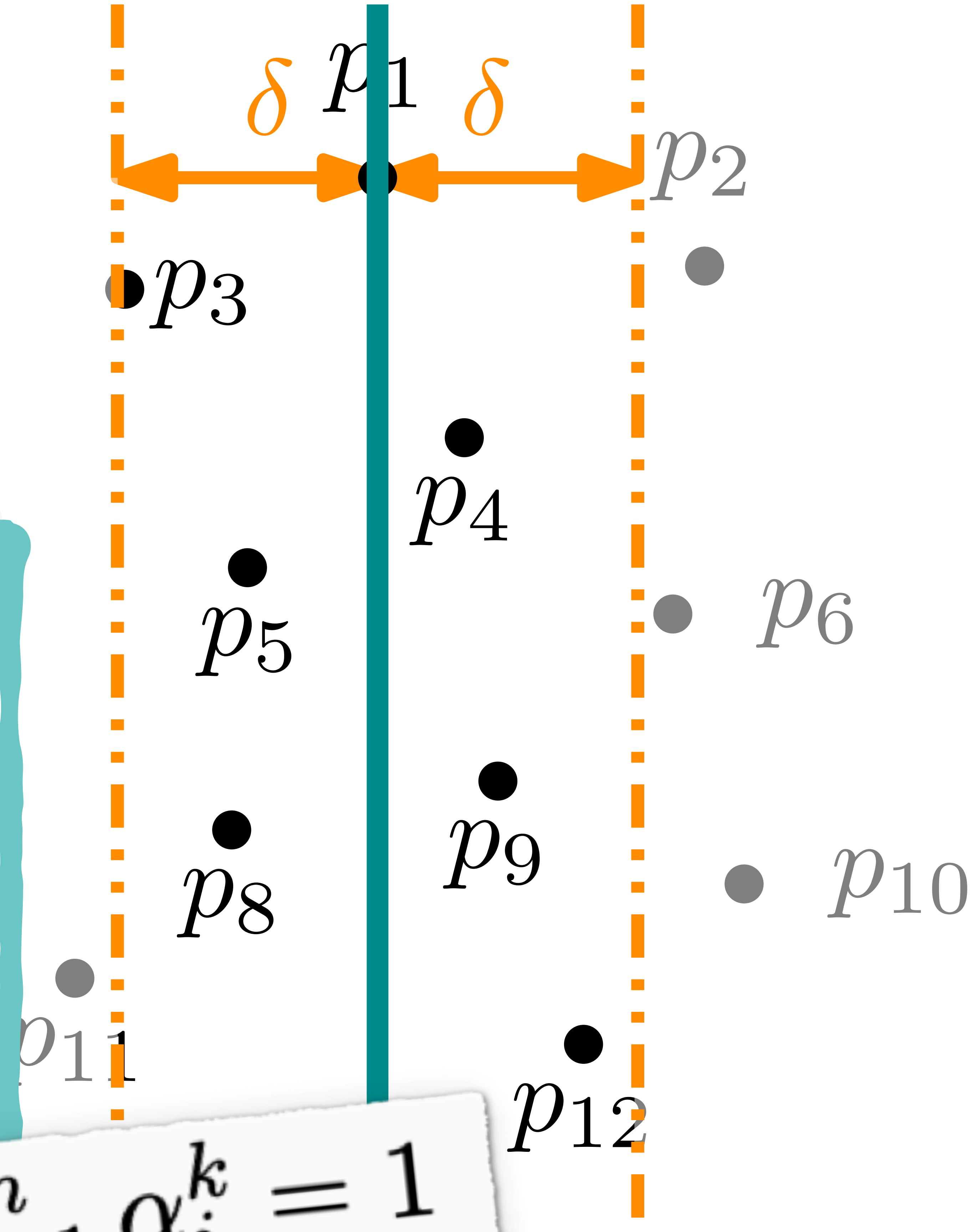
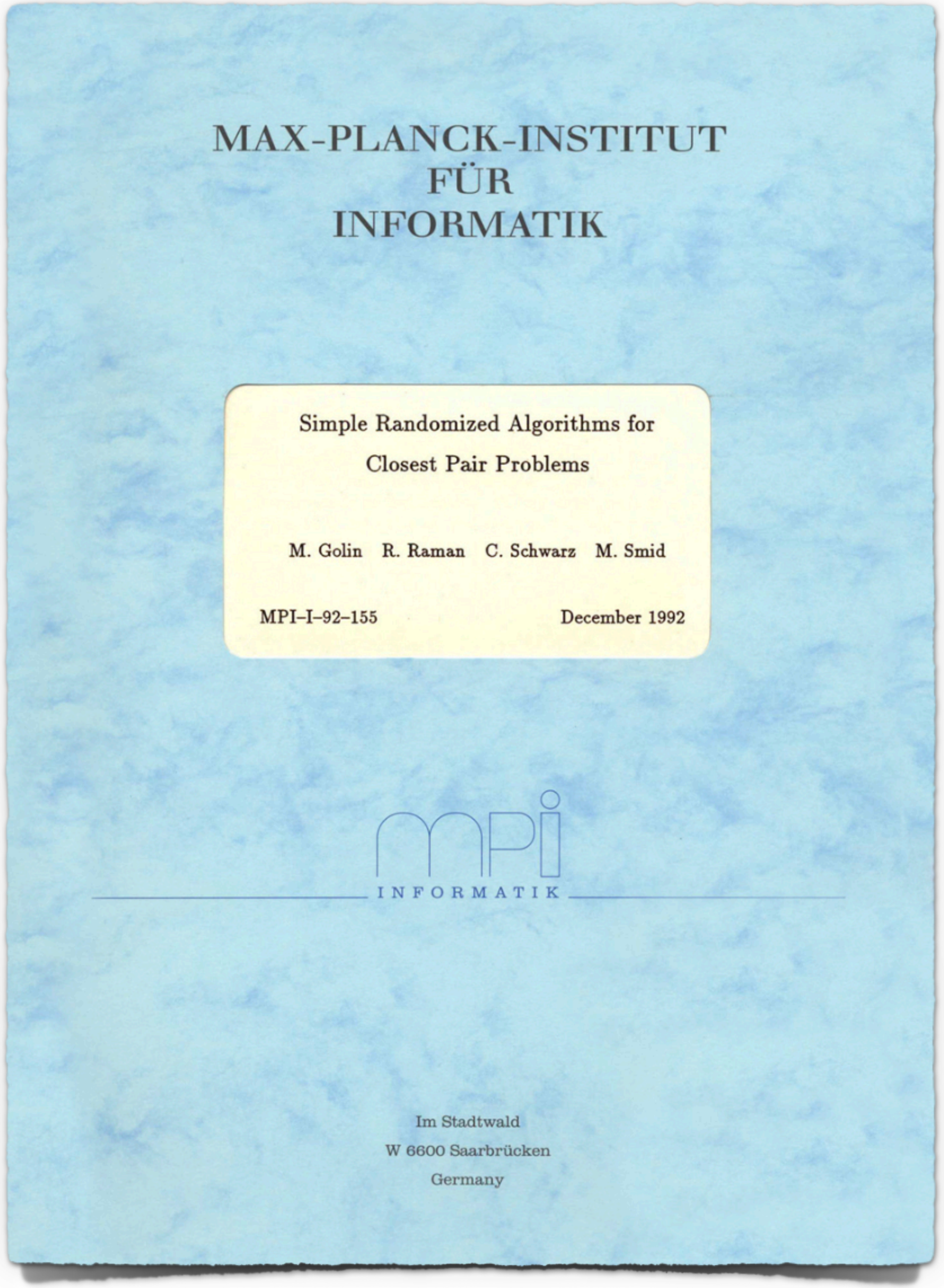
$$\Theta(n^k \log(n)) \quad \textit{for} \ \sum_{i=1}^{m} \alpha_i^k = 1$$



Closest Pair — Randomised Incremental Construction (RIC)

Peter Kramer | November 12th, 2024

Randomised Incremental Construction
Golin, Raman, Schwarz, Smid 1992/1995



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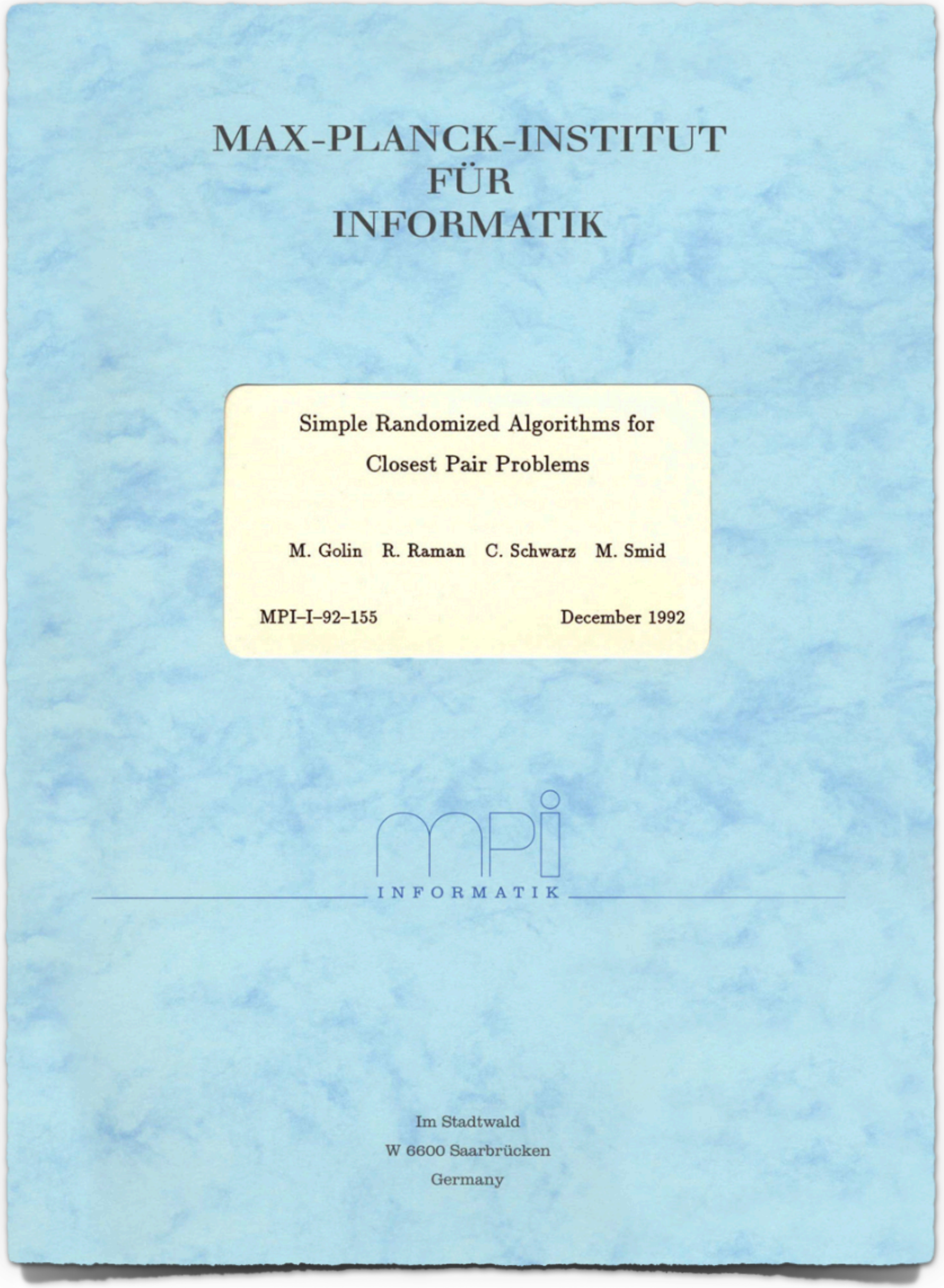
- **Idea:** Incrementally build a solution from a partial solution by refining it:



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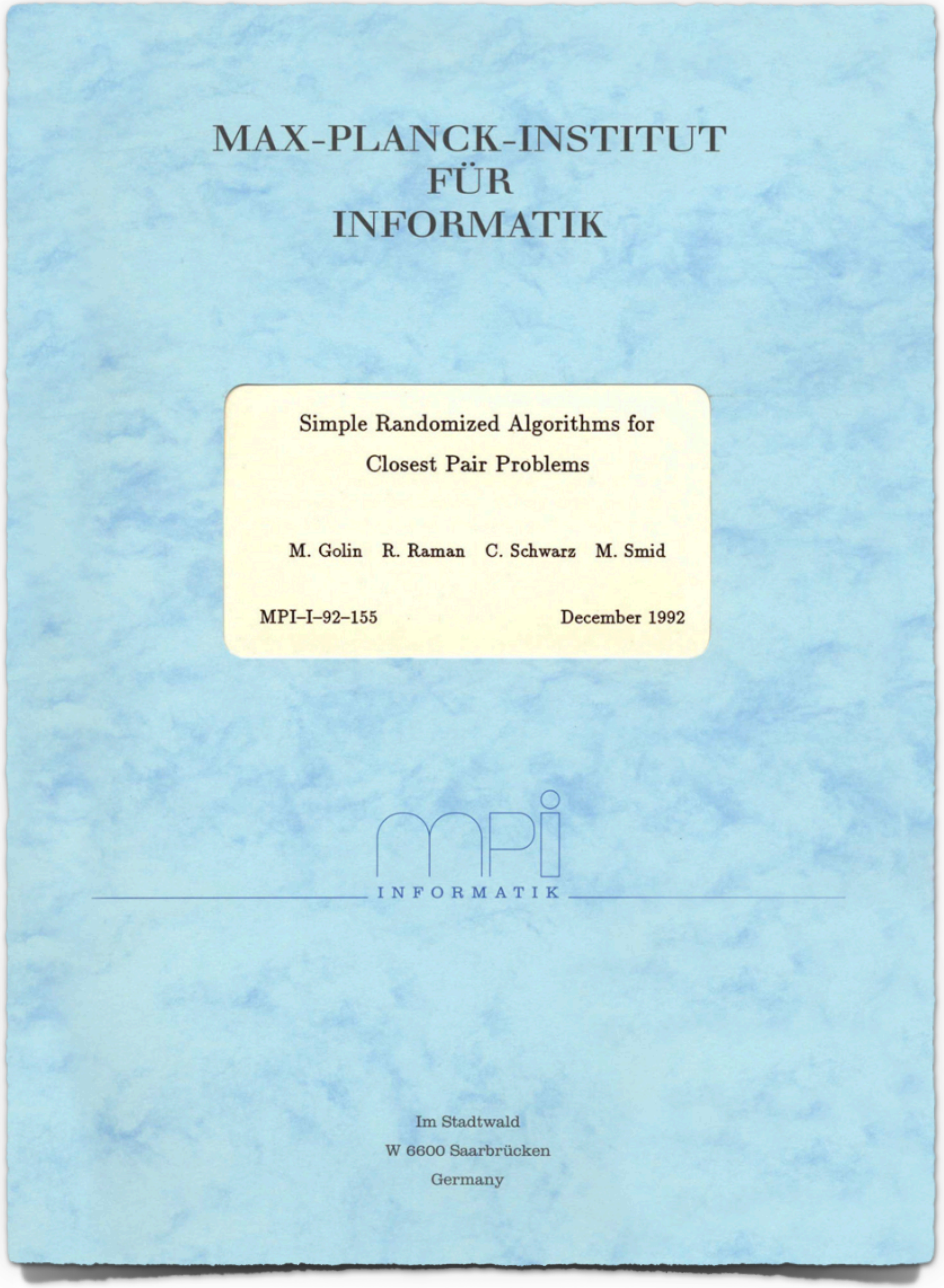

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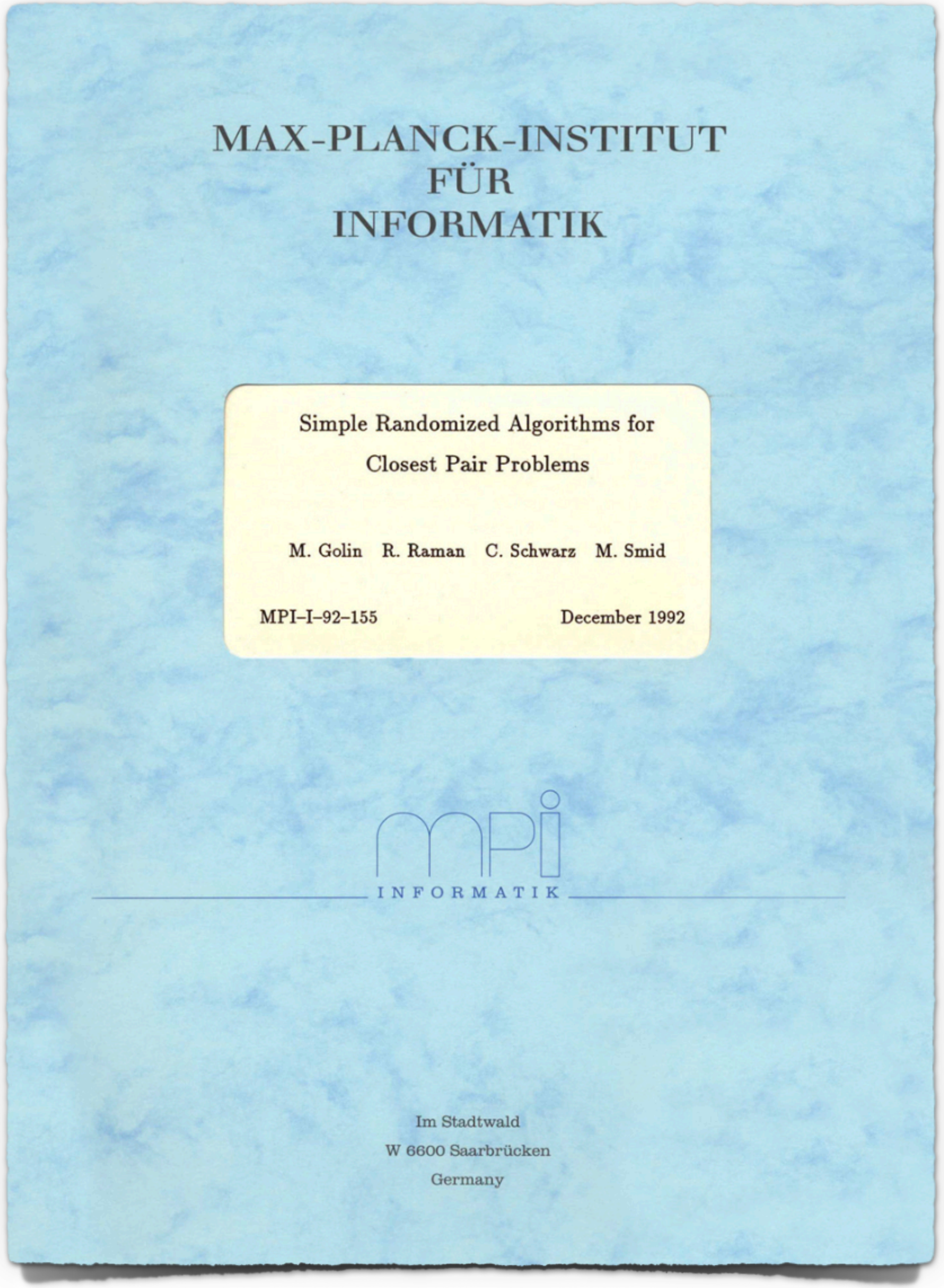

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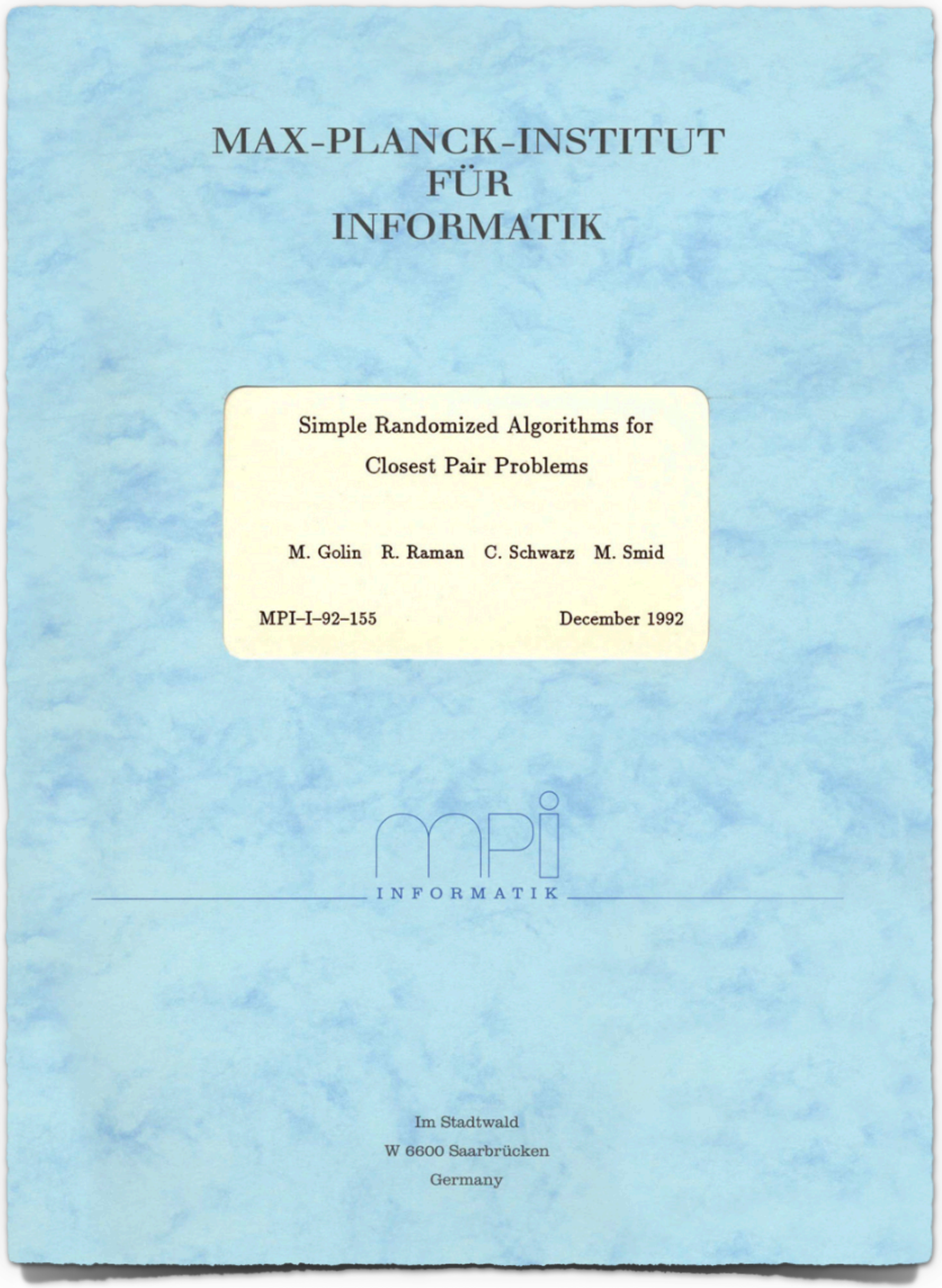
  - Crucial: Maintain sparsity, meaning just $\mathcal{O}(1)$ candidates to check!



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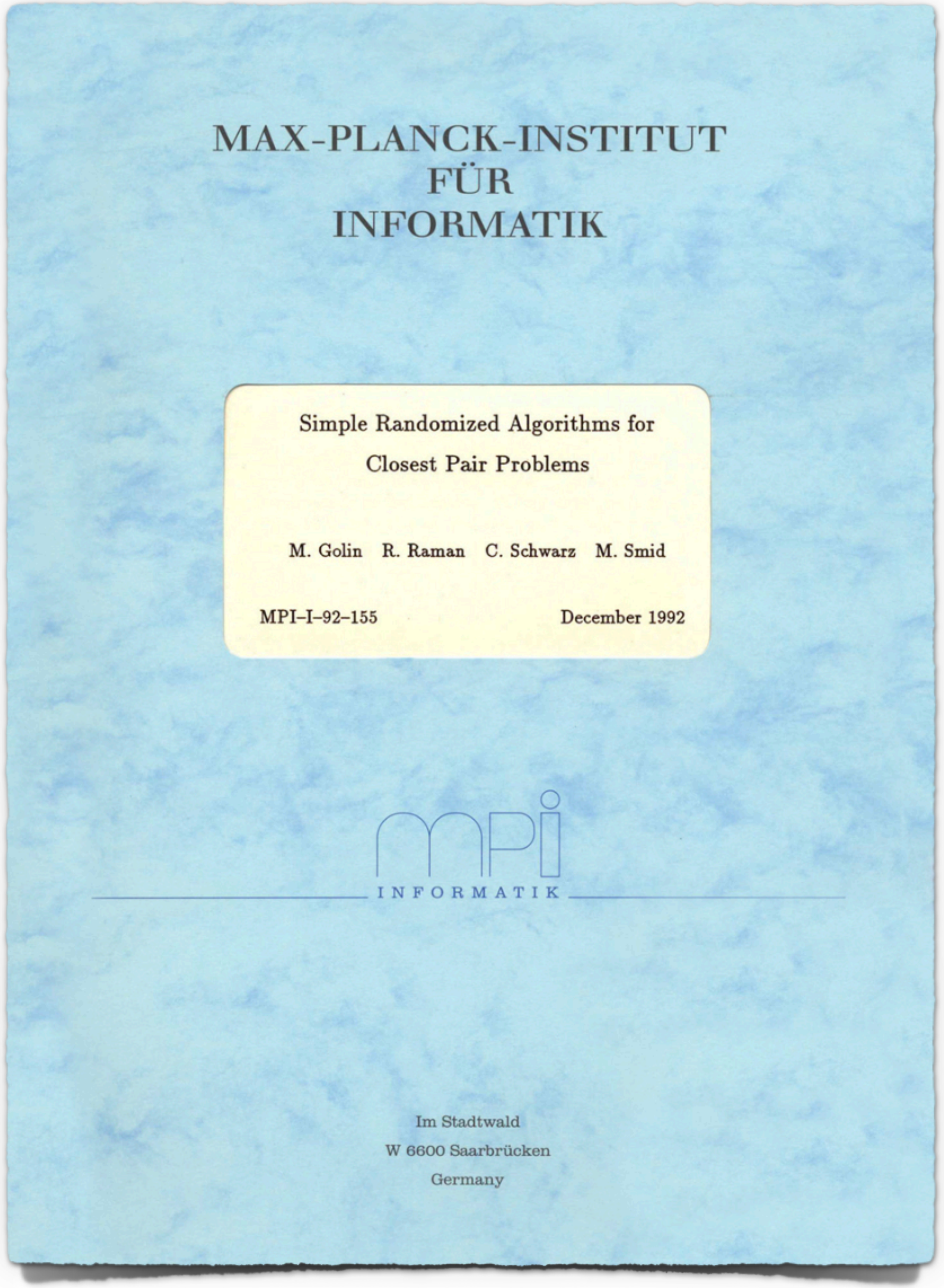
  - Crucial: Maintain sparsity, meaning just $\mathcal{O}(1)$ candidates to check!

- **Issue:** How do we identify candidates each step — efficiently!?



Randomised Incremental Construction
Golin, Raman, Schwarz, Smid 1992/1995

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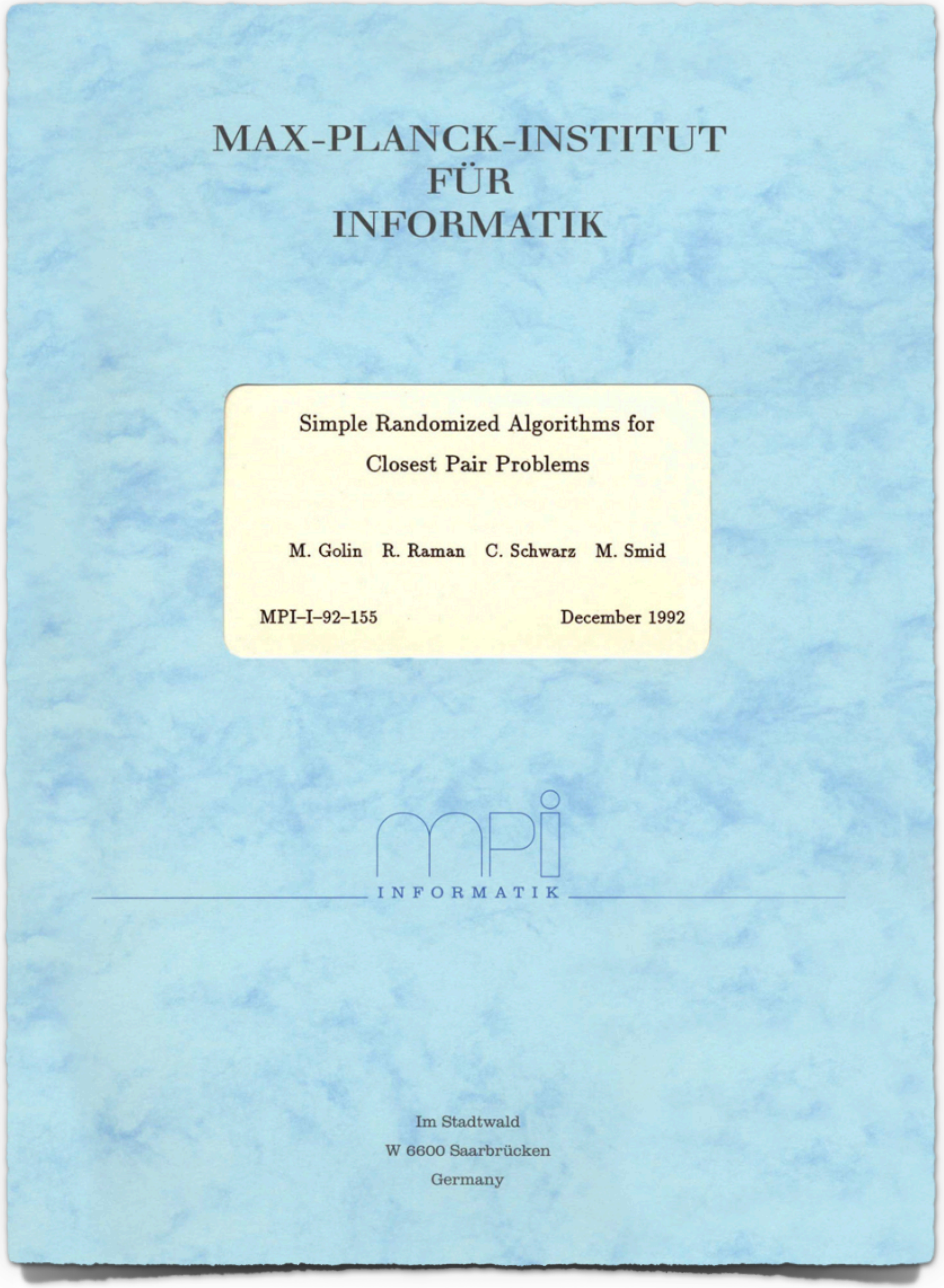
- Unsorted sequences $\Rightarrow$ No structure.



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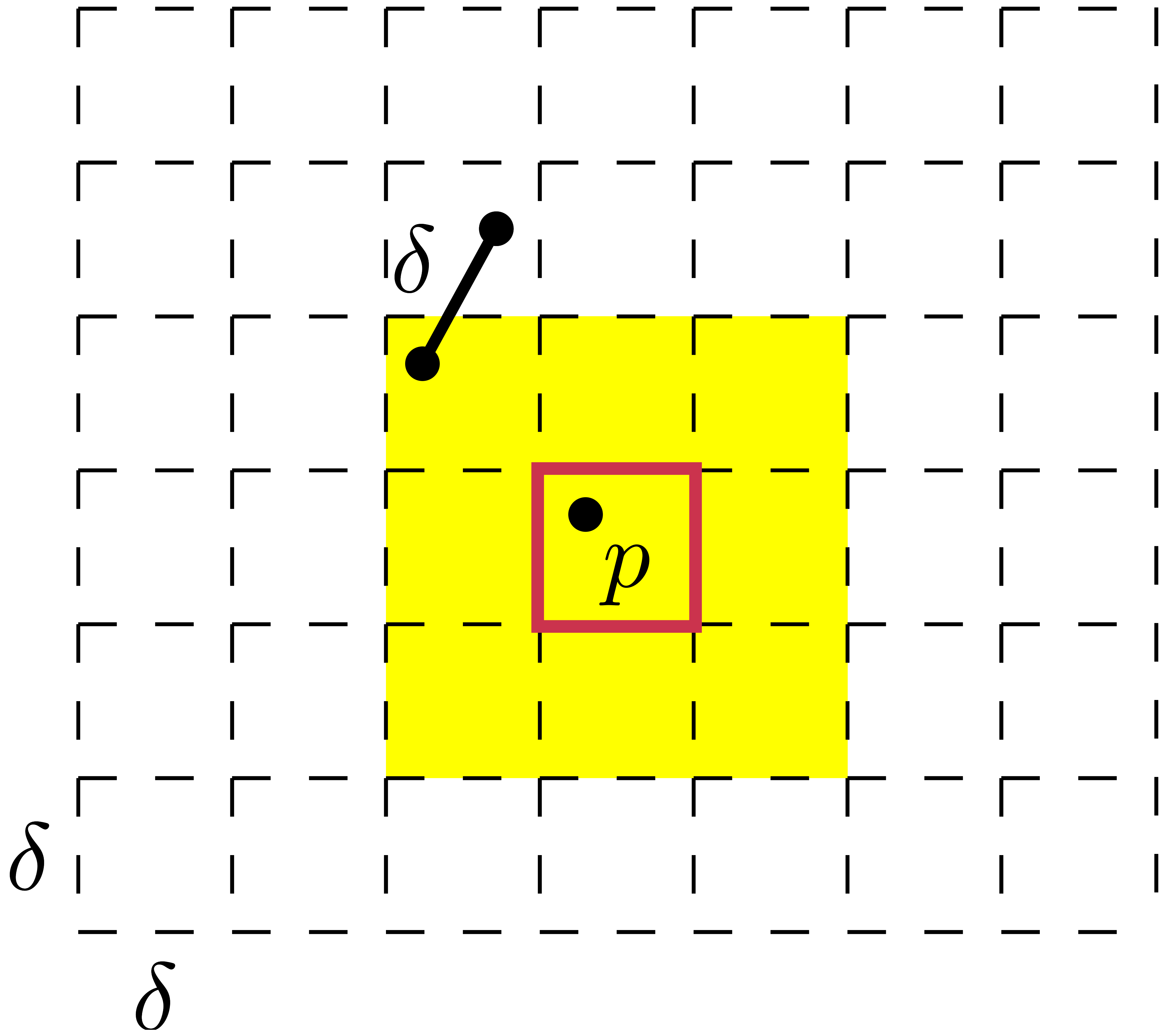

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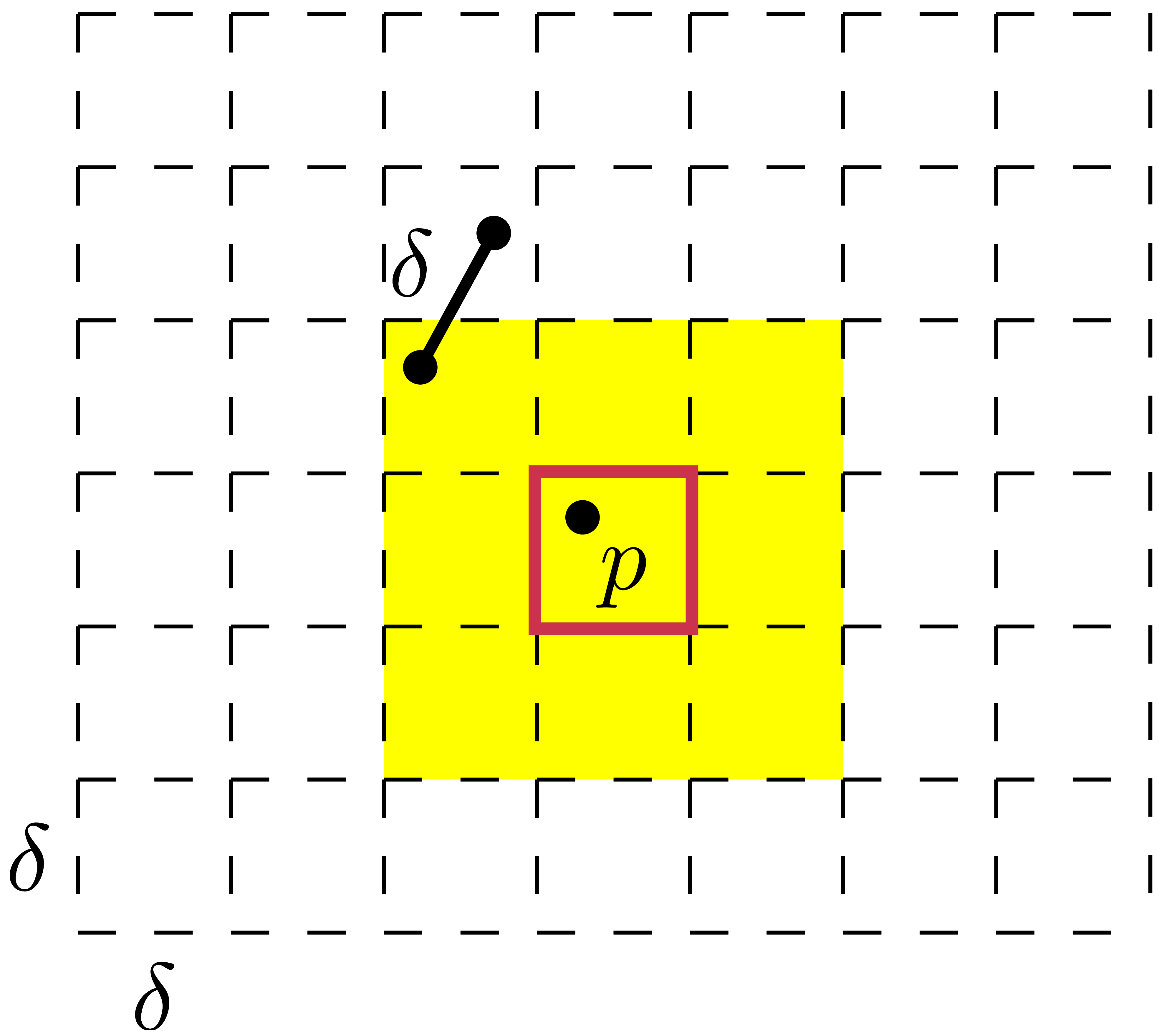

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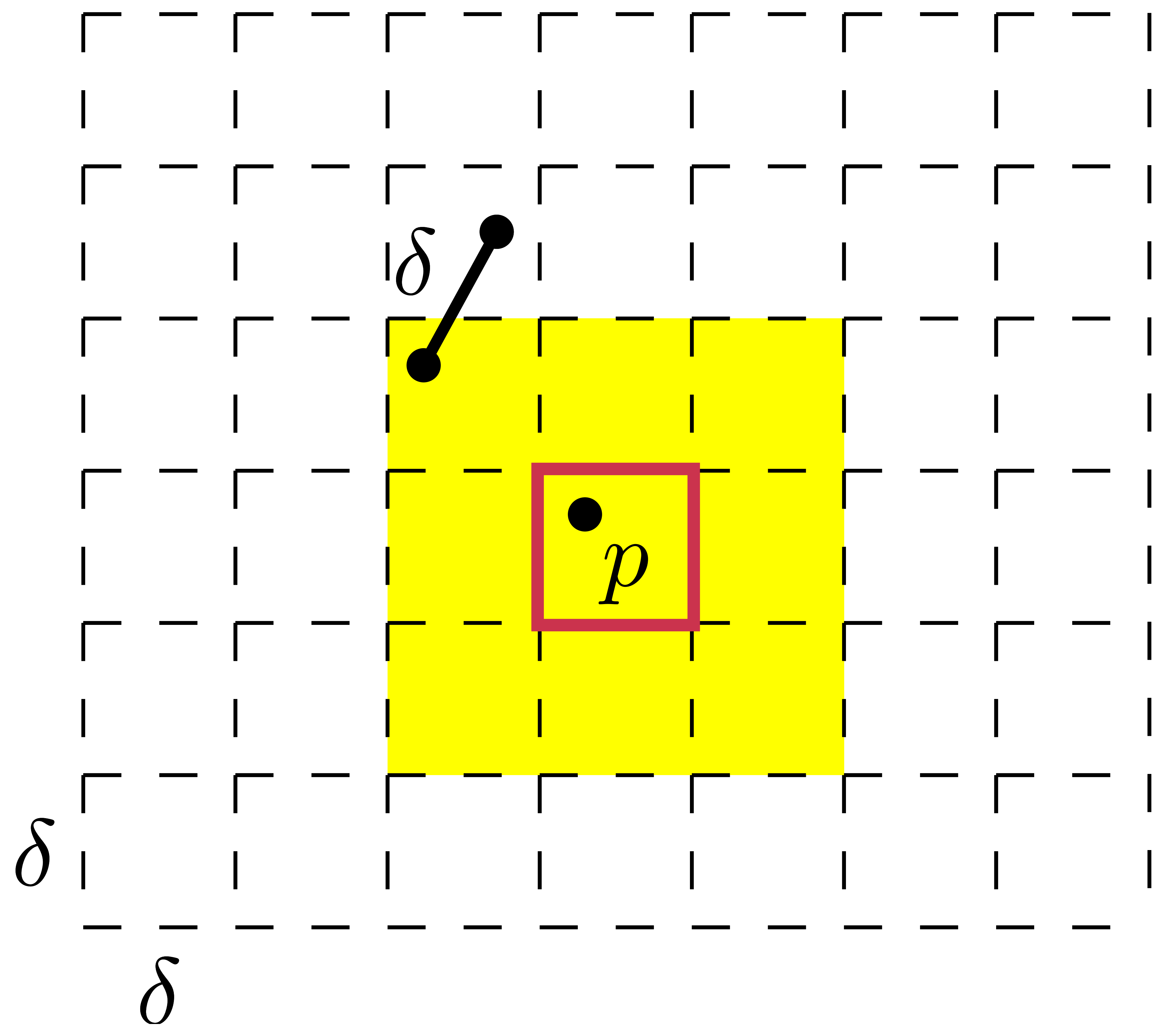

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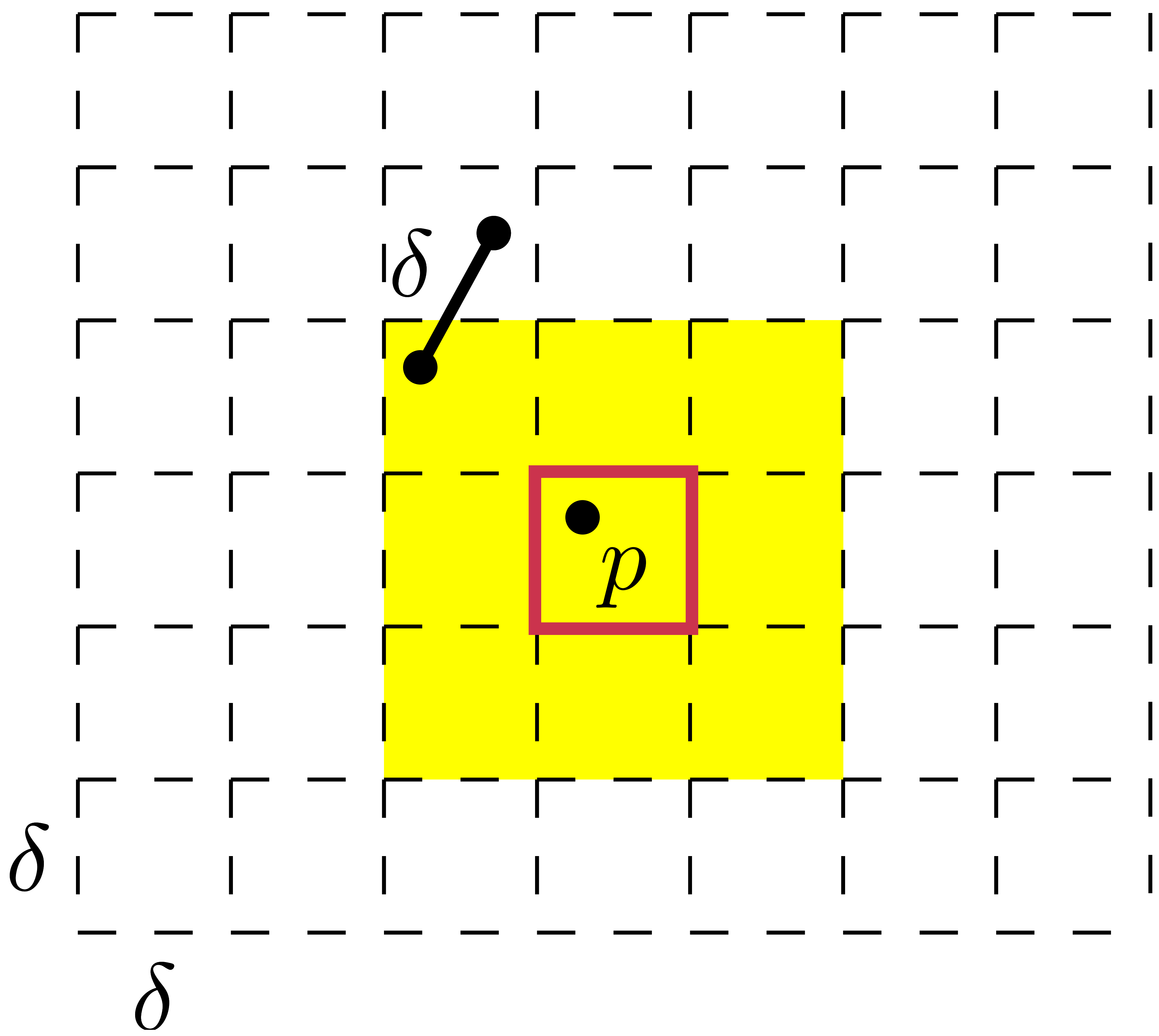
$$N_\delta(x, y) = \bigcup_{u,v \in [-1,1]} G_\delta[x - u, y - v].$$



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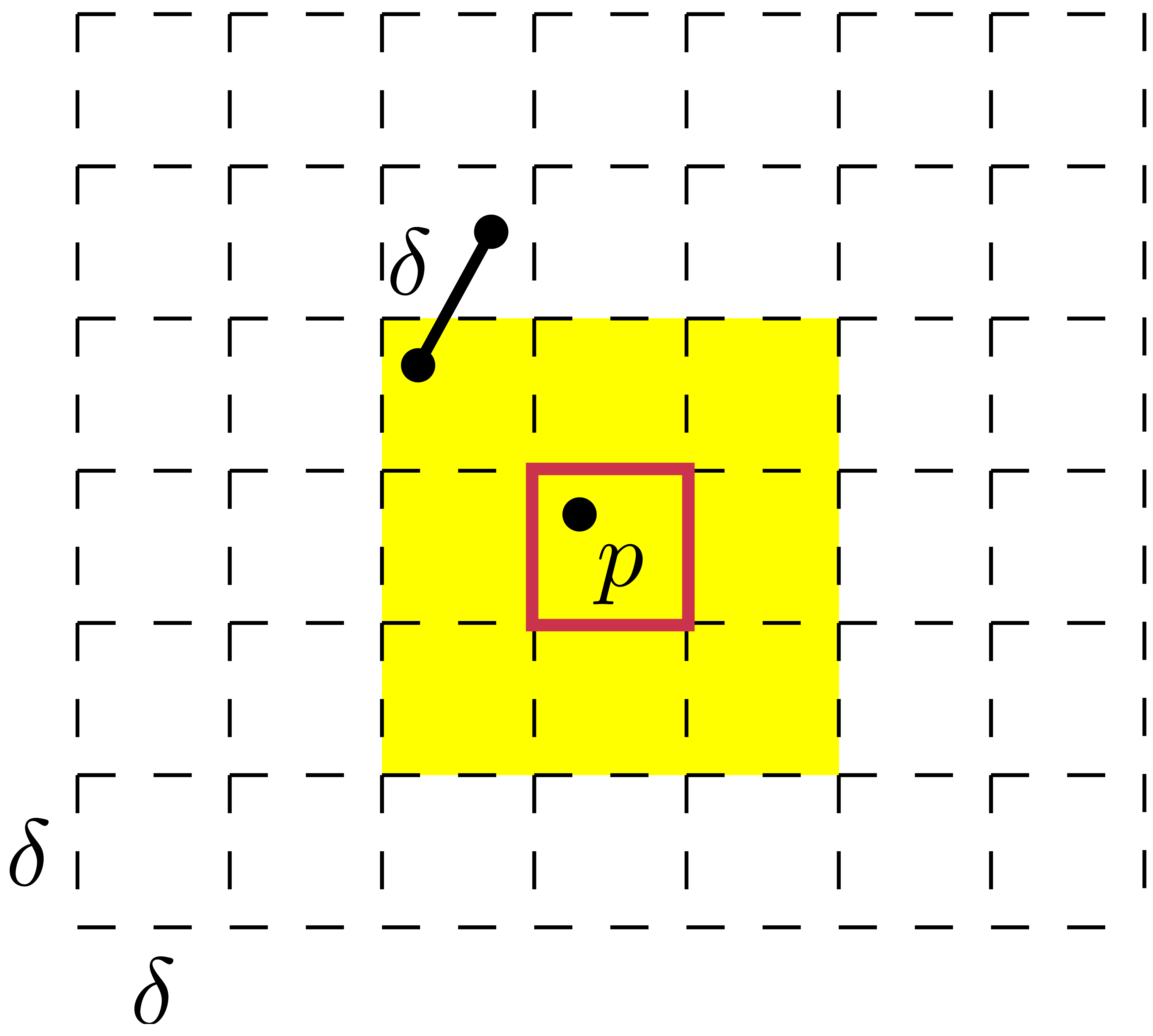
Lemma. For $p, q \in \mathscr{P}$ such that $p \in G_\delta[x, y]$:

$$d(p, q) \le \delta \implies q \in N_\delta(x, y)$$



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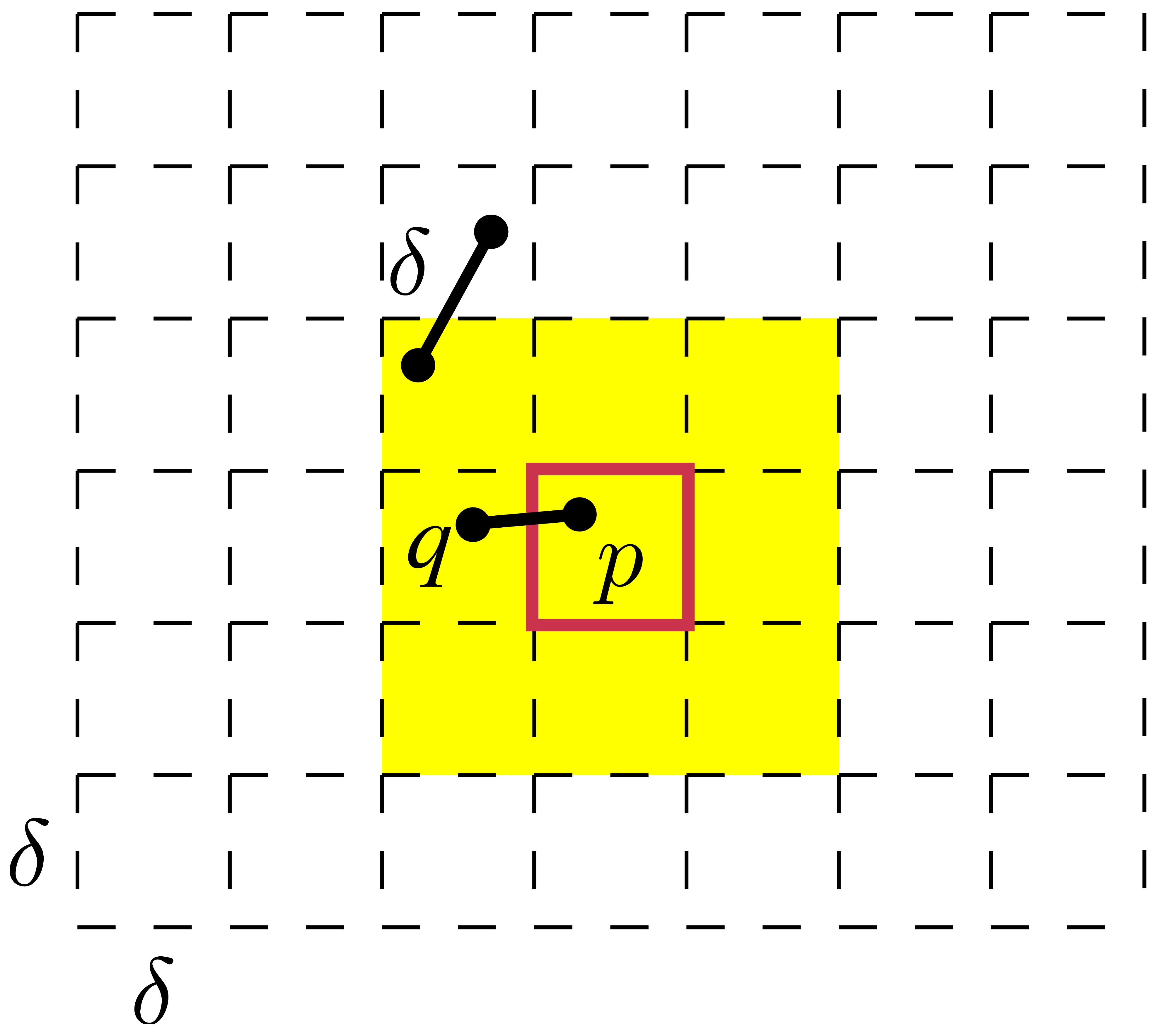

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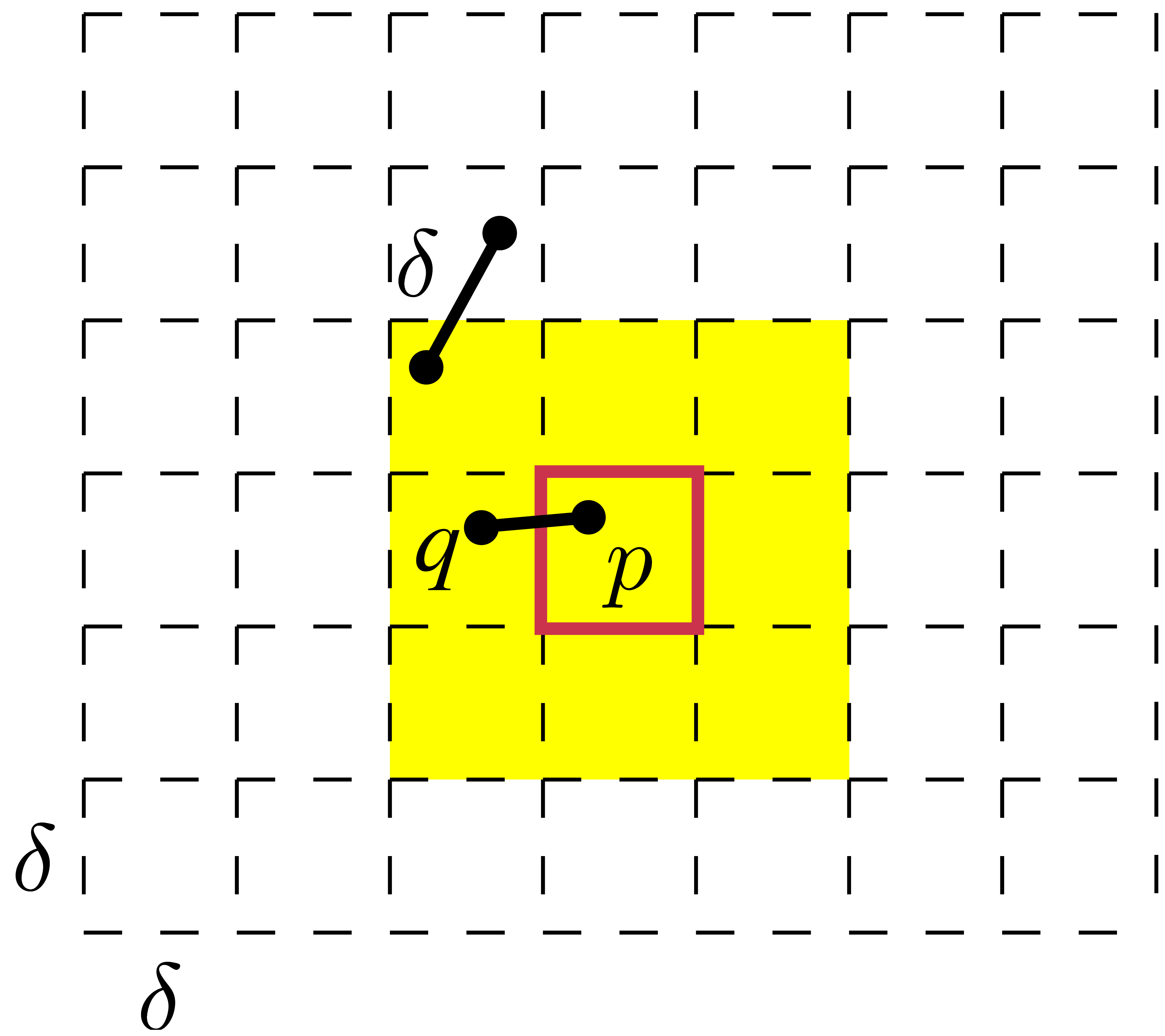

Randomised Incremental Construction
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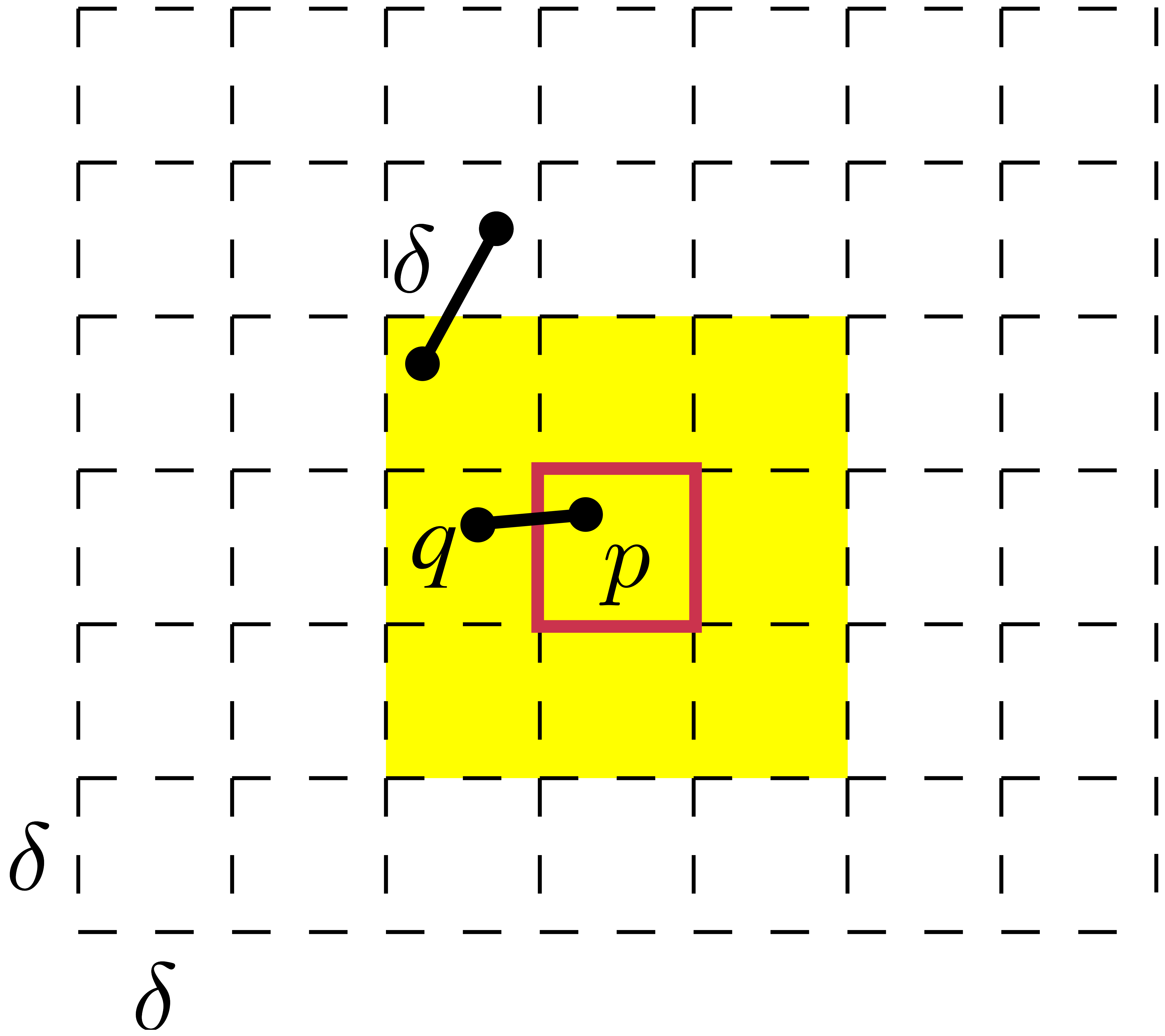
- **Idea:** Sparsity! Keep $|N_\delta(x,y)| \in \mathcal{O}(1)$.
- If each $N_\delta(x,y)$ contains constantly many points of pairwise distance at least δ, …
- … then $\mathcal{O}(1)$ comparisons for next $p \in \mathcal{P}$.



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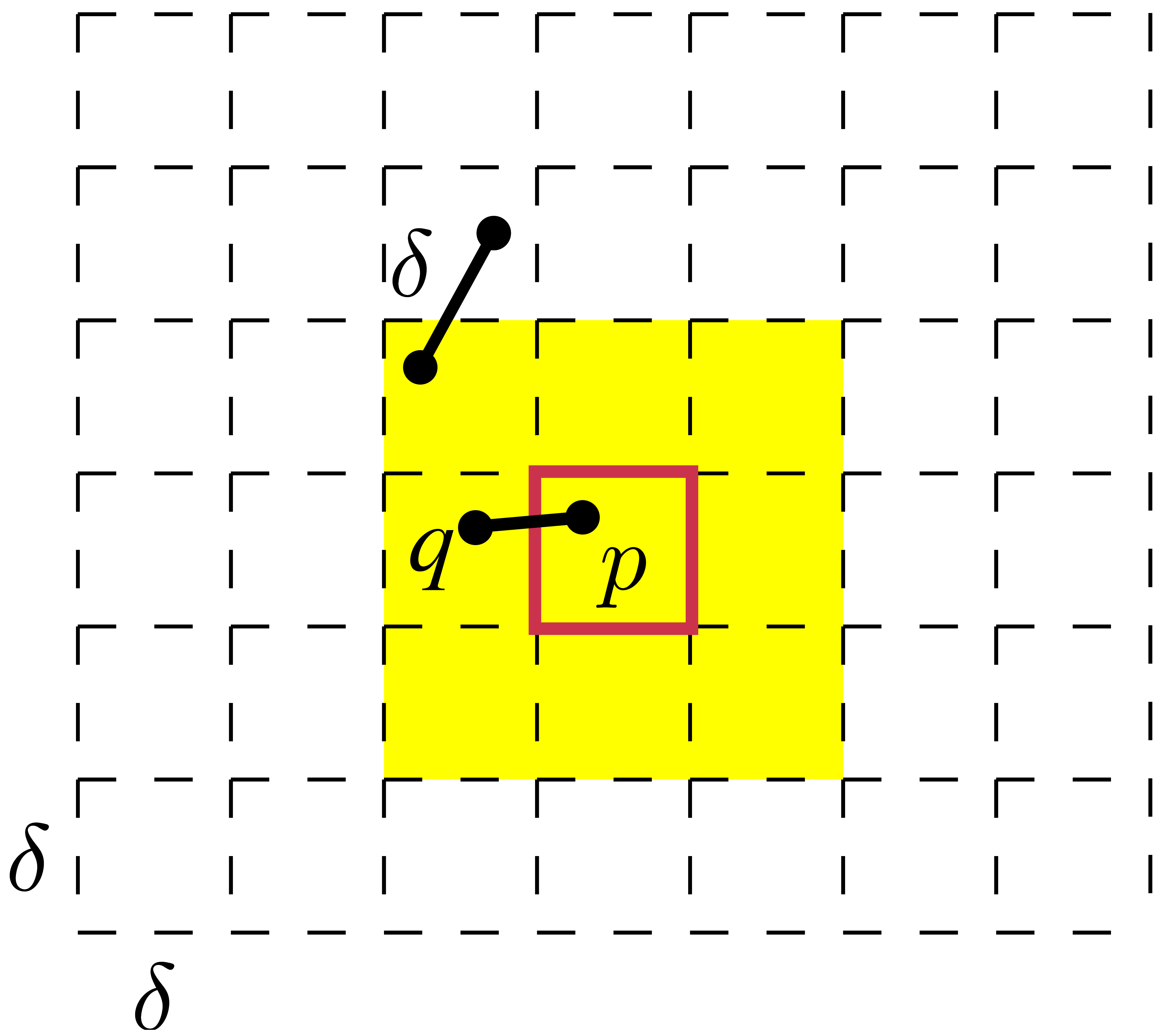
- … then $\mathcal{O}(1)$ comparisons for next $p \in \mathscr{P}$.

```
For each p ∈ 𝒫:
    Find x, y such that G_δ[x, y] contains p.
    Insert p to G_δ[x, y].
    If p is part of new closest pair {p, q}:
        Update δ = d(p, q).
        Rebuild sparse grid, G_δ'[x, y].
```



Randomised Incremental Construction
Golin, Raman, Schwarz, Smid 1992/1995

```
Algorithm   CP(p_1, p_2, ..., p_n)

δ := d(p_1, p_2); G := Build(S_2, δ);
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```
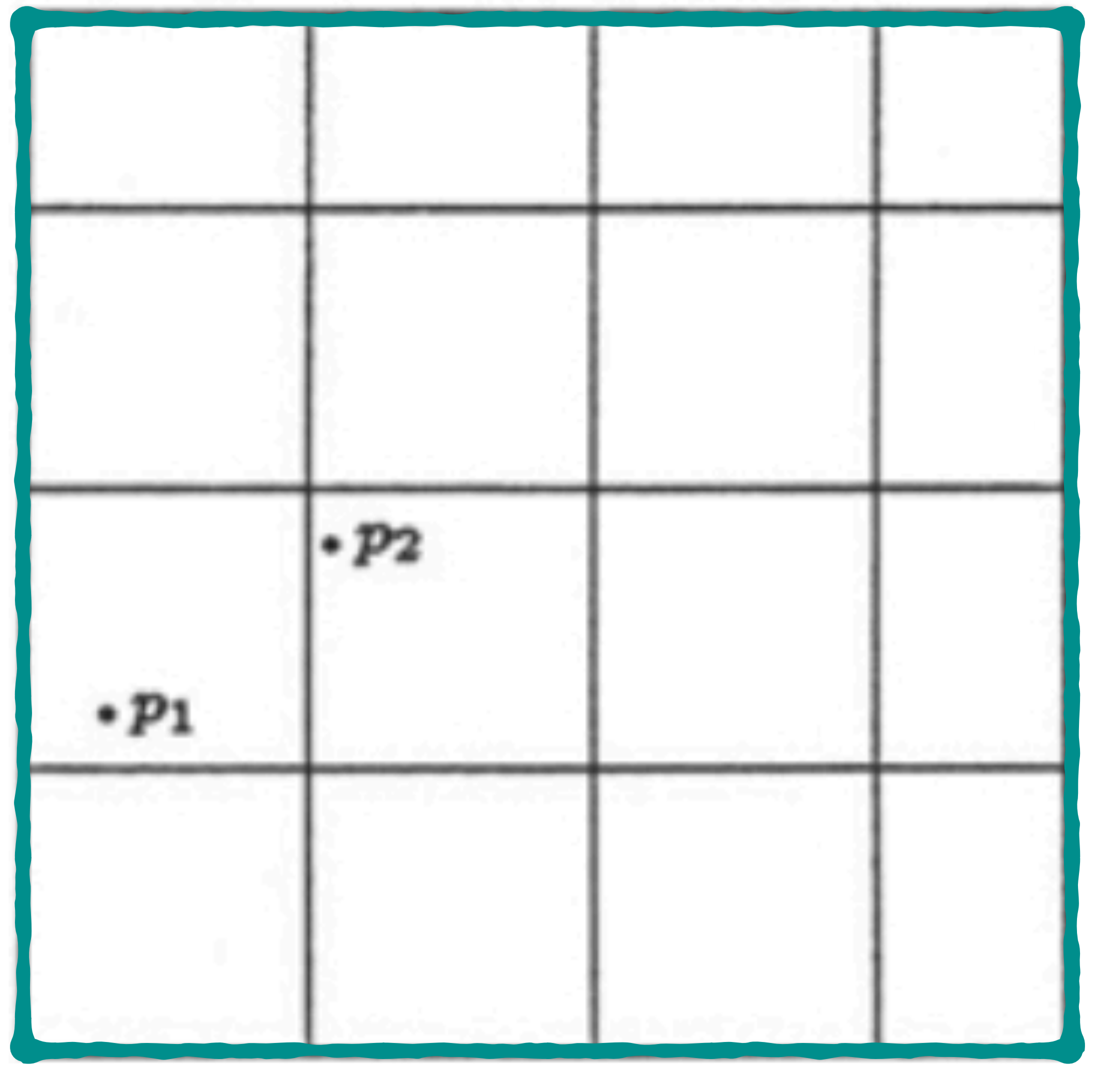

$\delta(S_2) = d(p_1, p_2)$



Randomised Incremental Construction
Golin, Raman, Schwarz, Smid 1992/1995

Algorithm $CP(p_1, p_2, \ldots, p_n)$

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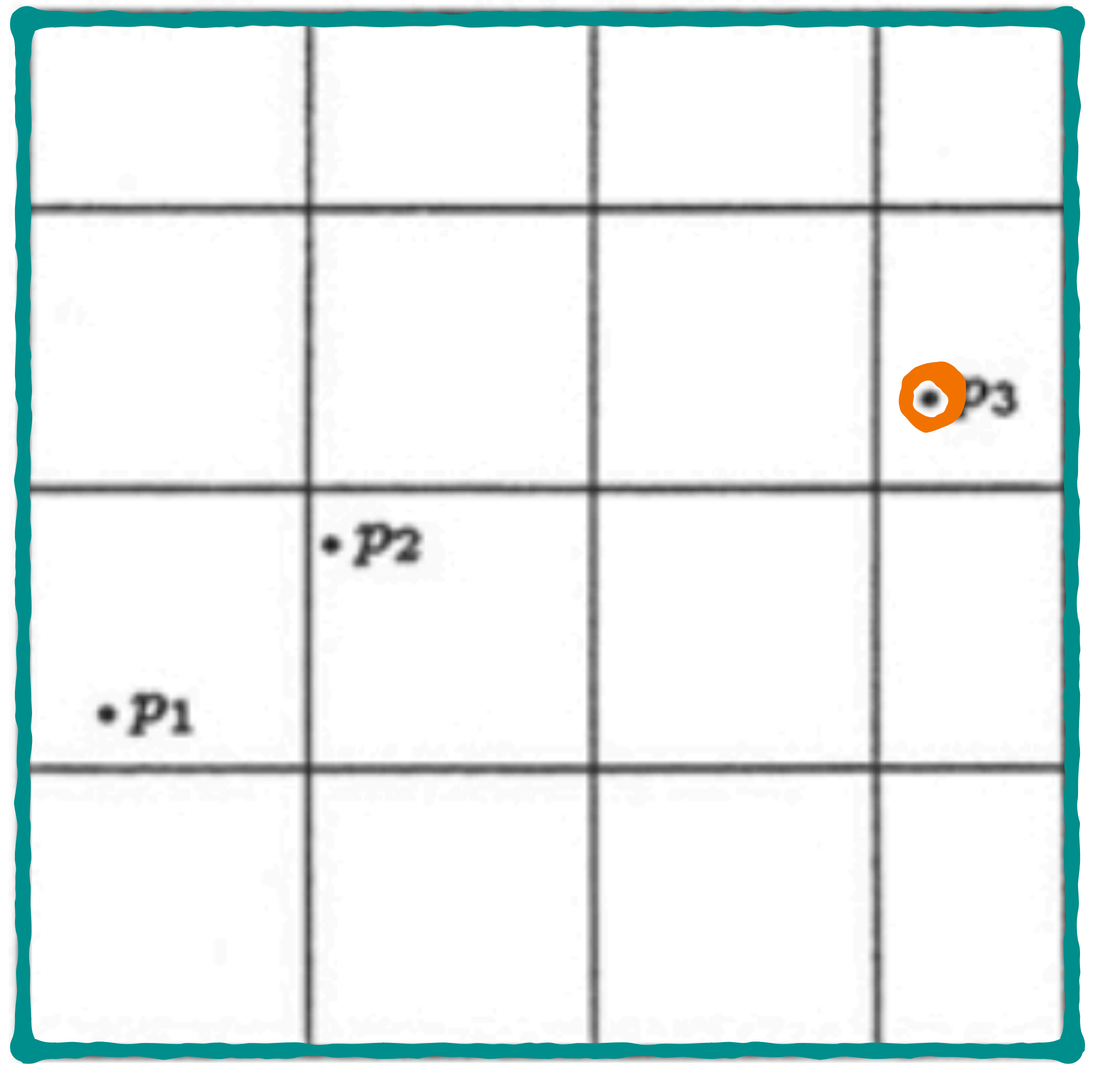

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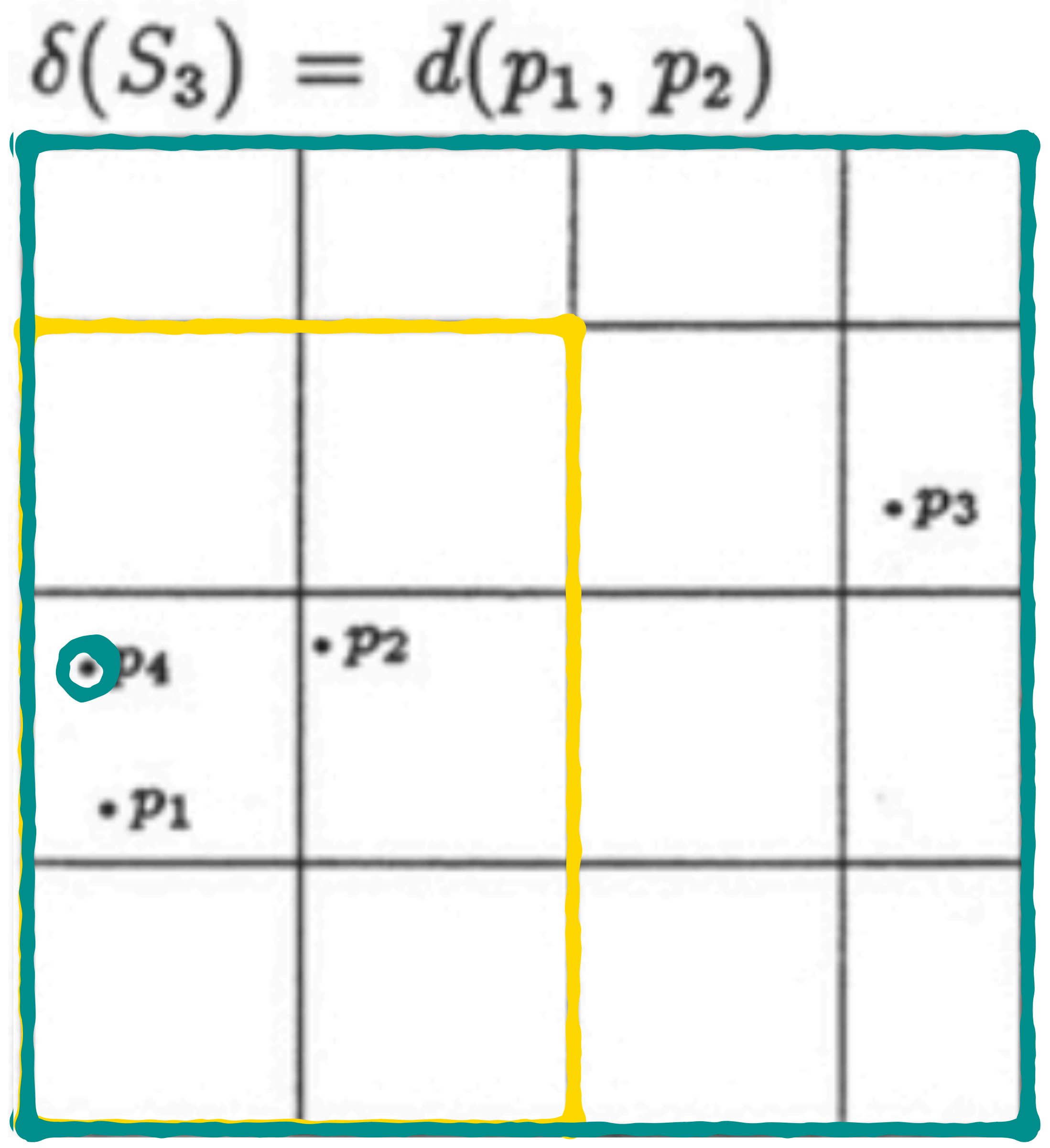

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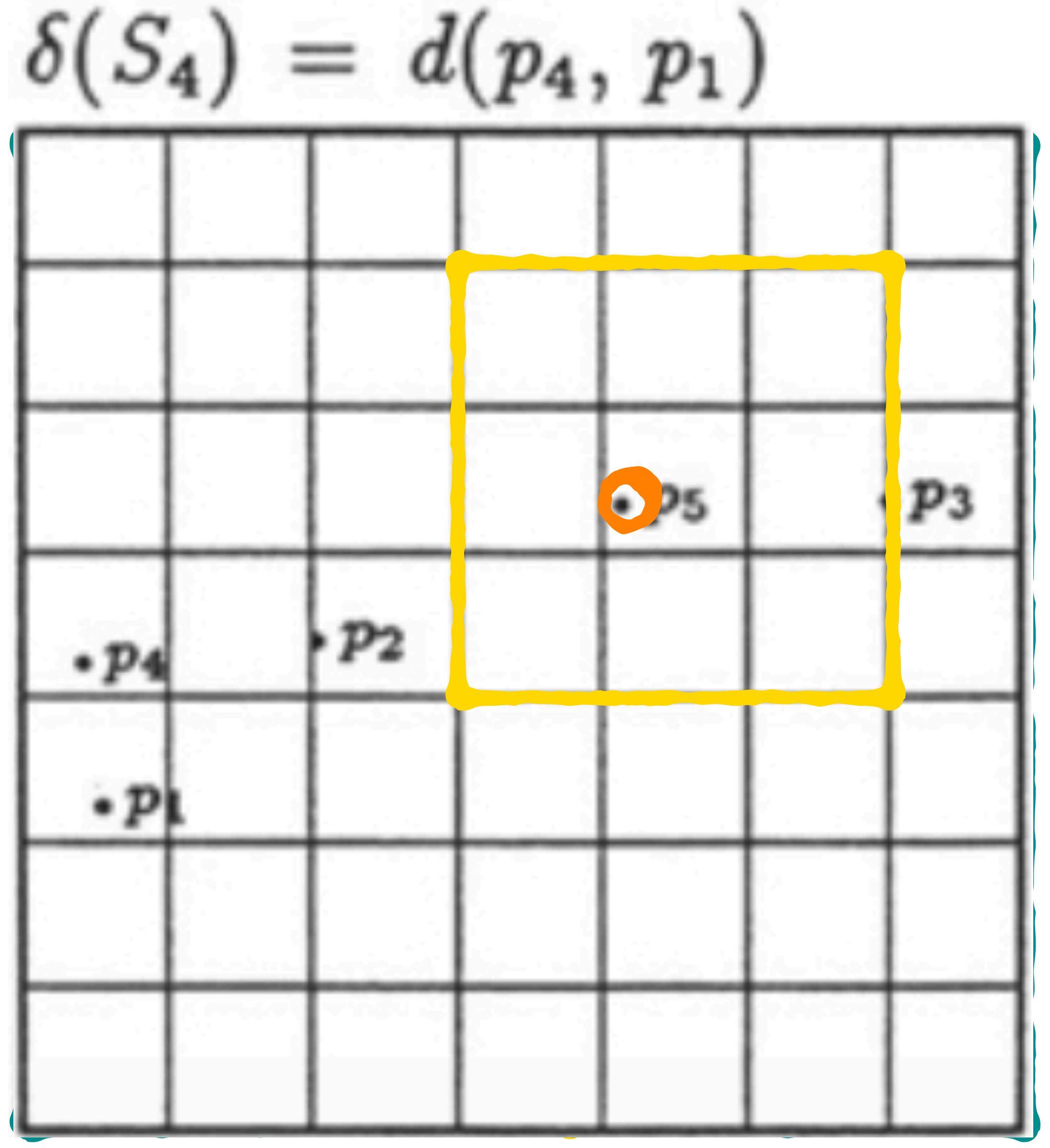

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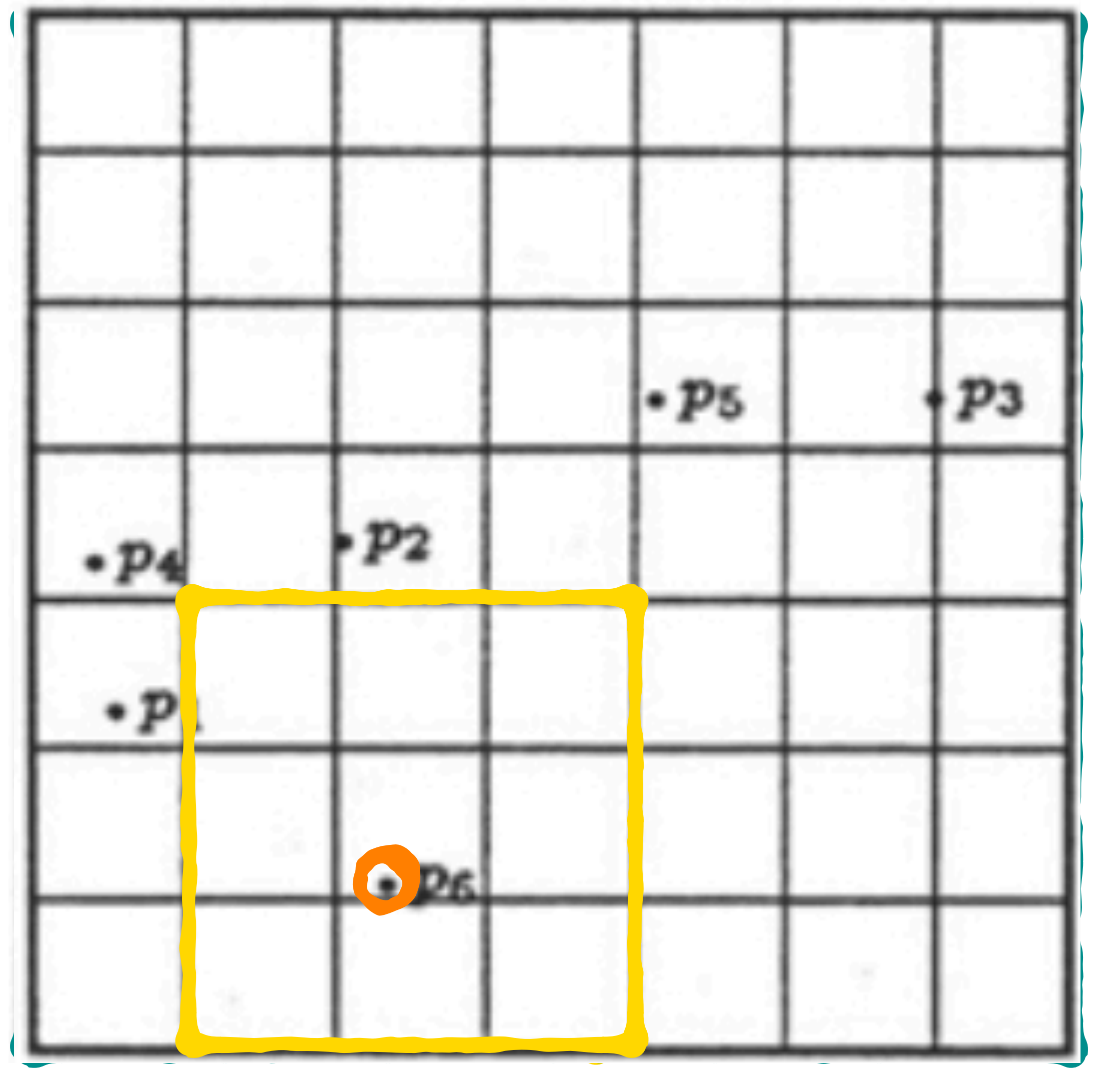
return(δ).

$\delta(S_5) = d(p_4, p_1)$



Randomised Incremental Construction
Golin, Raman, Schwarz, Smid 1992/1995

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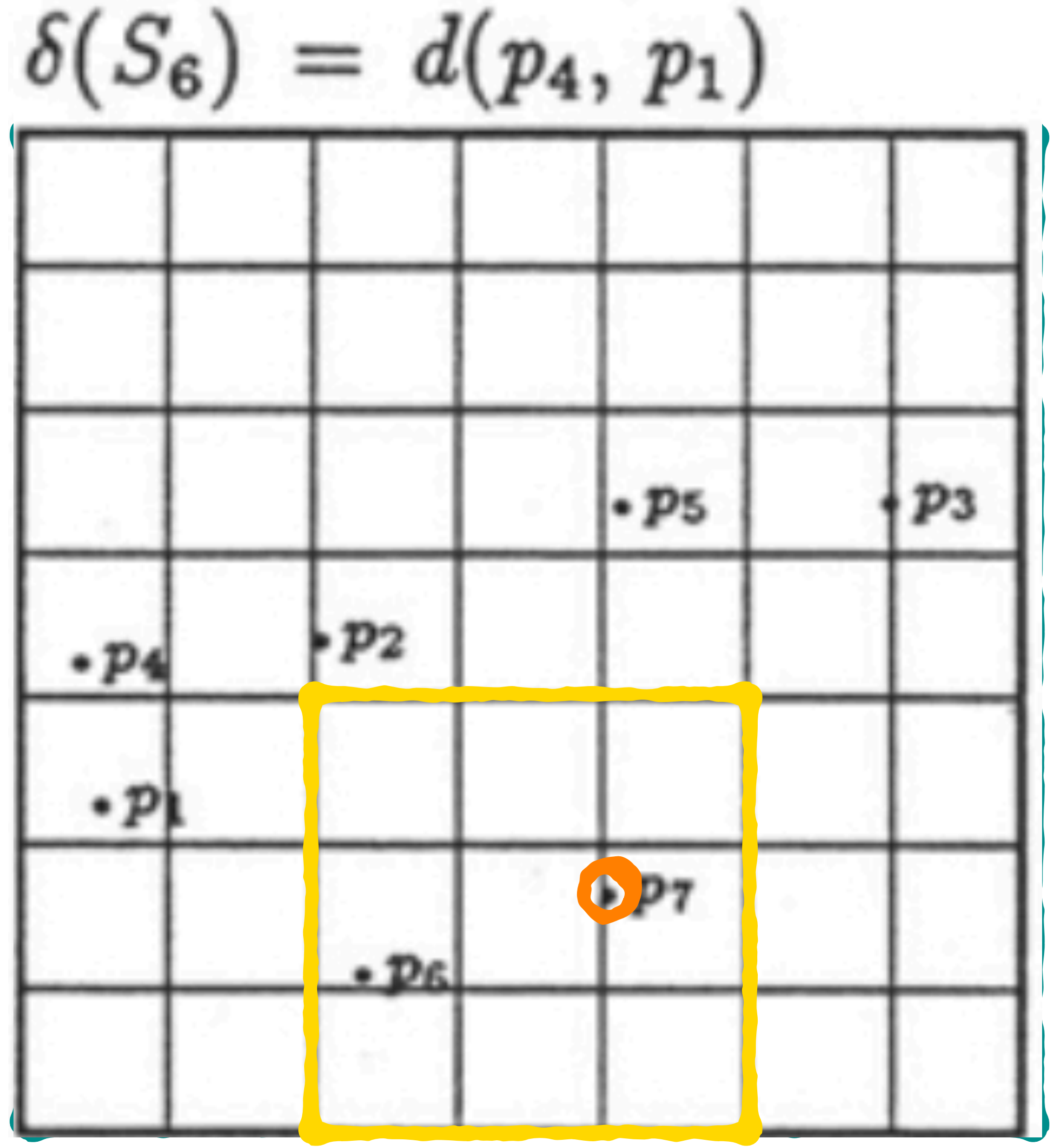

Randomised Incremental Construction
Golin, Raman, Schwarz, Smid 1992/1995

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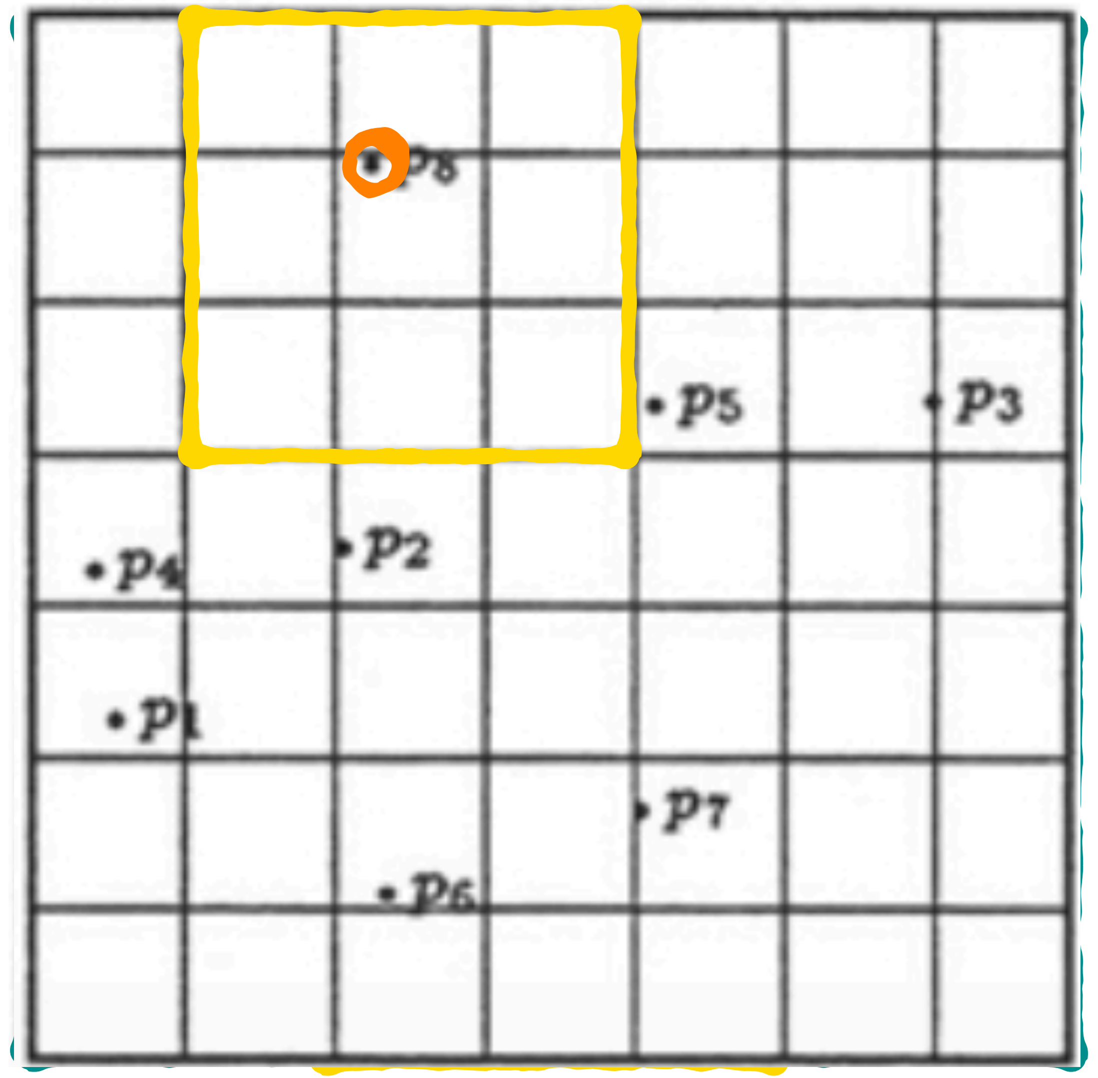
$\delta(S_7) = d(p_4, p_1)$



Randomised Incremental Construction
Golin, Raman, Schwarz, Smid 1992/1995

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  end;
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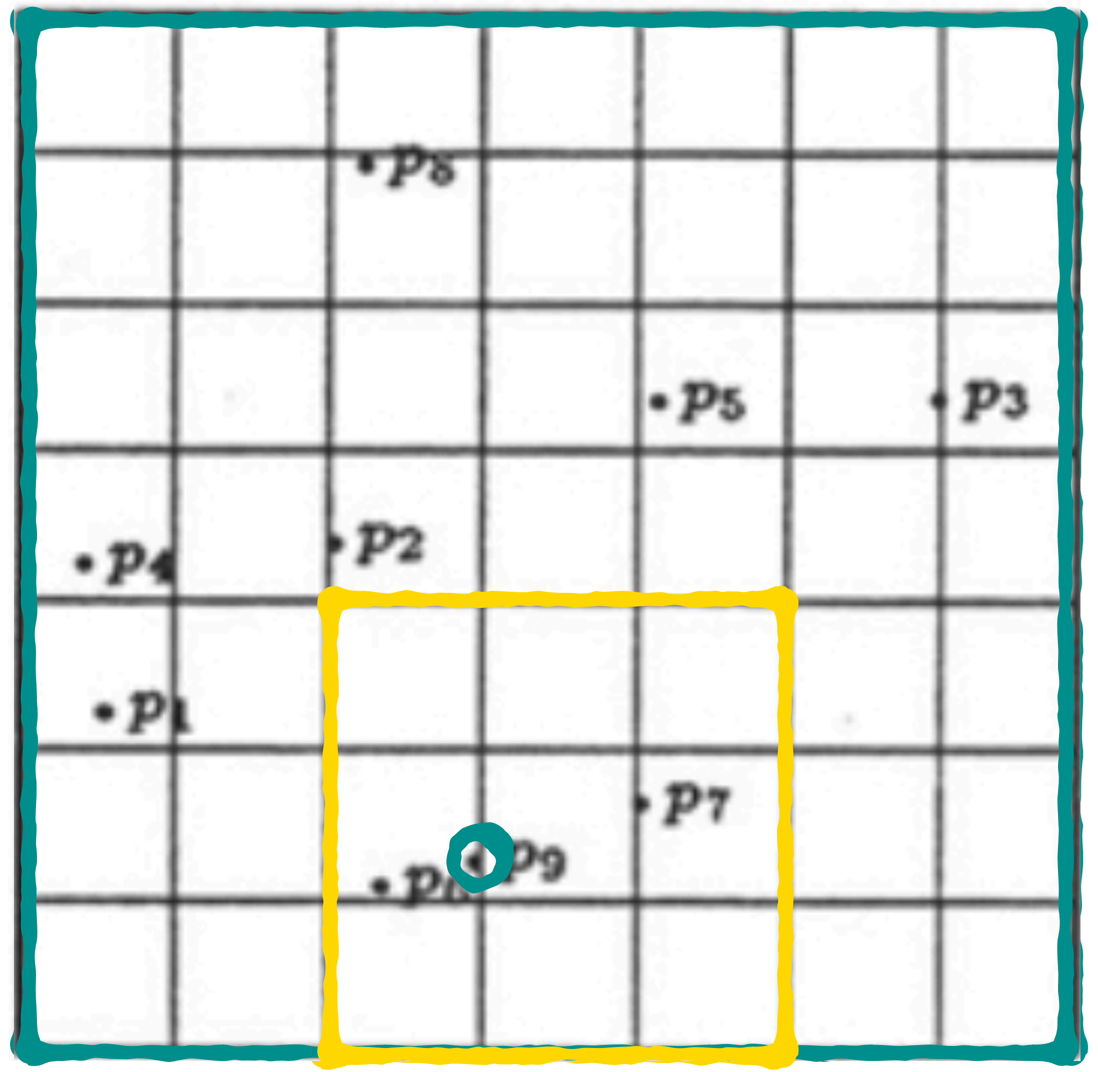
$\delta(S_8) = d(p_4, p_1)$



Randomised Incremental Construction
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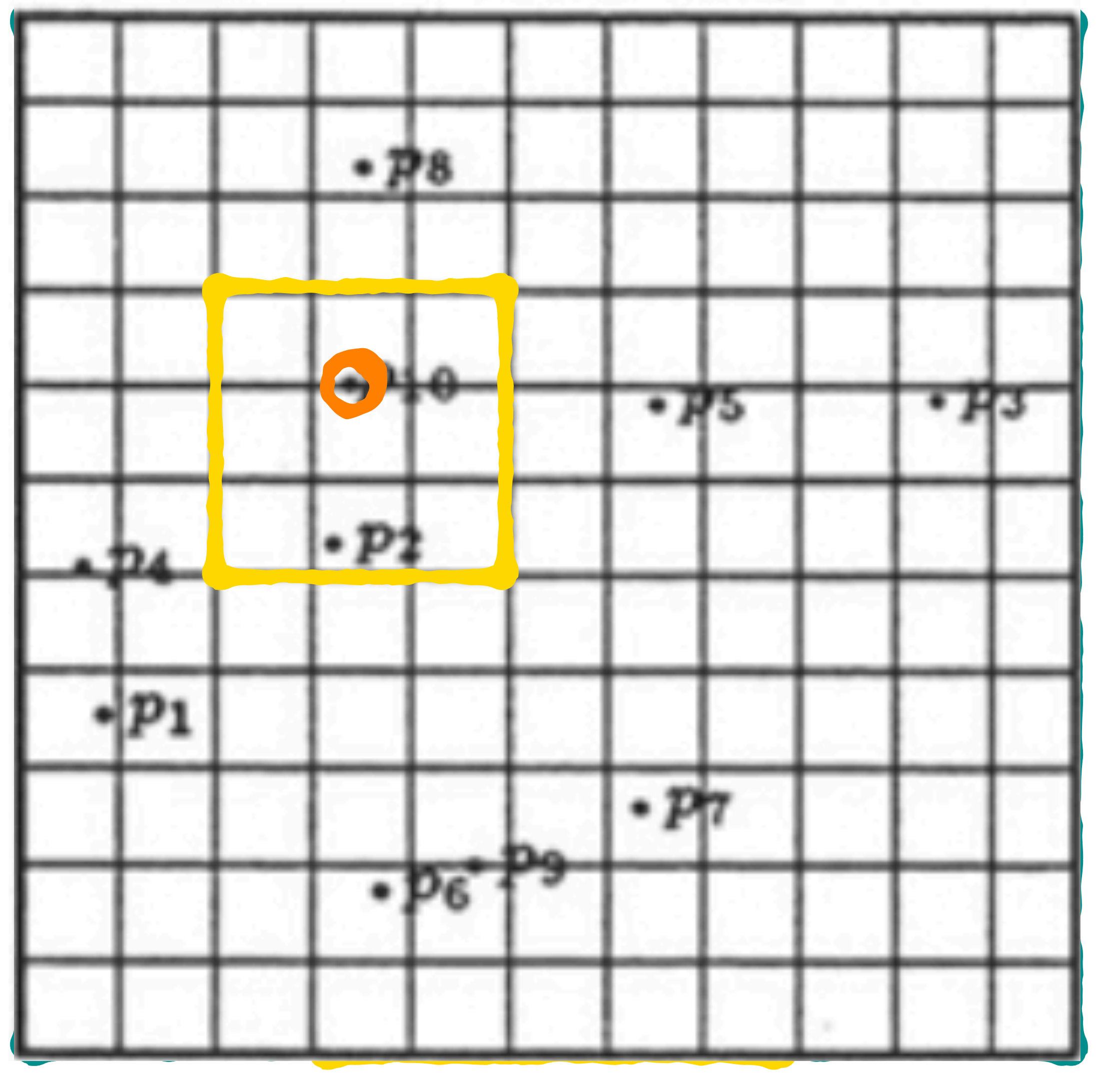
```

$\delta(S_9) = d(p_9, p_6)$



Randomised Incremental Construction
Golin, Raman, Schwarz, Smid 1992/1995

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```
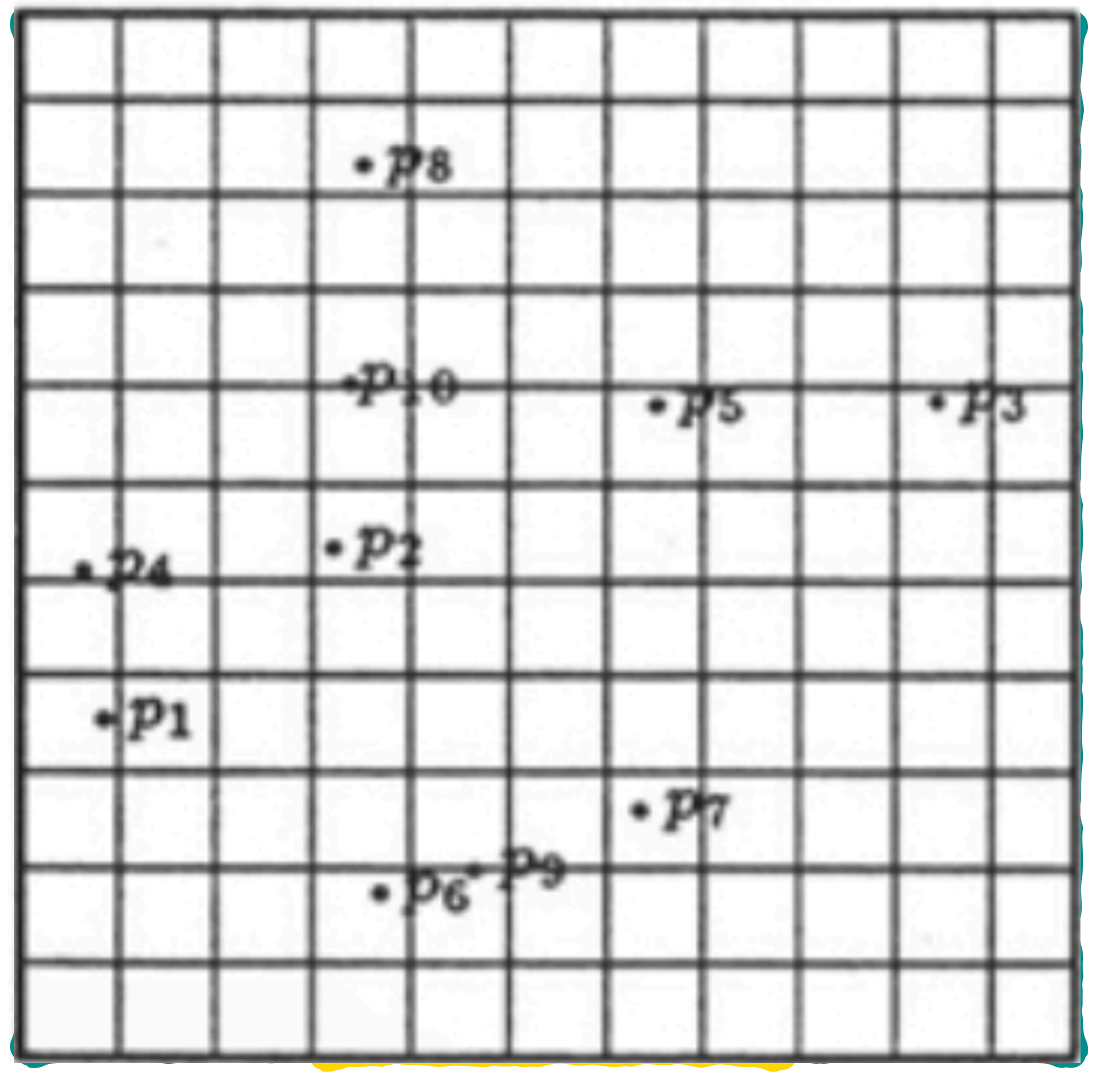
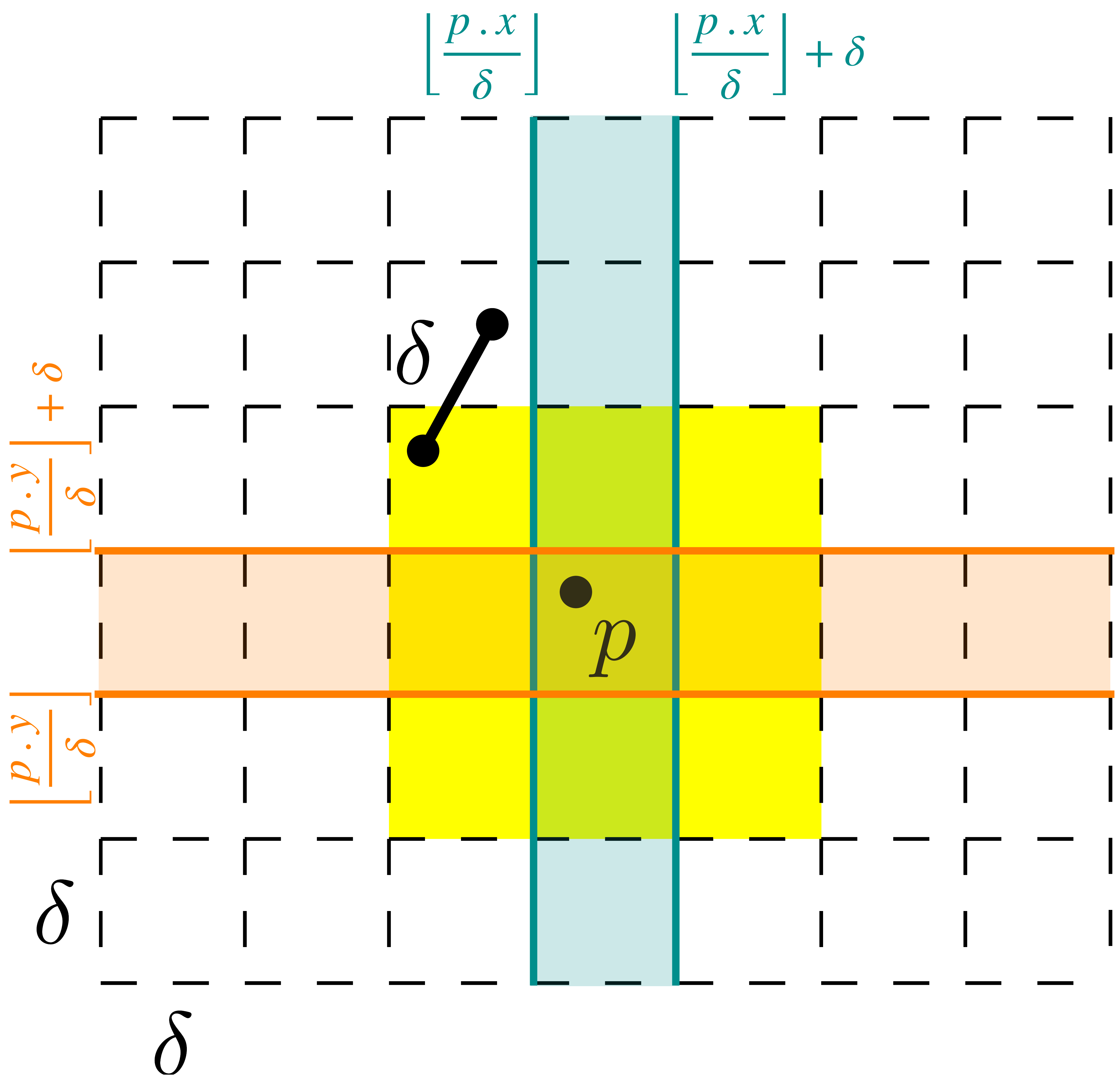

$\delta(S_{10}) = d(p_9, p_6)$



Fairly straightforward algorithm :)

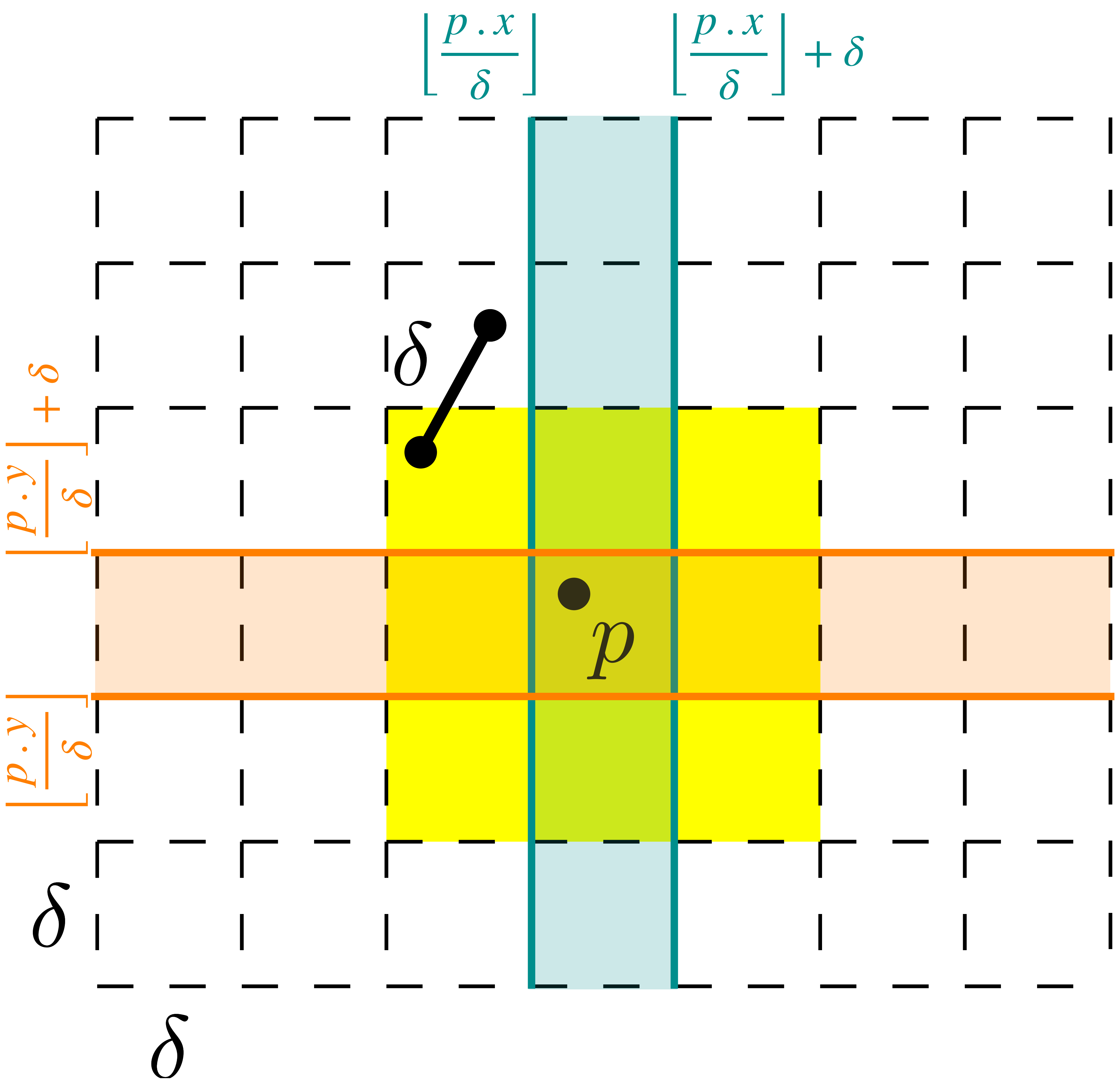
Randomised Incremental Construction

Golin, Raman, Schwarz, Smid 1992/1995



Randomised Incremental Construction
Golin, Raman, Schwarz, Smid 1992/1995

Representation

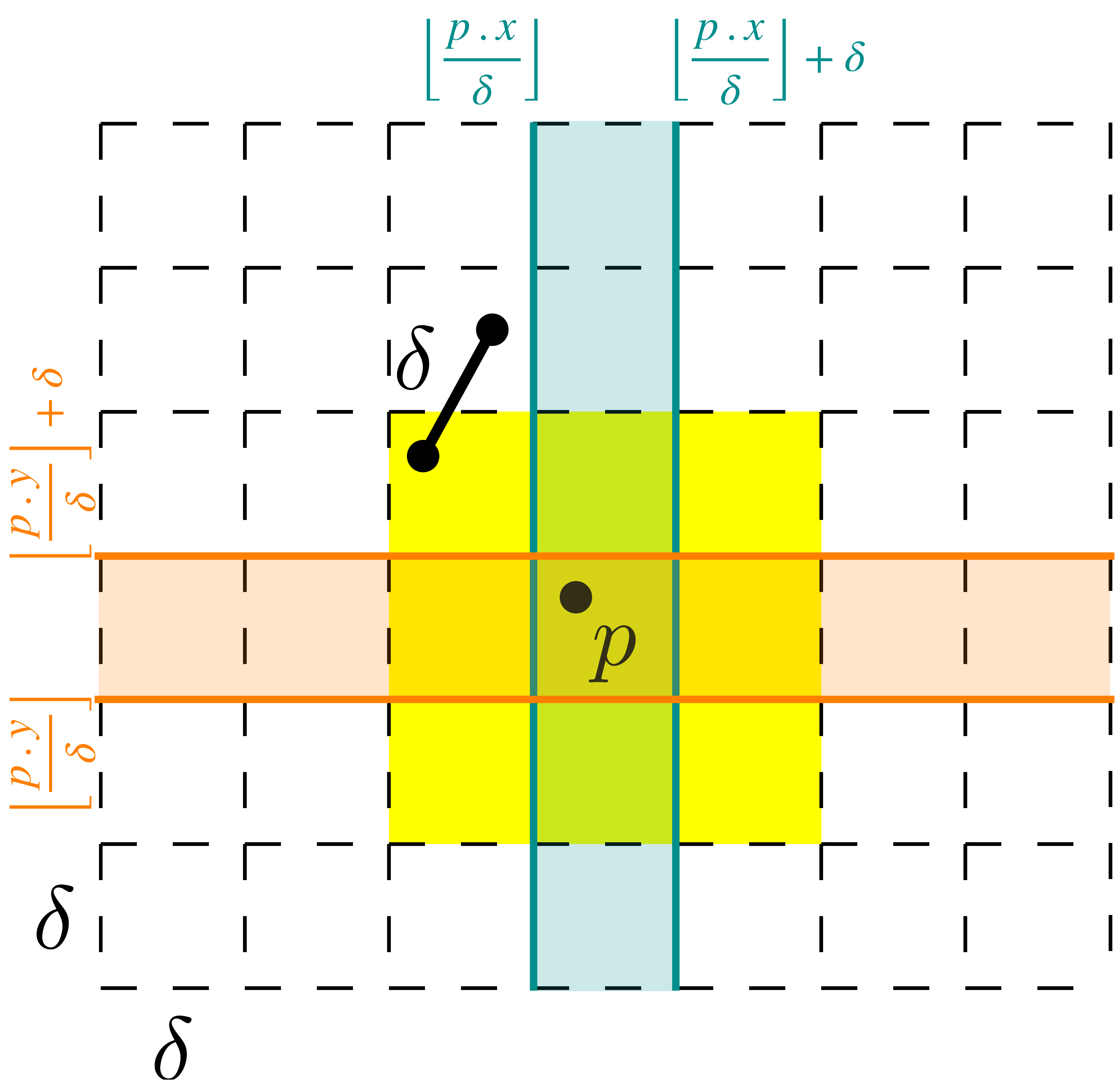


Randomised Incremental Construction
Golin, Raman, Schwarz, Smid 1992/1995

Representation

- Grid cell representation

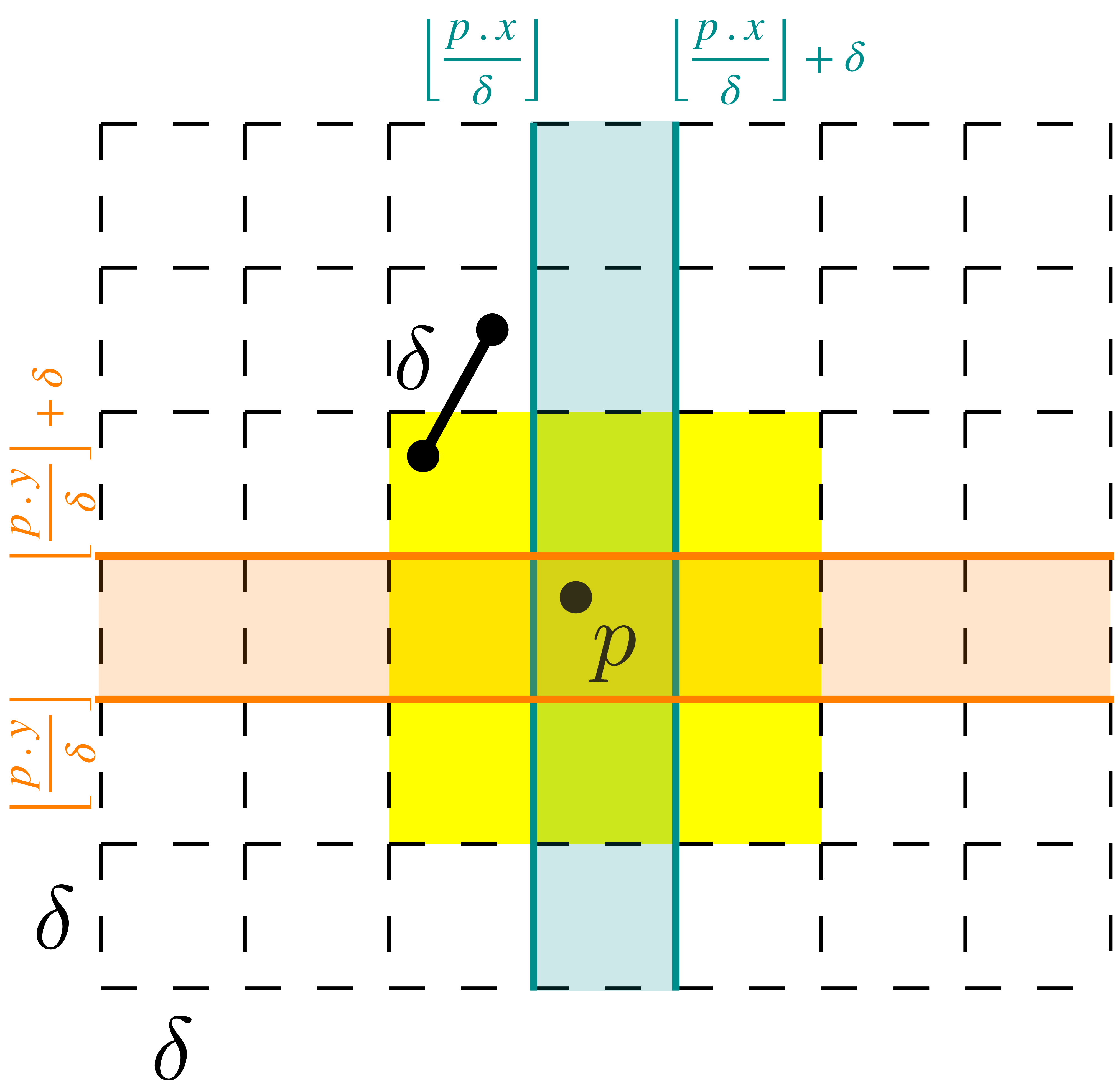
`(x,y)`



Randomised Incremental Construction
Golin, Raman, Schwarz, Smid 1992/1995

Representation

- Grid cell representation **`(x,y)`**
- Grid cell assignment G_δ **`OrderedDictionary`**



Randomised Incremental Construction
Golin, Raman, Schwarz, Smid 1992/1995

Representation

- Grid cell representation `(x,y)`
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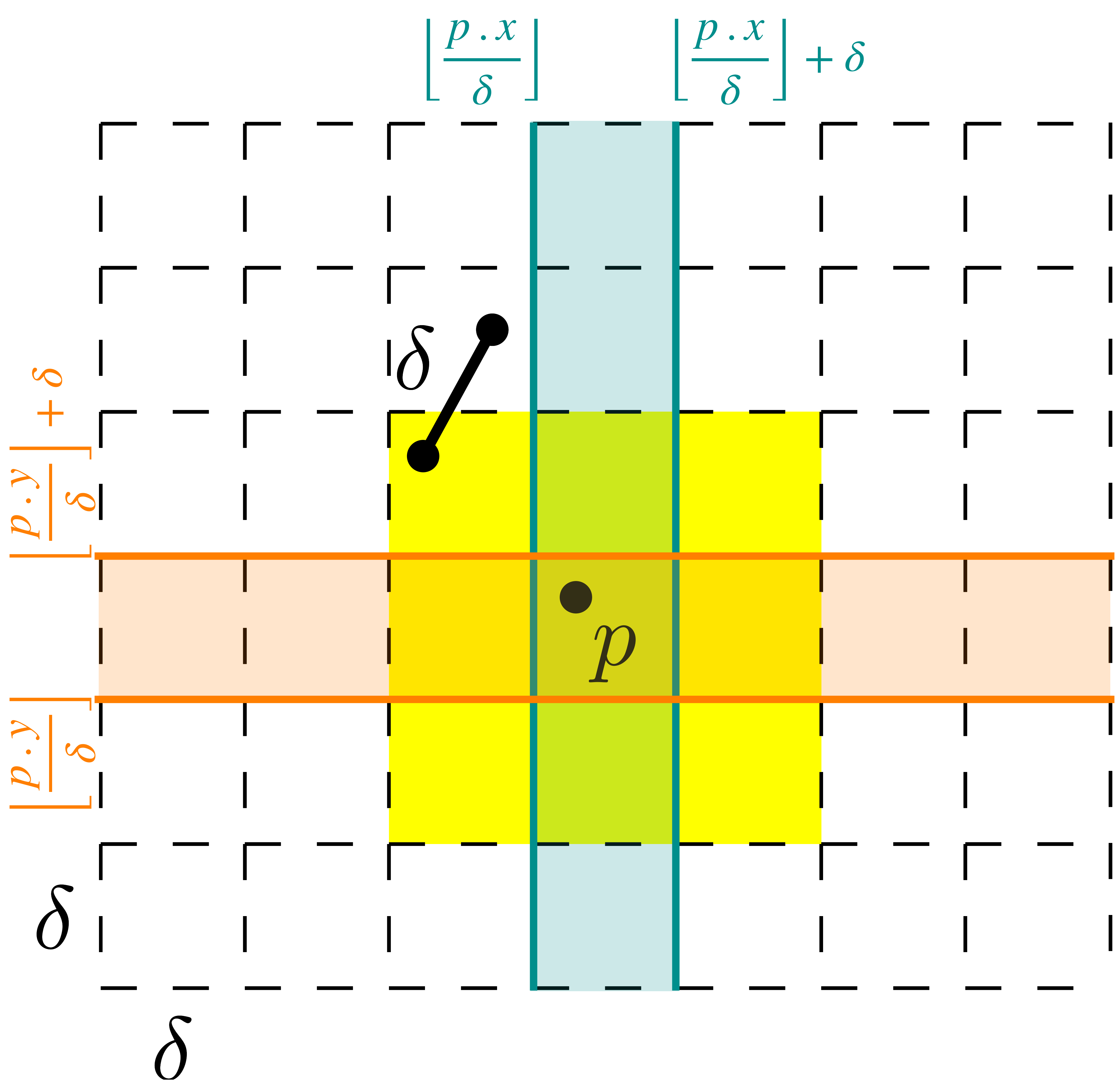
- Identifier/key for $p \in \mathscr{P}$ `(p.x,p.y)`



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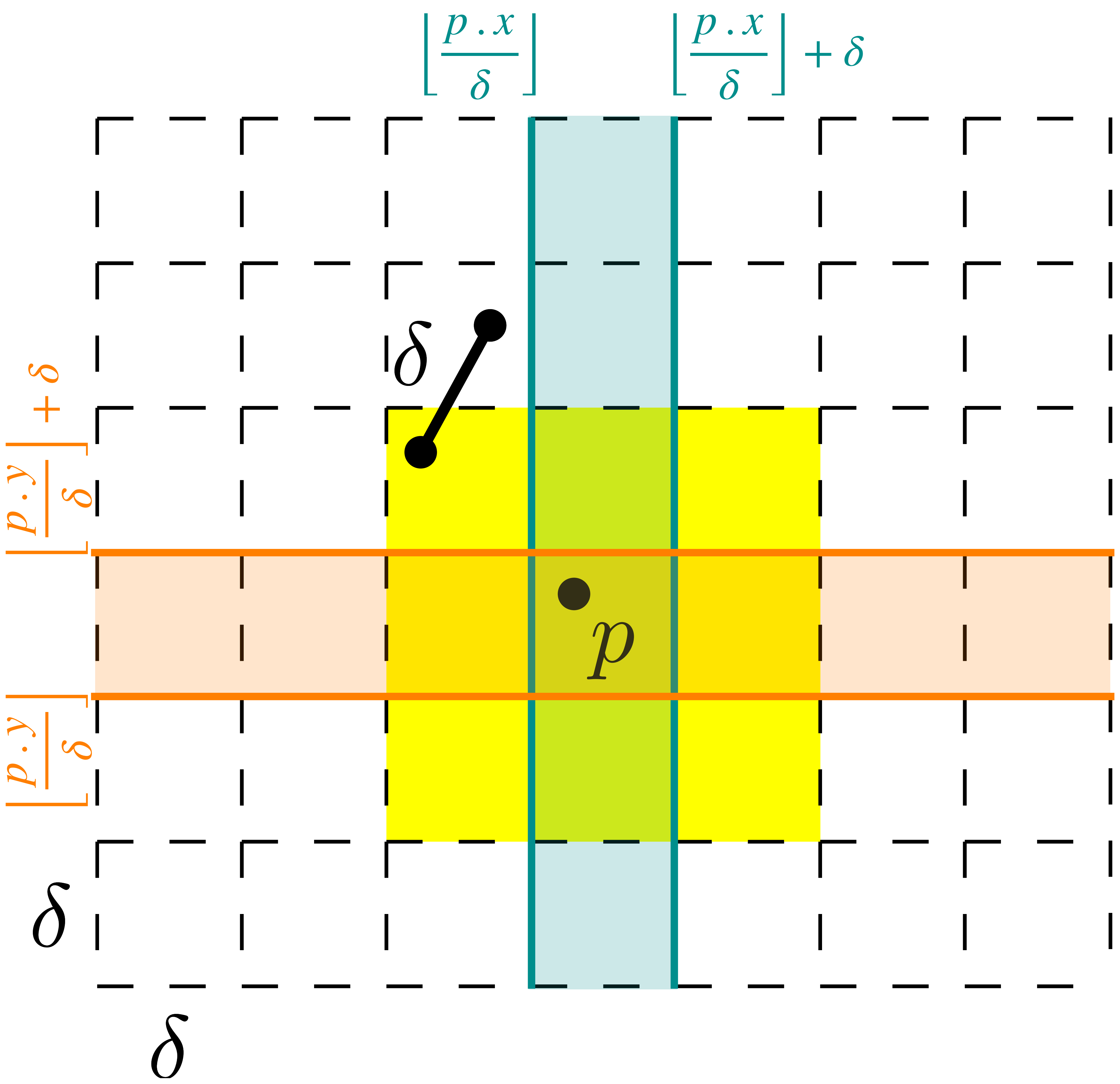

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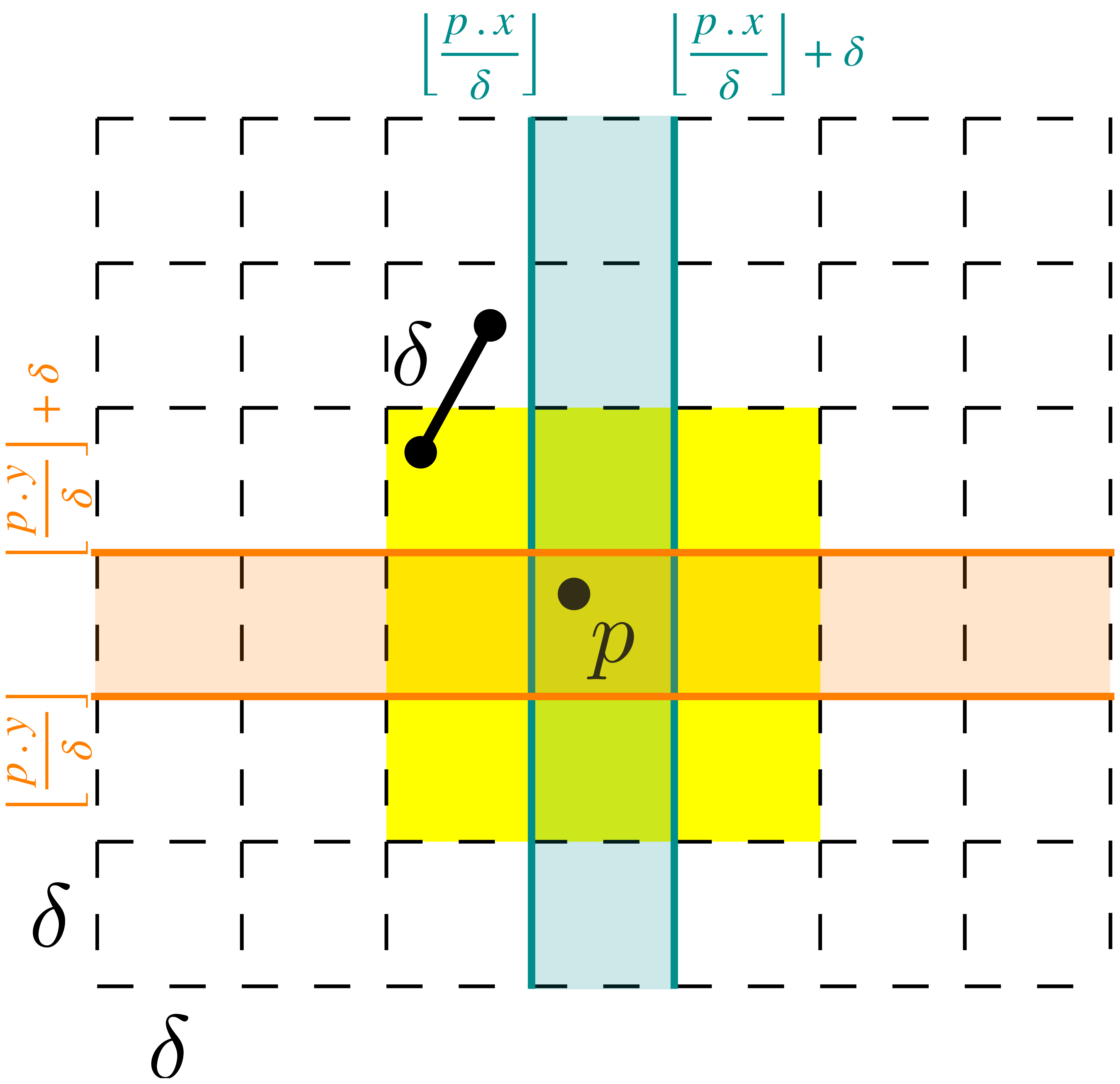
Implementation



Randomised Incremental Construction

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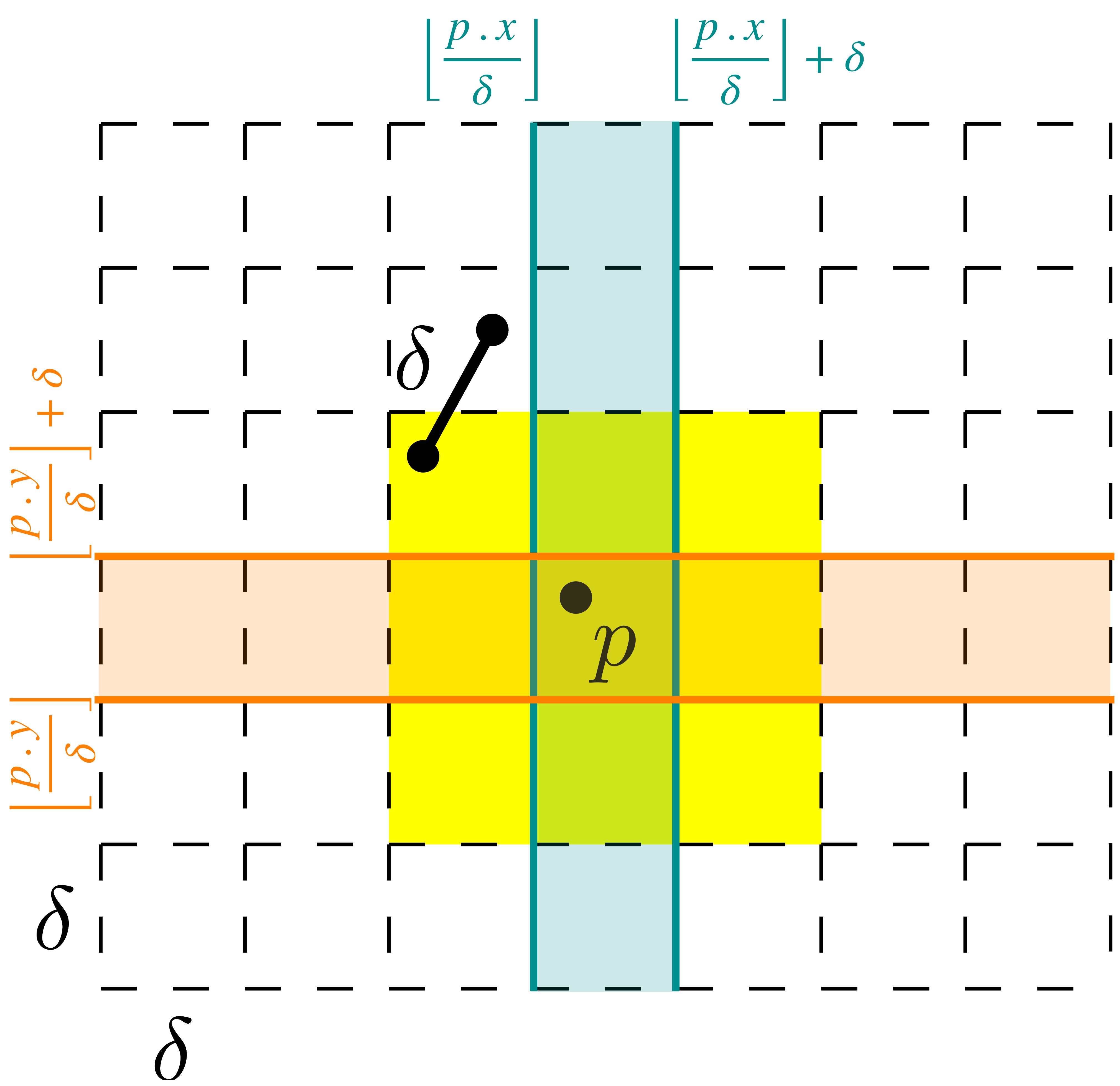
Implementation

- $T_{build}(n)$ Construction cost



Randomised Incremental Construction
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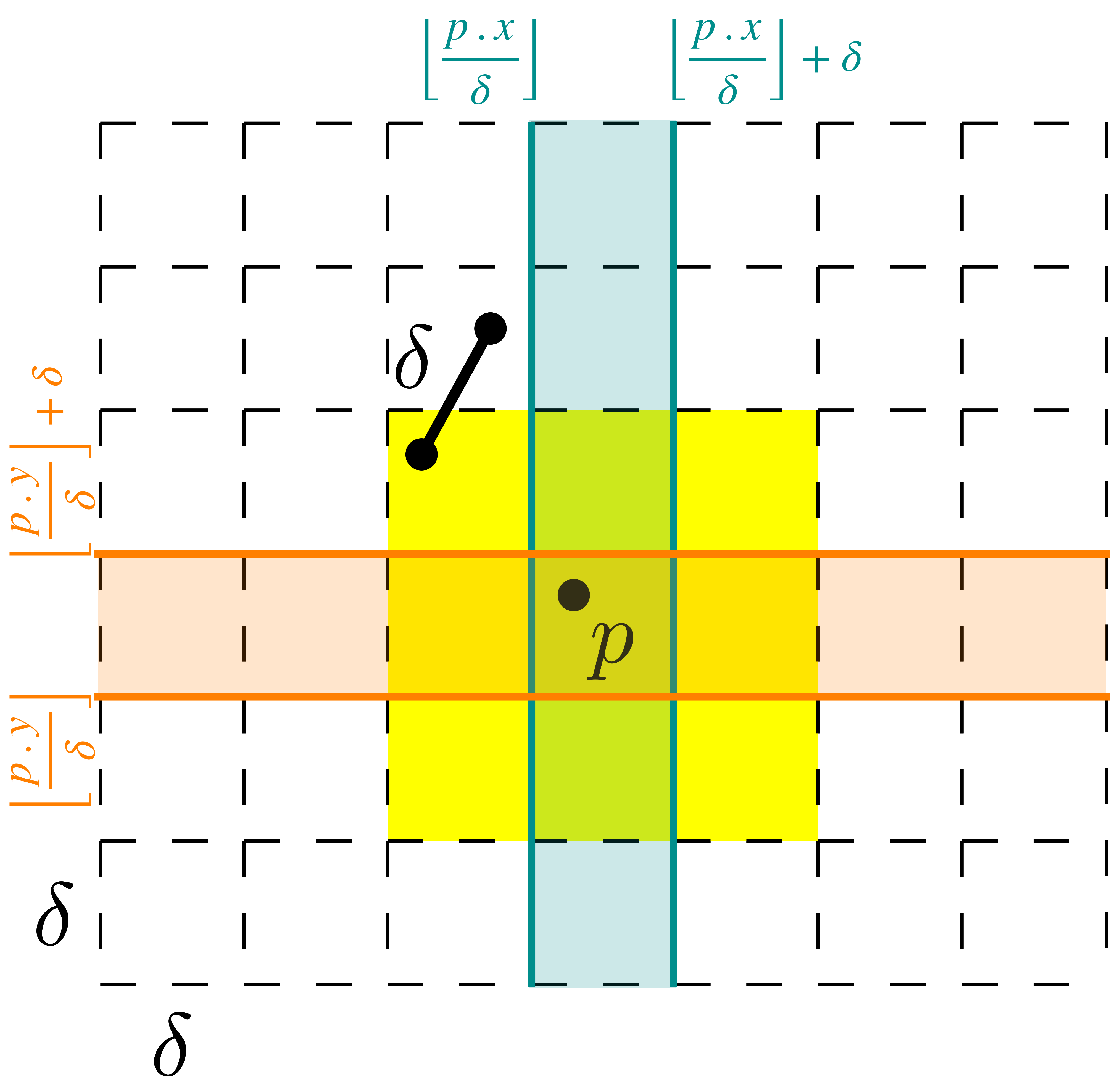
Implementation

- $T_{build}(n)$ Construction cost
- $T_{insert}(n)$ Insertion cost



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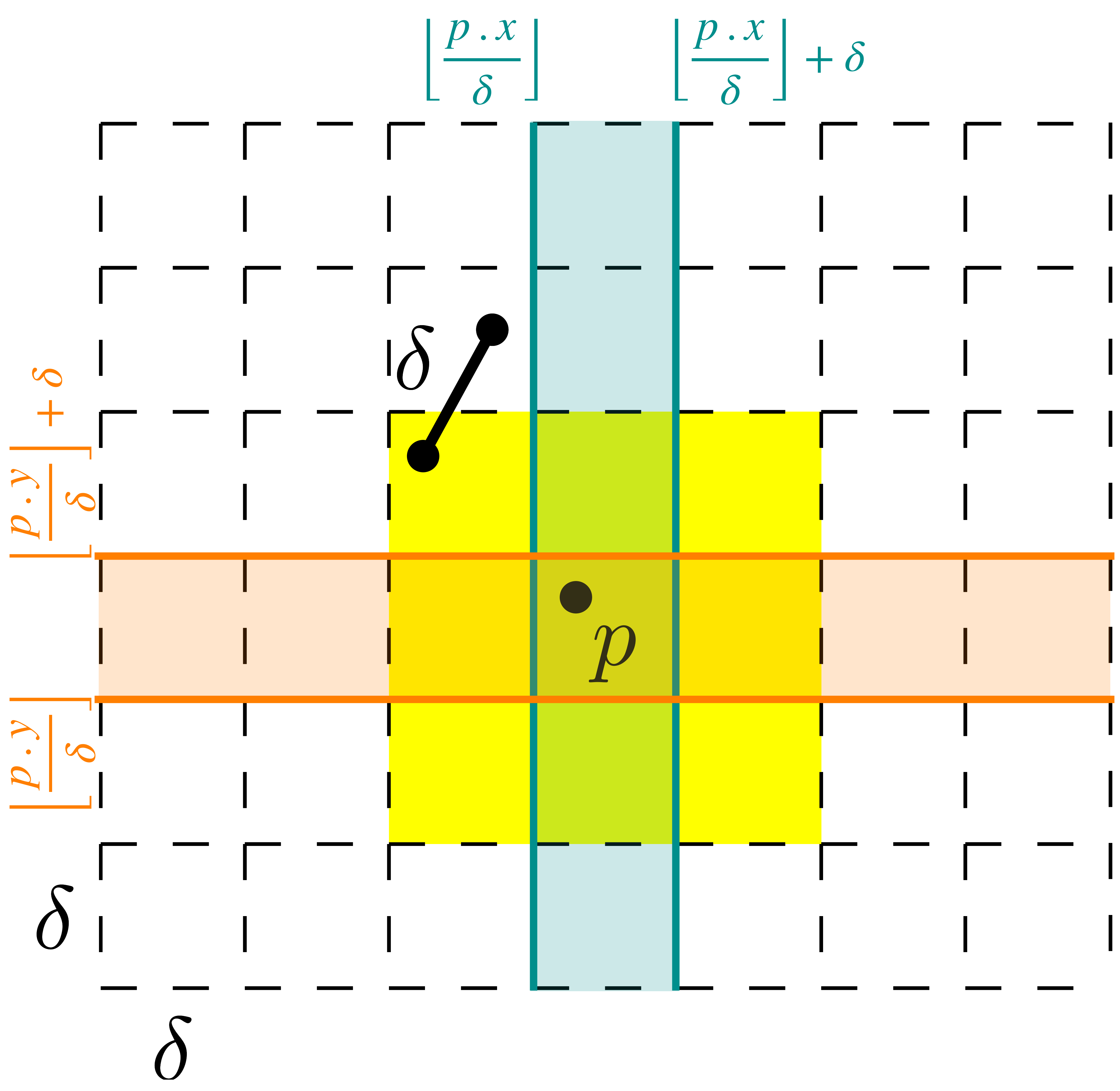
Implementation

- $T_{build}(n)$ Construction cost
- $T_{insert}(n)$ Insertion cost
- $T_{query}(n)$ Query cost



Randomised Incremental Construction
Golin, Raman, Schwarz, Smid 1992/1995

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Randomised Incremental Construction
Golin, Raman, Schwarz, Smid 1992/1995

Terminology

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- Randomised: $X(p_i, \mathscr{P}_{i-1}) = \begin{cases} 1, \text{ if } \delta_i < \delta_{i-1} \\ 0, \text{ otherwise.} \end{cases}$

Randomised Incremental Construction

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Lemma 1 *Let $p_1, p_2, \ldots, p_n$ be a random permutation of the points of S. Let $S_i := \{p_1, p_2, \ldots, p_i\}$. Then* $\Pr[\delta(S_{i+1}) < \delta(S_i)] \leq 2/(i+1)$.

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    - Case 2: No point is shared among the all pairs. — **Impossible — one such pair must have been known.**

      $$Pr\left[\{\delta_{i+1} < \delta_i\}\right] = 0$$

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Thank you.

References

[Bentley und Shamos, 1976] Bentley, Jon Louis und Michael Ian Shamos:
Divide-and-Conquer in multidimensional space. In: Proceedings of the Eighth Annual AXM Symposium on the Theory of Computation, S. 220-230, Association for Computing Machinery, 1976.

[Golin et al., 1995] Golin, Mordecai J., Rajeev Raman, Christian Schwarz Michiel Smid:
Simple randomized algorithms for closest pair problems. Nordic Journal of Computing 2(1):3-27, 1995.

[Hinrichs et al., 1988] Hinrichs, Klaus Helmer, Jürg Nievergelt und Peter Schom:
Plane-sweep solves the closest problem elegantly. Information Processing Letters 26(5): 255-261, 1988.

[Preparata und Shamos, 1988] Preparata, Franco P. und Michael Ian Shamos:
Computational Geometry: An Introduction. Springer, Berlin, 2. Edition, 1988.

[Shamos und Hoey, 1976] Shamos, Michael Ian und Daniel J. Hoey:
Geometric intersection problems. In: Proceedings of the 17th Annual Symposium on the Foundation of Computer Science, S. 208-215, IEEE Computer Press, 1976.