Computational Geometry
Chapter 4: Voronoi Diagrams

Prof. Dr. Sándor Fekete

Algorithms Division
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TU Braunschweig

Technische Universität Braunschweig


The 1850s map that changed how we fight outbreaks
Vox
GREAT MARLBOROUGH STREET
WORK HOUSE
PUMP
BROAD STREET
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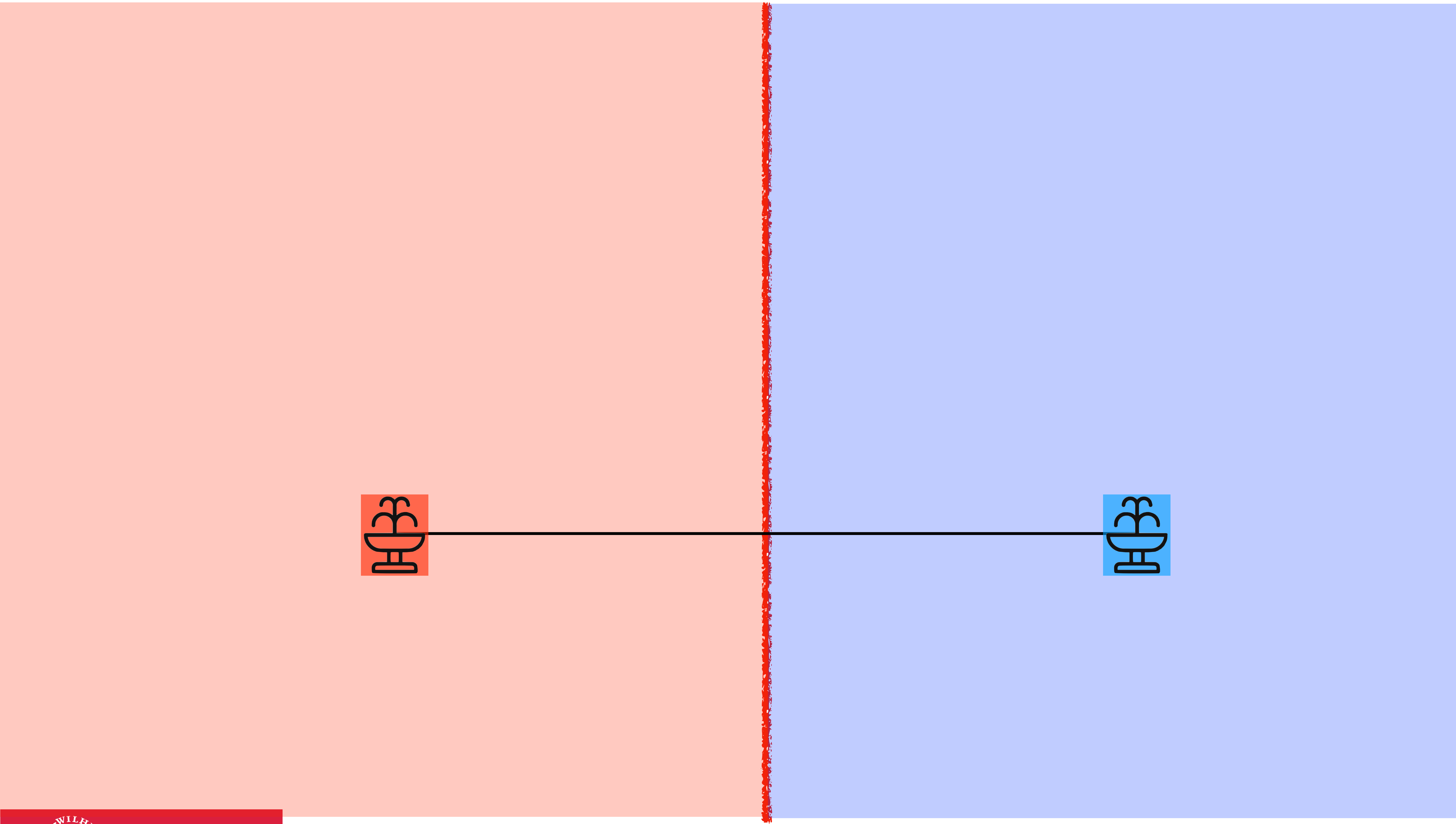
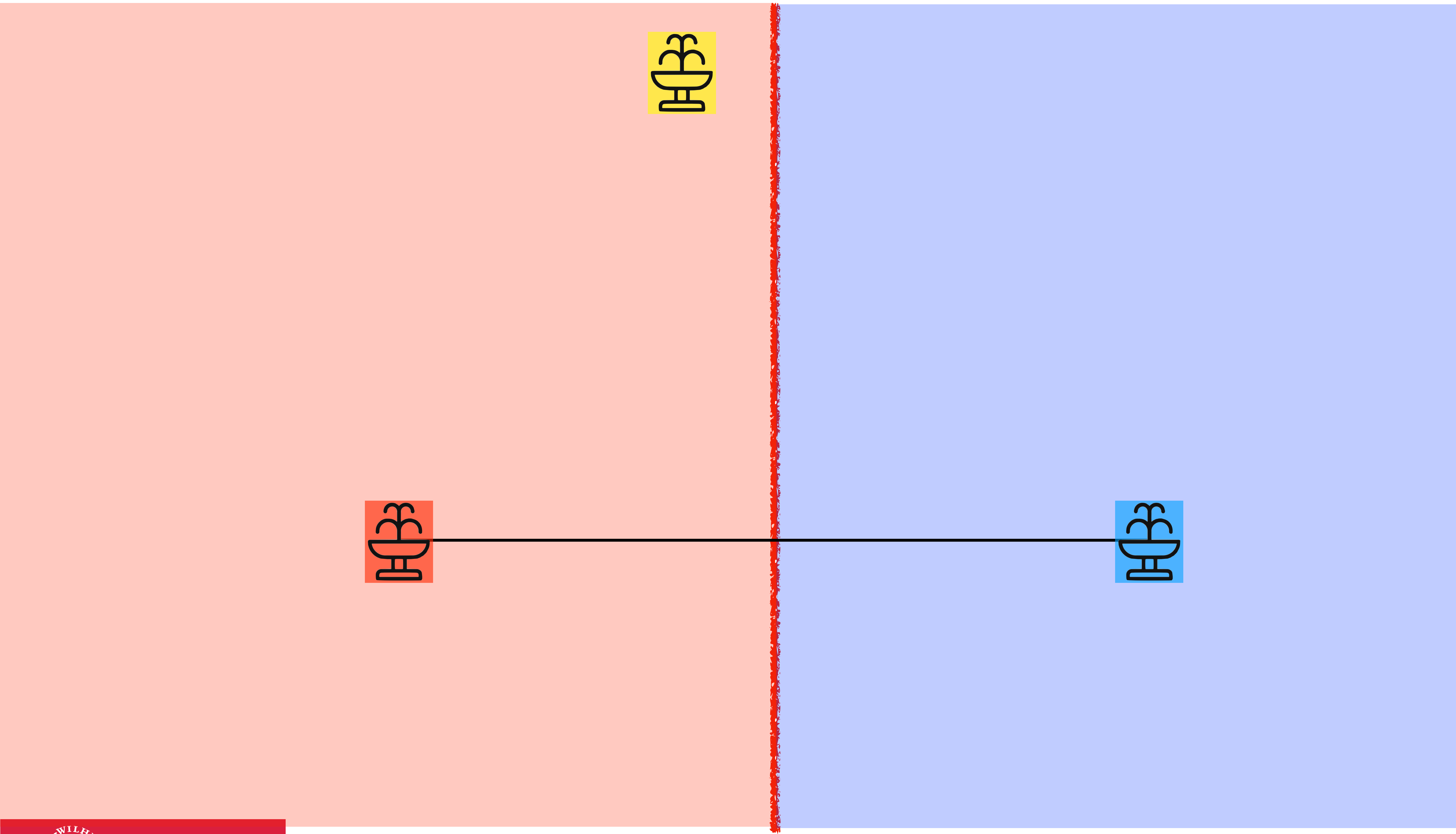
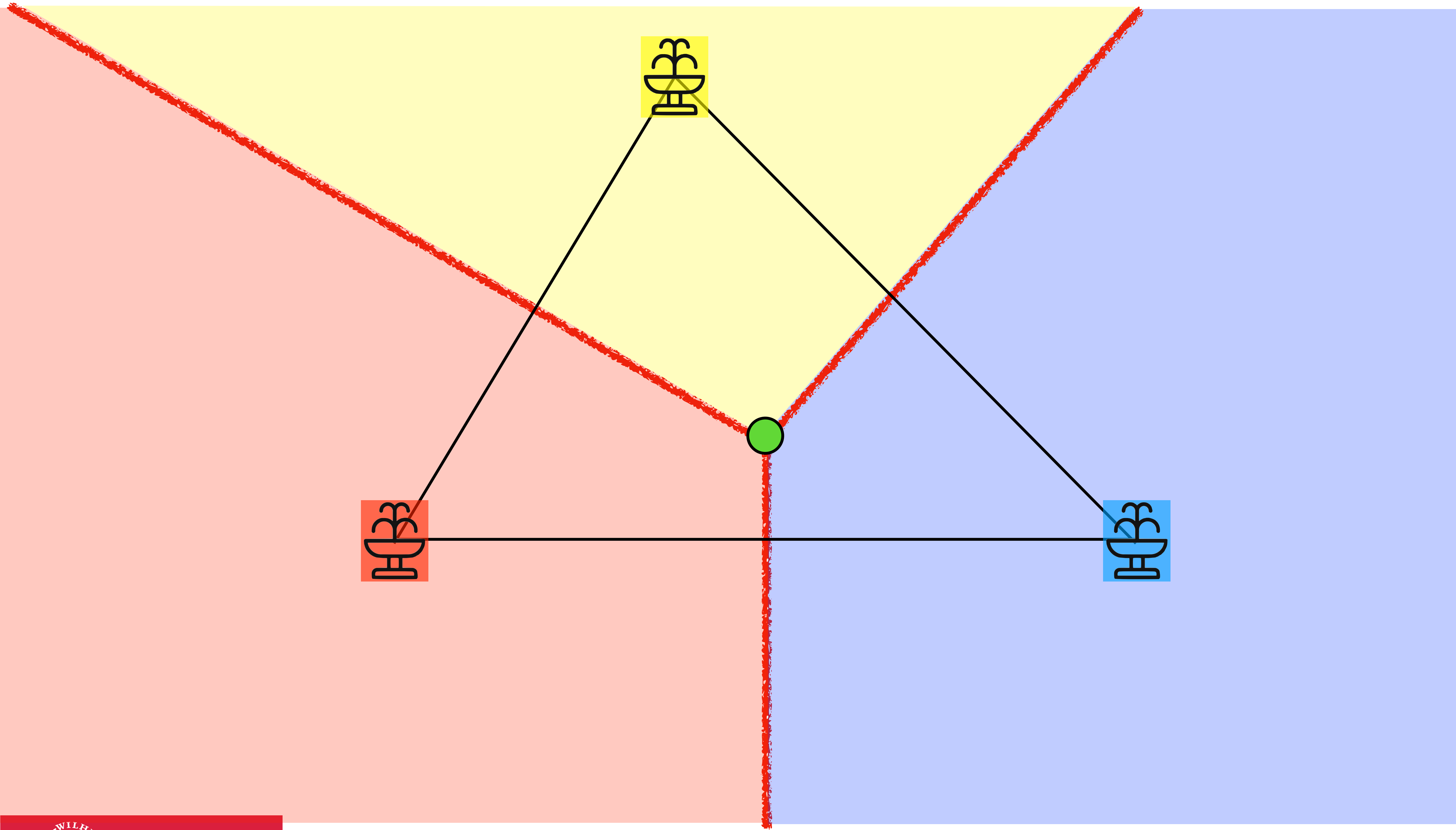
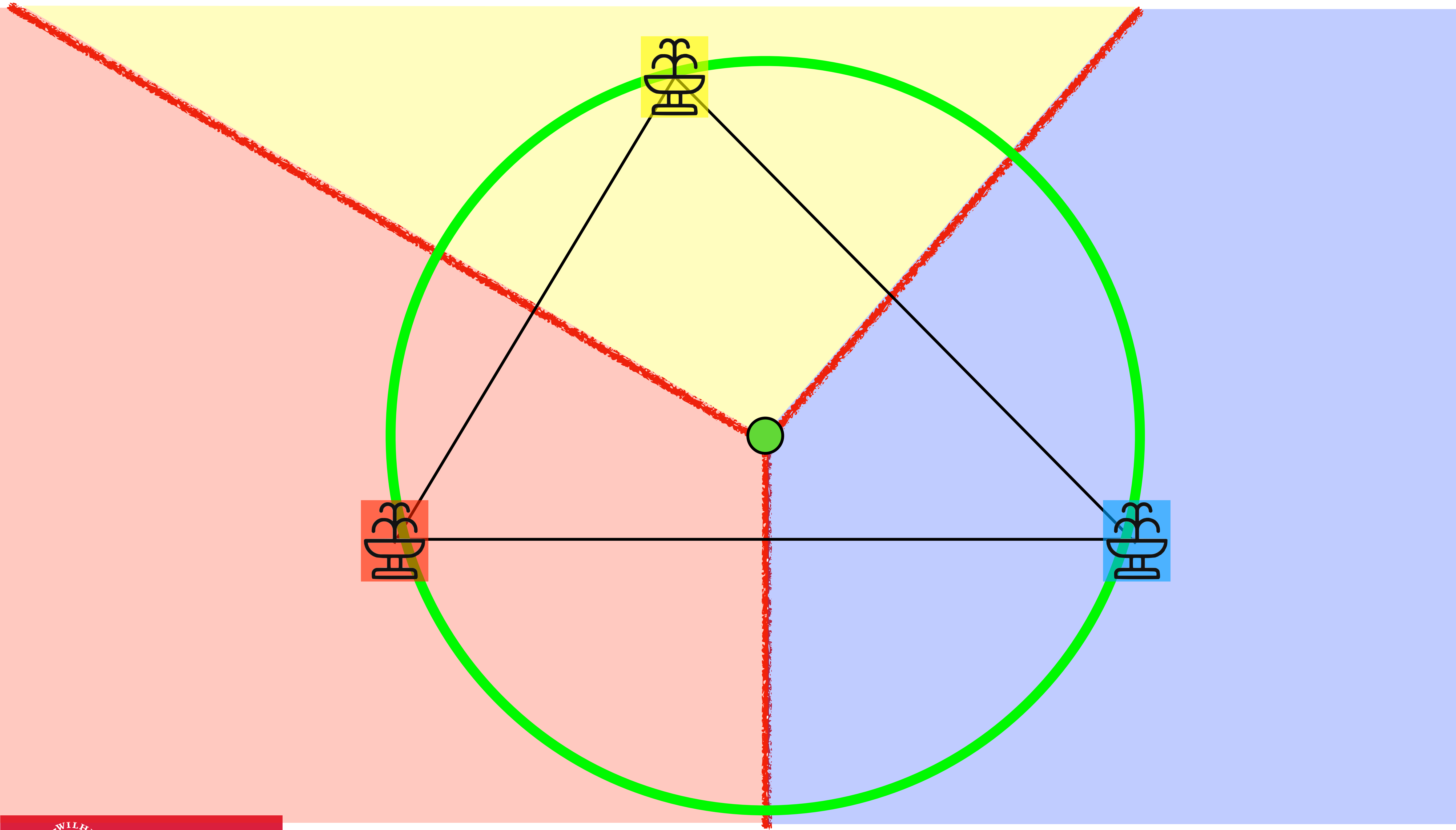


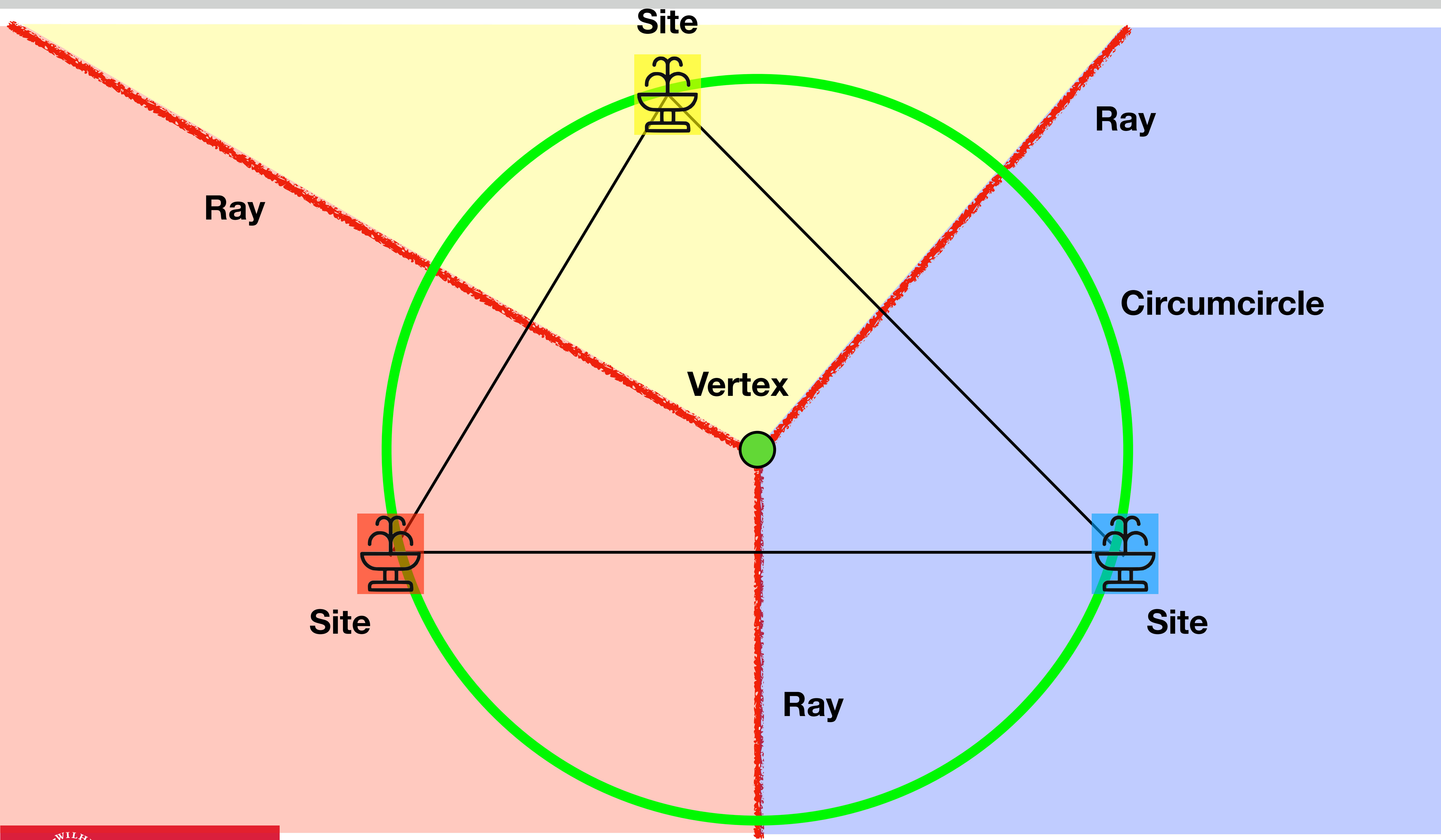
Euclidean Distances





Euclidean Distances

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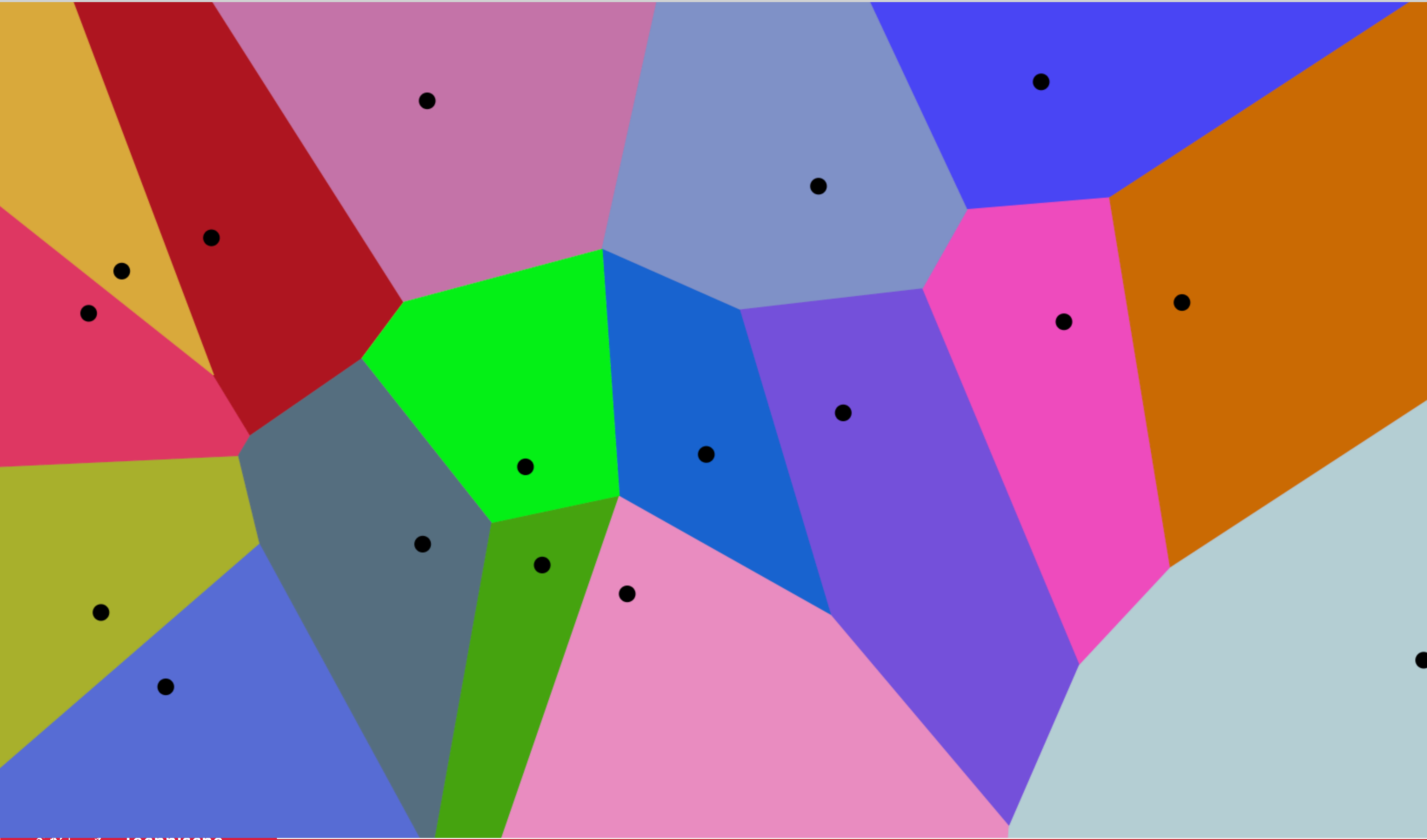
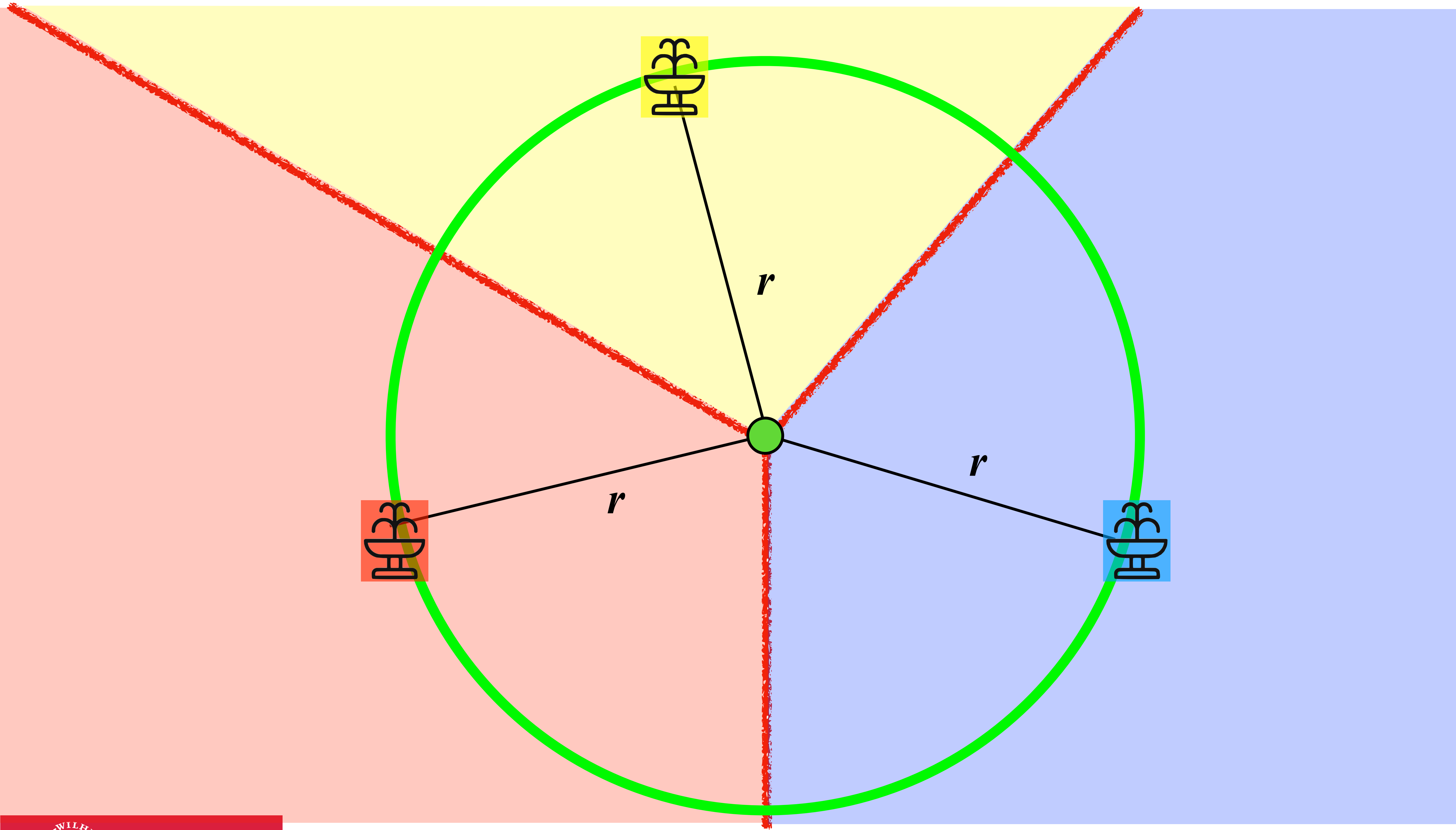
Euclidean Distances

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Vironoi Man

Nouvelles applications des paramètres continus à la théorie des formes quadratiques.

Premier Mémoire.

Sur quelques propriétés des formes quadratiques positives parfaites.

Par M. *Georges Voronoï* à Varsovie.

Introduction.

Hermite avait introduit dans la théorie des nombres un principe nouveau et fécond, à savoir: étant donné un ensemble (X) des systèmes $(x_1, x_2, \ldots x_n)$ des valeurs entières de $x_1, x_2, \ldots x_n$, on fait correspondre à l'ensemble (X) un ensemble (R) composé des domaines déterminés à l'aide des paramètres continus $\varrho_1, \varrho_2, \ldots \varrho_m$ de manière qu' en étudiant l'ensemble (R) on étudie en même temps l'ensemble (X).

Hermite a montré*) de nombreuses applications du nouveau principe pour la généralisation des fractions continues, pour la recherche des unités algébriques etc.

Les idées d'*Hermite* ont été développées dans les travaux de M. M. *Zolotareff*, *Charve*, *Selling*, *Minkowski*.**)

*) *Hermite*. Extraits de lettres de M. *Ch. Hermite* à M. *Jacobi* sur différents objets de la théorie des nombres. (Ce Journal t. 40, p. 261.)

Hermite. Sur l'introduction des variables continues dans la théorie des nombres. (Ce Journal t. 41, p. 191.)

Hermite. Sur la théorie des formes quadratiques. (Ce Journal t. 47, p. 313.)

**) *Zolotareff*. Sur une équation indéterminée du troisième degré. (Pétersbourg, 1869, *en russe*.)

Zolotareff. Théorie des nombres entiers complexes avec des applications au calcul intégral. (Pétersbourg, 1874, *en russe*.)

20.

Über die Reduction der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen.

(Von Herrn Prof. *G. Lejeune Dirichlet* zu Berlin.)

(Vorgetragen in der Sitzung der physikalisch-mathematischen Classe der Akademie am 31ten Juli 1848 *)).

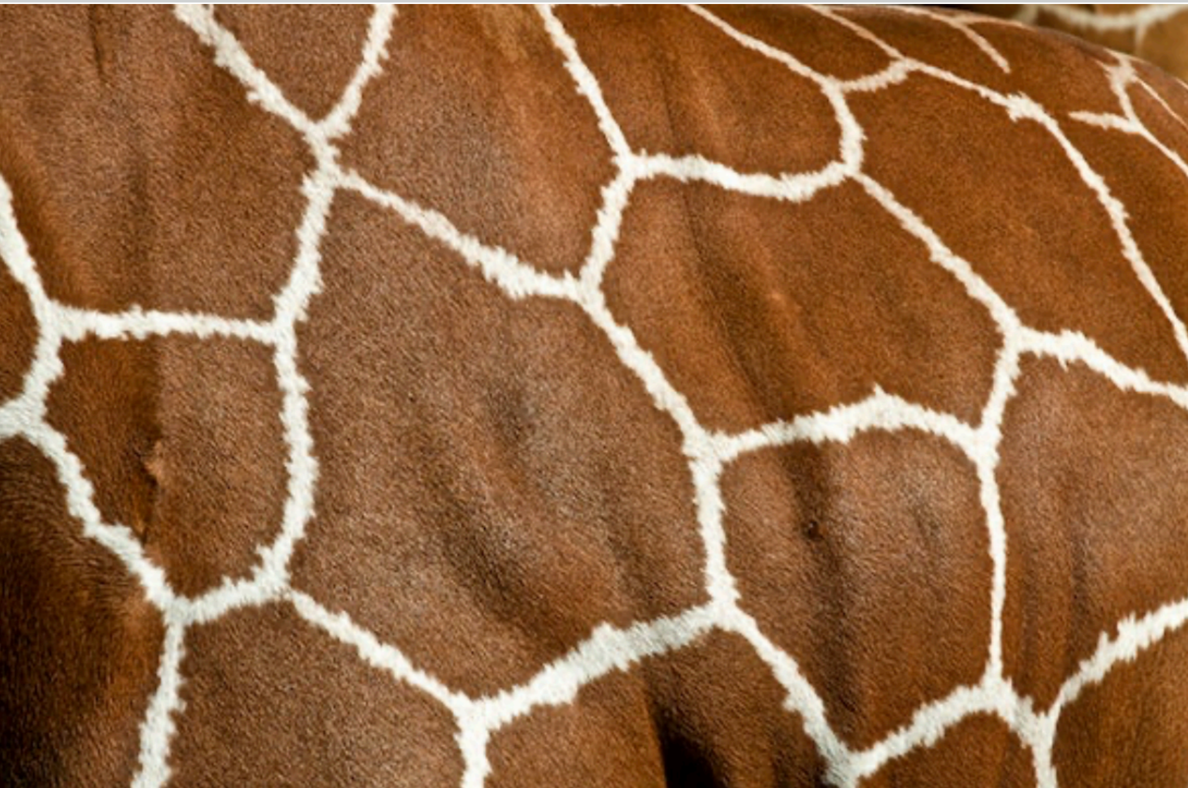
Bekanntlich hat *Lagrange* zuerst gezeigt, dafs jede binäre quadratische Form reducirt, d. h. in eine andere äquivalente verwandelt werden kann, deren Coëfficienten gewisse Ungleichheitsbedingungen erfüllen, und zugleich nachgewiesen, dafs in jeder Classe positiver Formen immer nur eine einzige solche Form existirt, so dafs für diesen Fall die verschiedenen, einer gegebenen Determinante entsprechenden reducirten Formen als die Repräsentanten der verschiedenen Classen dienen können. Nachdem später in den „Disquisitiones arithmeticae" die ternären Formen aus einem allgemeinen Gesichtspunct betrachtet worden waren, wurde es für die weitere Ausbildung dieser Theorie erforderlich, die von *Lagrange* für die positiven binären Formen ausgeführte Untersuchung auf die ternären derselben Art auszudehnen, d. h. solche Ungleichheitsbedingungen zwischen den Coëfficienten aufzufinden, dafs dieselben in jeder Classe von einer und nur von einer Form erfüllt werden. Diese mit grofsen Schwierigkeiten verbundene Erweiterung ist von *Seeber* in einem speciell den positiven ternären Formen gewidmeten Werke geleistet worden, dessen Hauptinhalt sie ausmacht und welches *Gaufs* in einer höchst interessanten Anzeige **) wie folgt characterisirt: „Dem Geiste der Gründlich-„keit, womit diese Gegenstände (die Auflösung der Aufgabe nämlich, in jeder „Classe eine reducirte Form zu finden, und der Beweis, dafs es in jeder nur „eine giebt) durchgeführt sind, müssen wir volle Gerechtigkeit widerfahren „lassen, und wenn wir es dabei bedauern müssen, dafs damit eine grofse und

*) Von dieser Abhandlung ist bereits ein Auszug im Monatsbericht der Akademie gegeben worden, worin das Princip dieser neuen Behandlung der Reduction der positiven ternären Formen, die Betrachtung successiver Minima, angedeutet und der Beweis des ersten der beiden *Seeber*'schen Resultate nach diesem Princip vollständig durchgeführt ist.

**) *Crelle's* Journal Band 20. pag. 312.

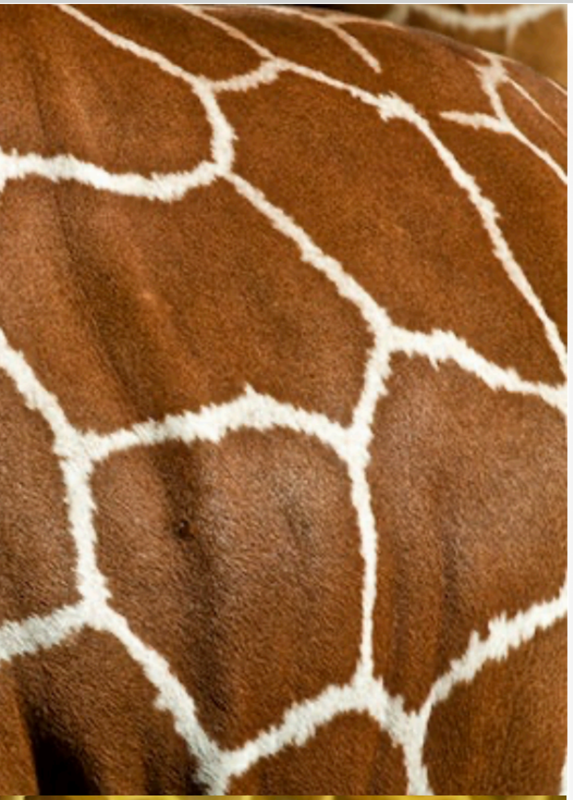
Crelle's Journal f. d. M. Bd. XL. Heft 3. 28

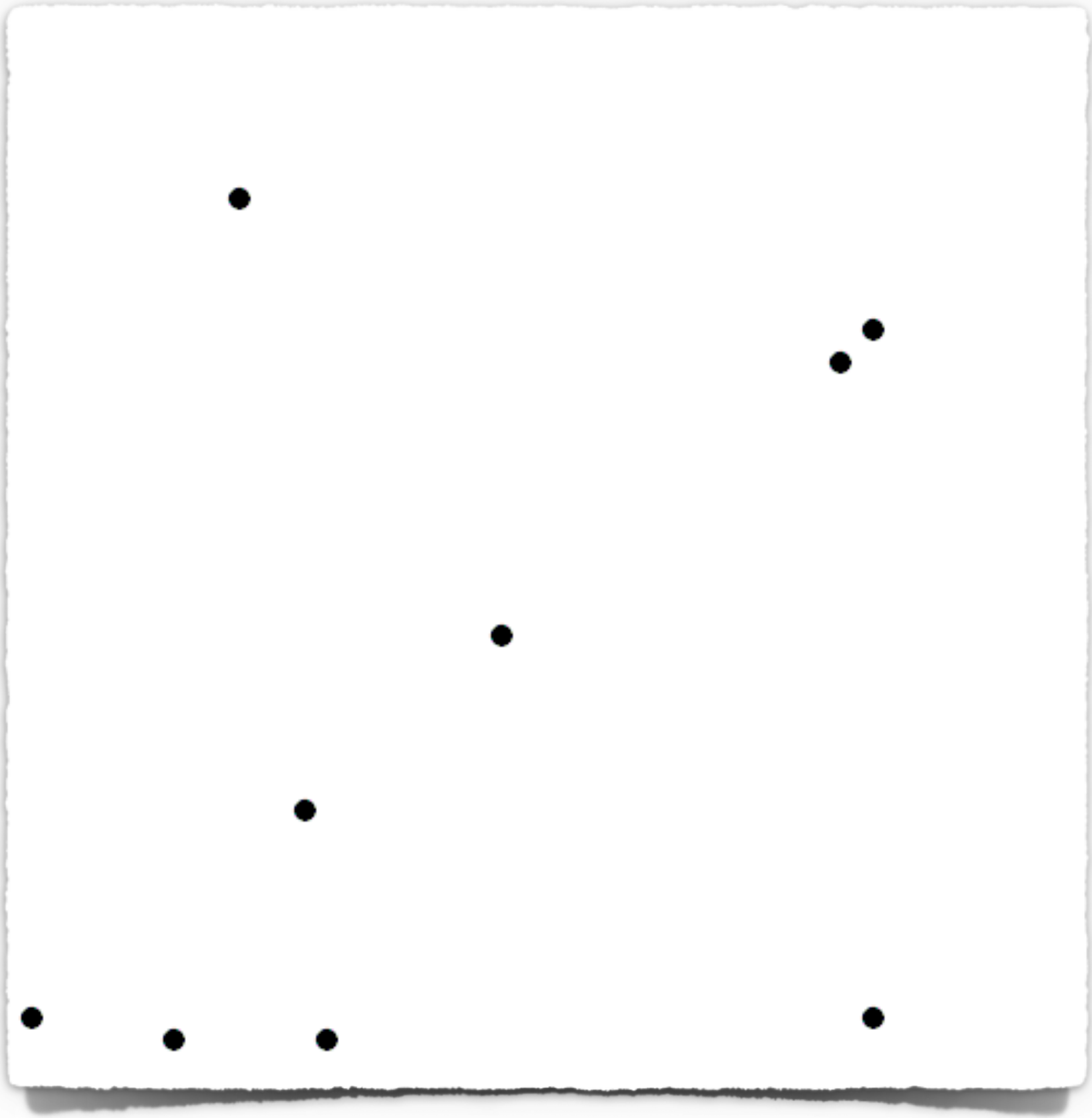
Nature

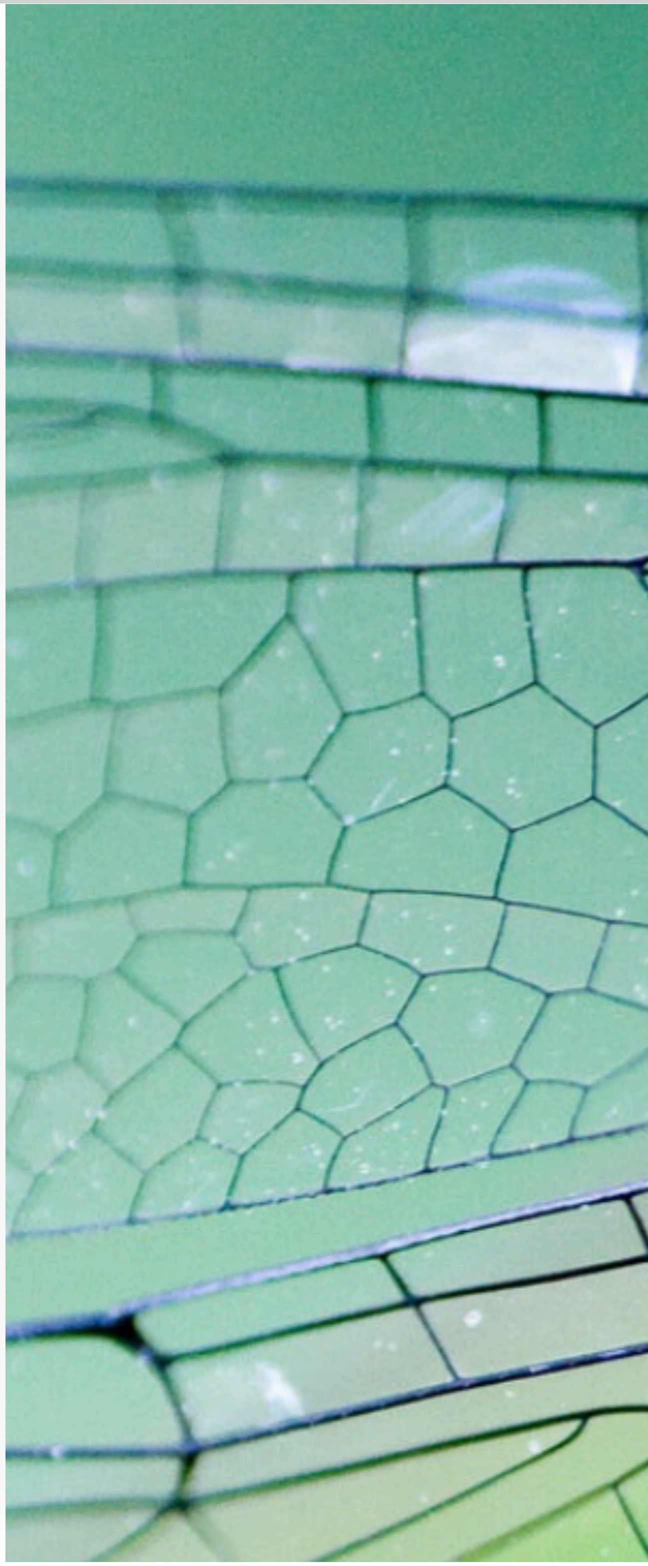


Scientific American

Nature







Scientific American

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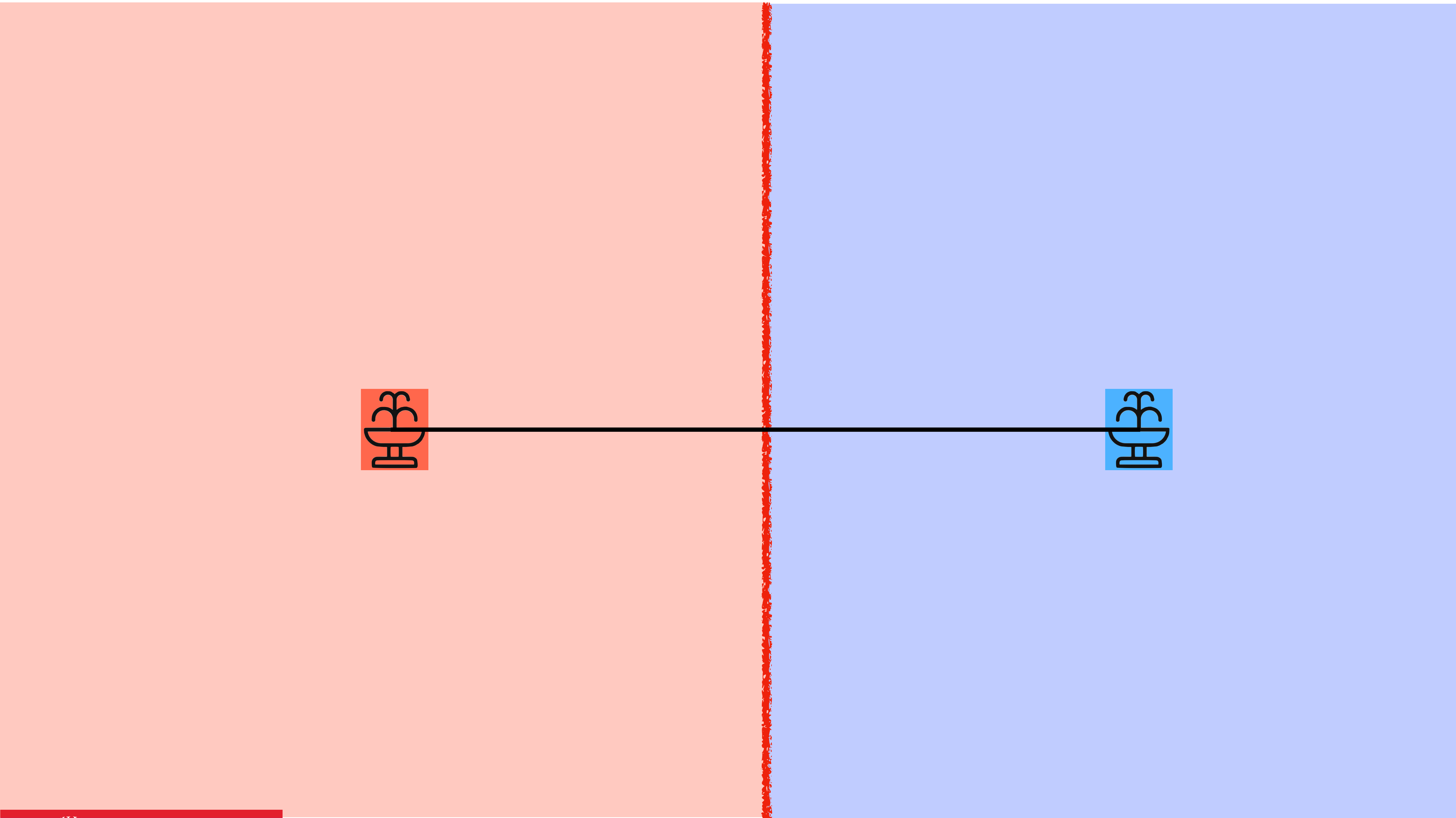
Manhattan Distances



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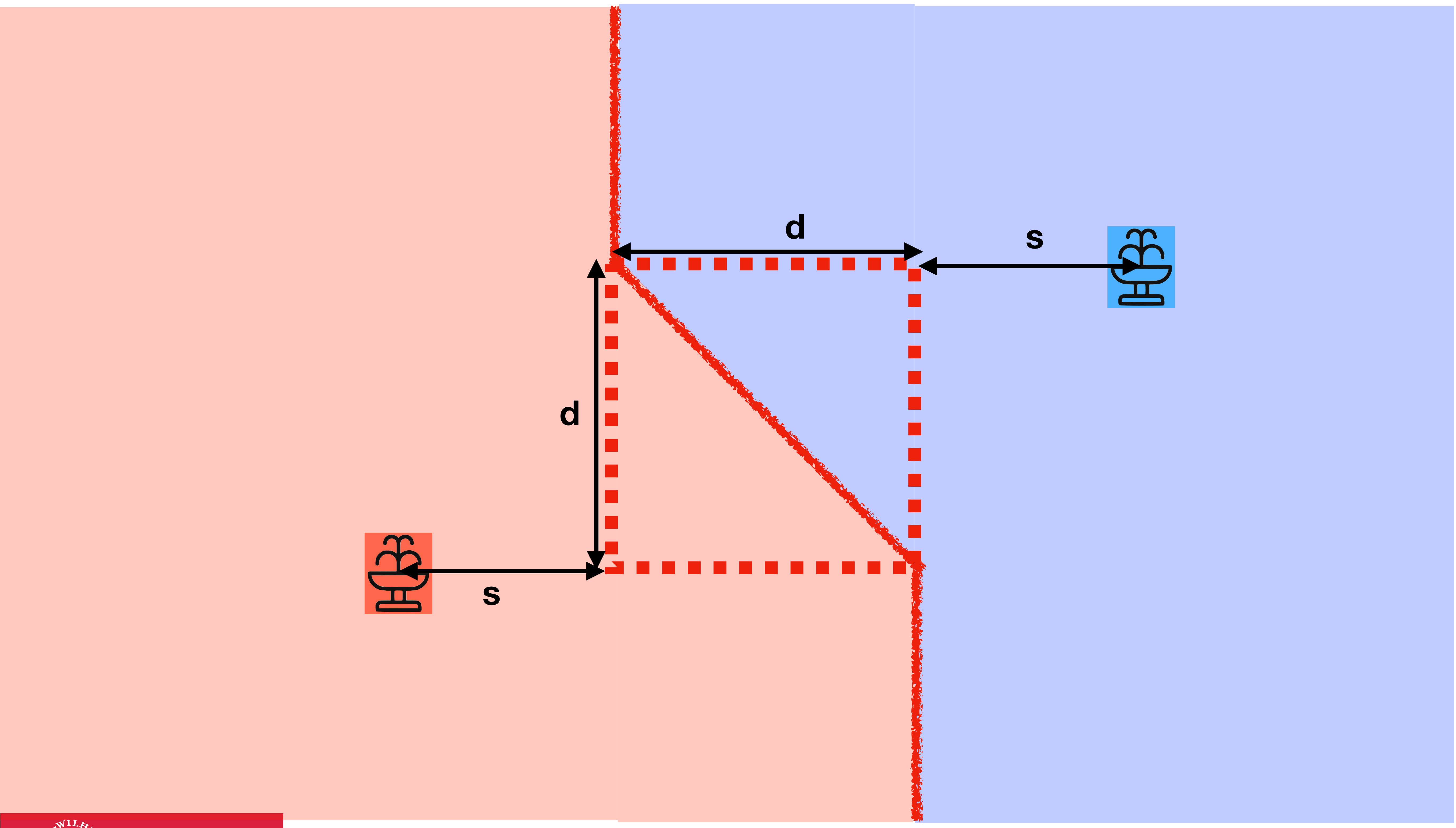
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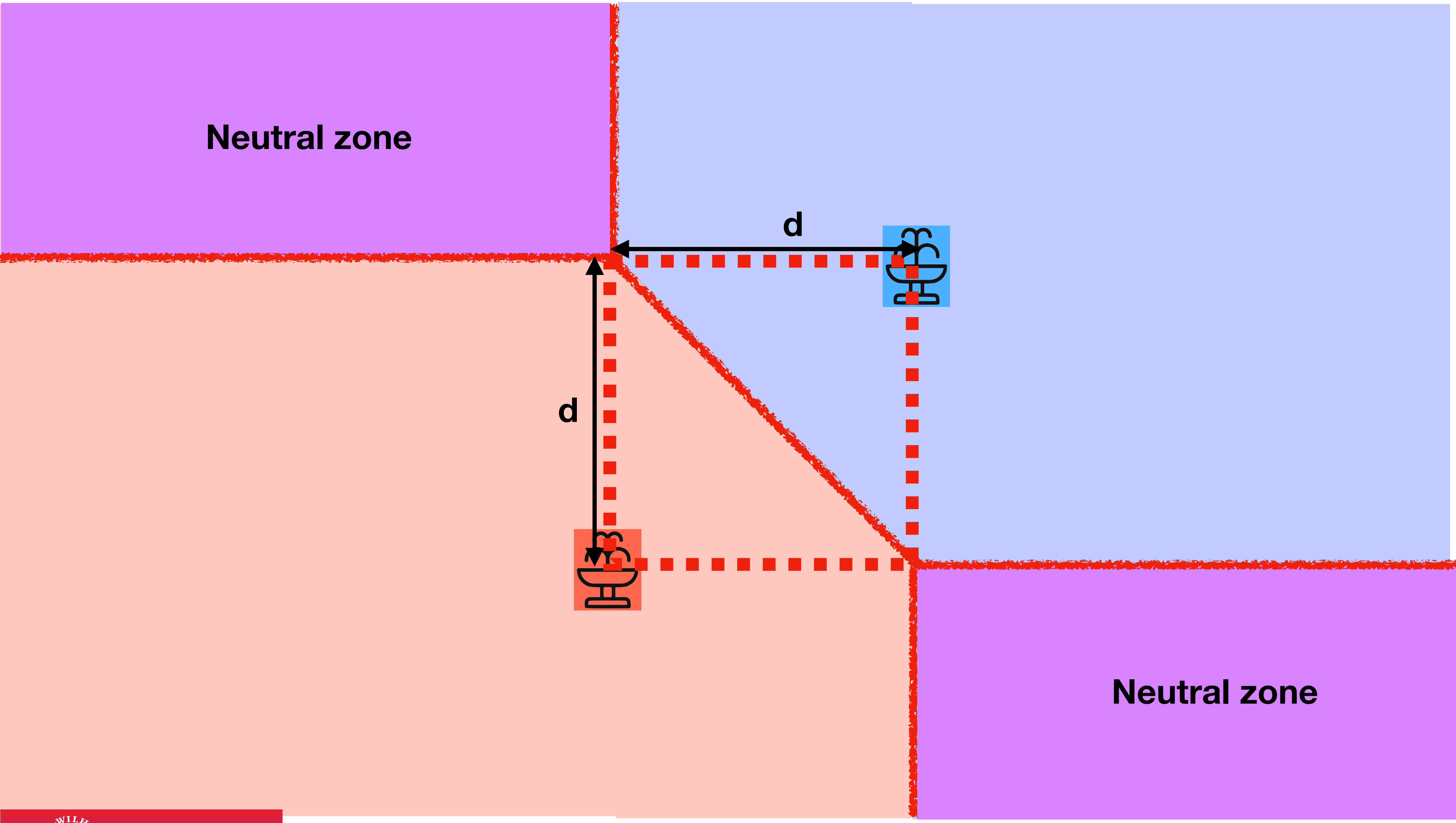

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Manhattan Distances

Neutral z

Neutral zone

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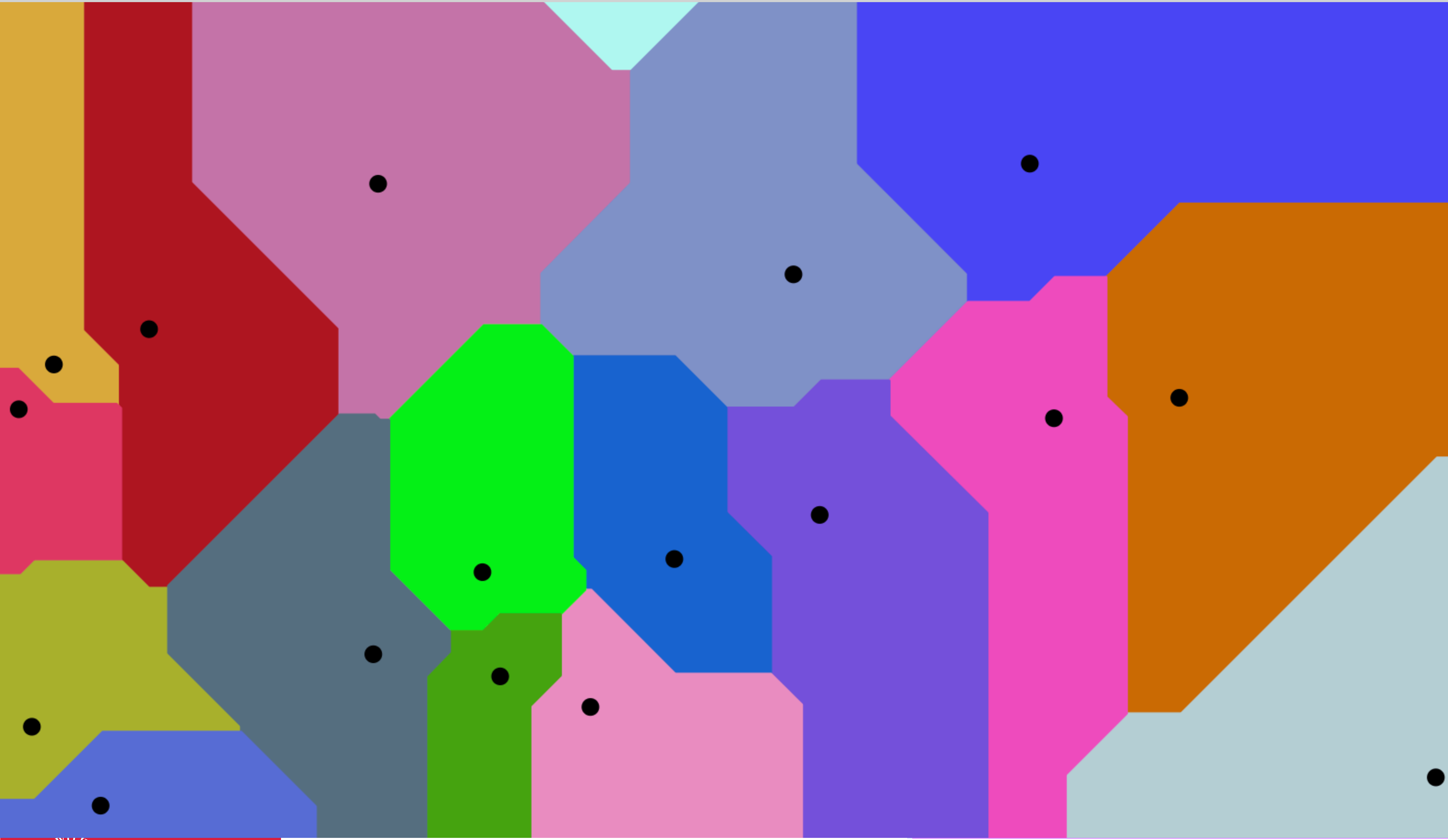
Manhattan Distances



Competitive Location Problems: Balanced Facility Location and the One-Round Manhattan Voronoi Game

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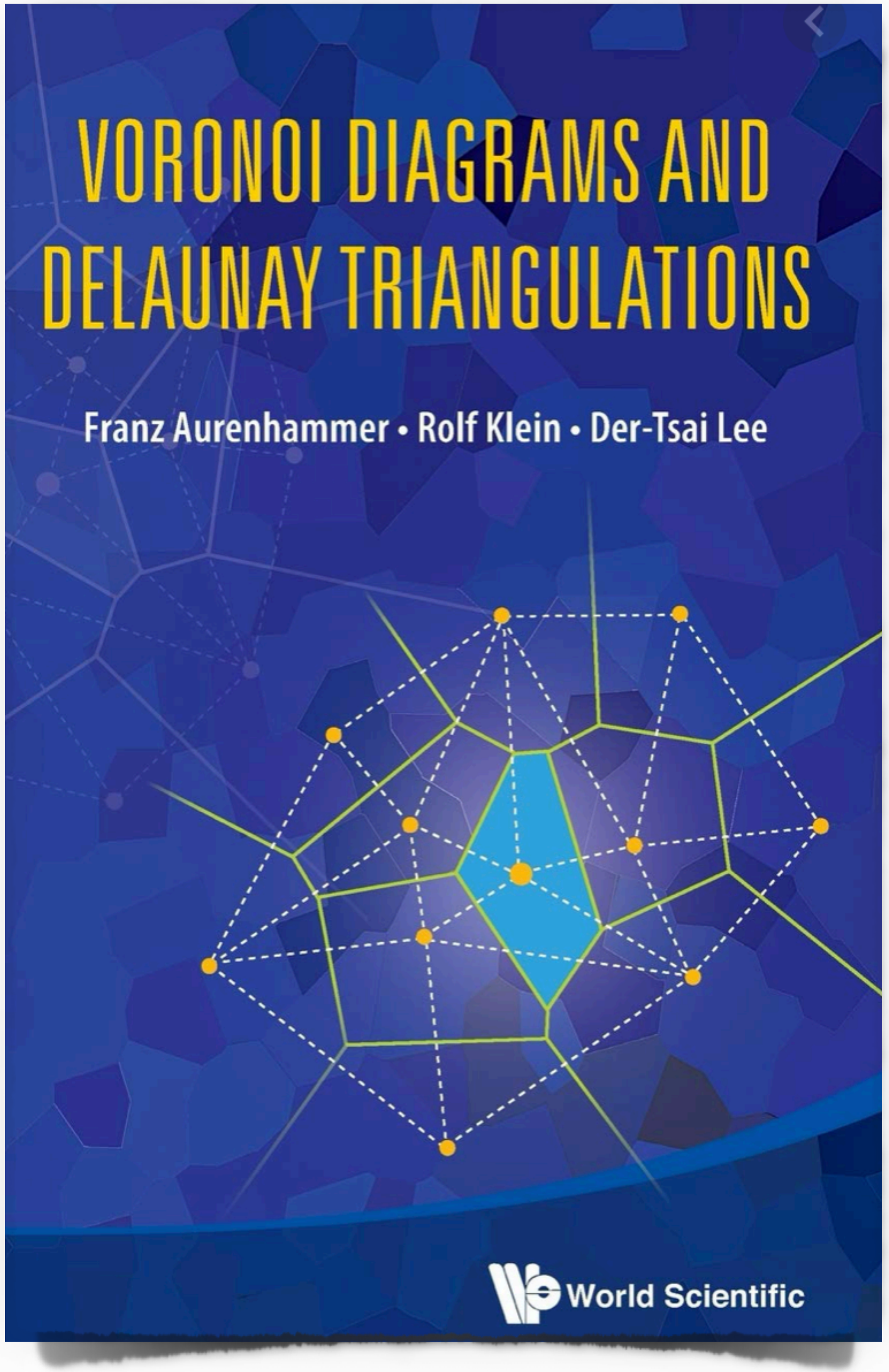
Abstract

We study competitive location problems in a continuous setting, in which facilities have to be placed in a rectangular domain R of normalized dimensions of 1 and $\rho \geq 1$, and distances are measured according to the Manhattan metric. We show that the family of *balanced* facility configurations (in which the Voronoi cells of individual facilities are equalized with respect to a number of geometric properties) is considerably richer in this metric than for Euclidean distances. Our main result considers the *One-Round Voronoi Game* with Manhattan distances, in which first player White and then player Black each place n points in R; each player scores the area for which one of its facilities is closer than the facilities of the opponent. We give a tight characterization: White has a winning strategy if and only if $\rho \geq n$; for all other cases, we present a winning strategy for Black.


Lecture Notes in
Computer Science
400
Rolf Klein
Concrete and Abstract
Voronoi Diagrams


VORONOI DIAGRAMS AND
DELAUNAY TRIANGULATIONS
Franz Aurenhammer • Rolf Klein • Der-Tsai Lee
World Scientific

1. Introduction and Motivation
2. Definitions
3. Representing planar partitions
4. Properties
5. Fortune's algorithm
6. The Voronoi game
7. Summary and conclusions

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In the following: Distances and visualization in **Euclidean metric,** other metrics possible.

Definition 4.1

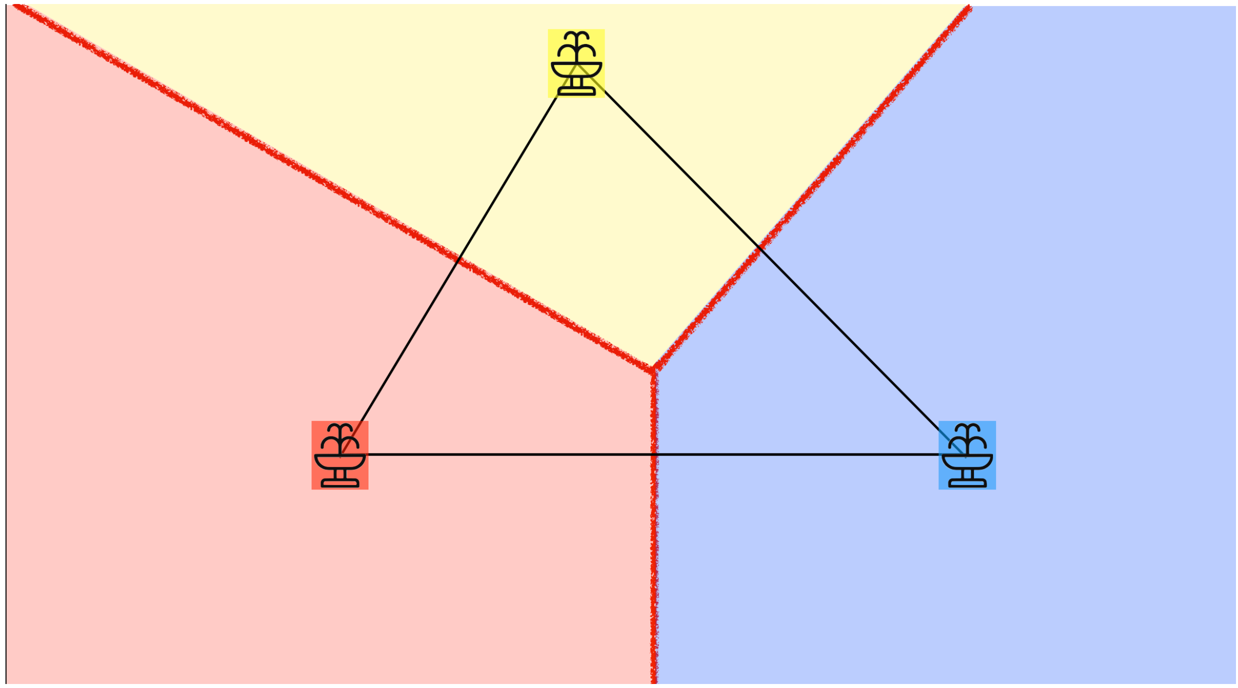
Voronoi region $V(p)$ of $p \in \mathcal{P}$:

$$V(p) := \{x \in \mathbb{R}^2 \mid \forall q \in \mathcal{P} : d(x,p) \leq d(x,q)\}$$

Problem 4.2

Given: Finite set of points $\mathcal{P}$ in $\mathbb{R}^2$

Wanted: For any $p \in \mathcal{P}$ find its Voronoi region



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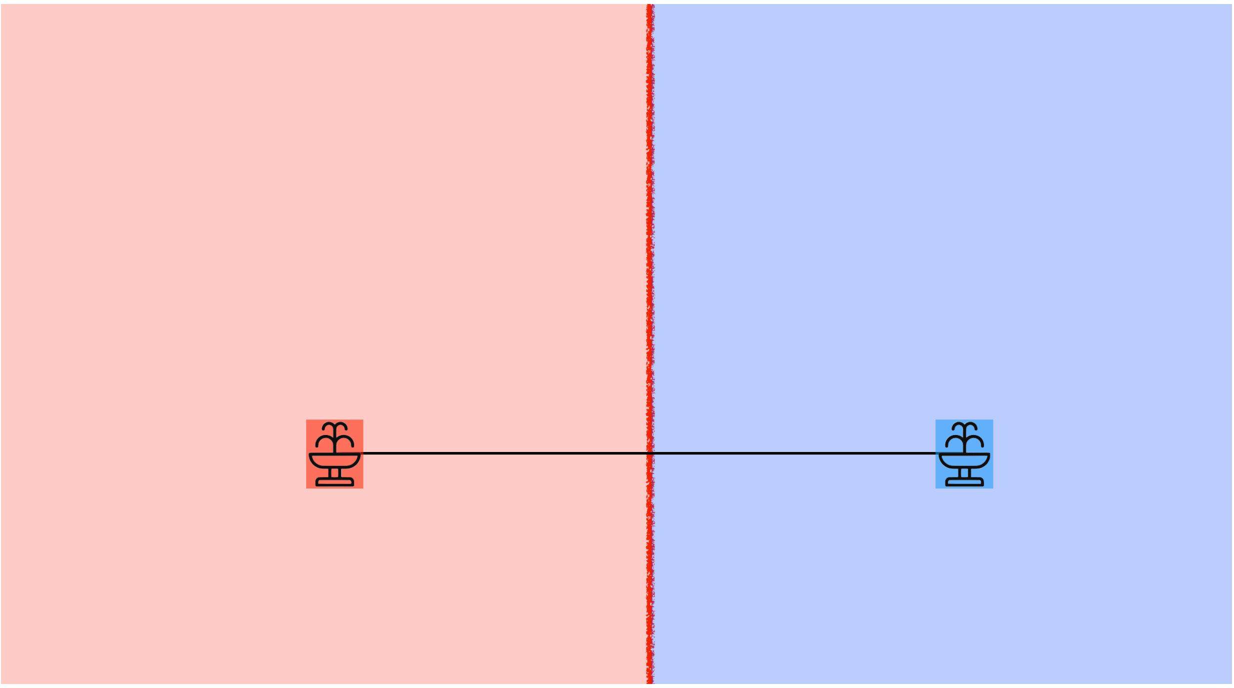
Definition 4.3

For $p \neq q \in \mathcal{P}$ the halfspace of p is $H(p,q) = \{x \in \mathbb{R}^2 \mid d(x,p) \leq d(x,q)\}$

For $p \neq q \in \mathcal{P}$ the bisector $B(p,q)$ with $p \in H(p,q), q \in H(q,p)$

is $B(p,q) = B(q,p) = H(p,q) \cap H(q,p)$

- i.e., the set of all points with equal distance from p and q.



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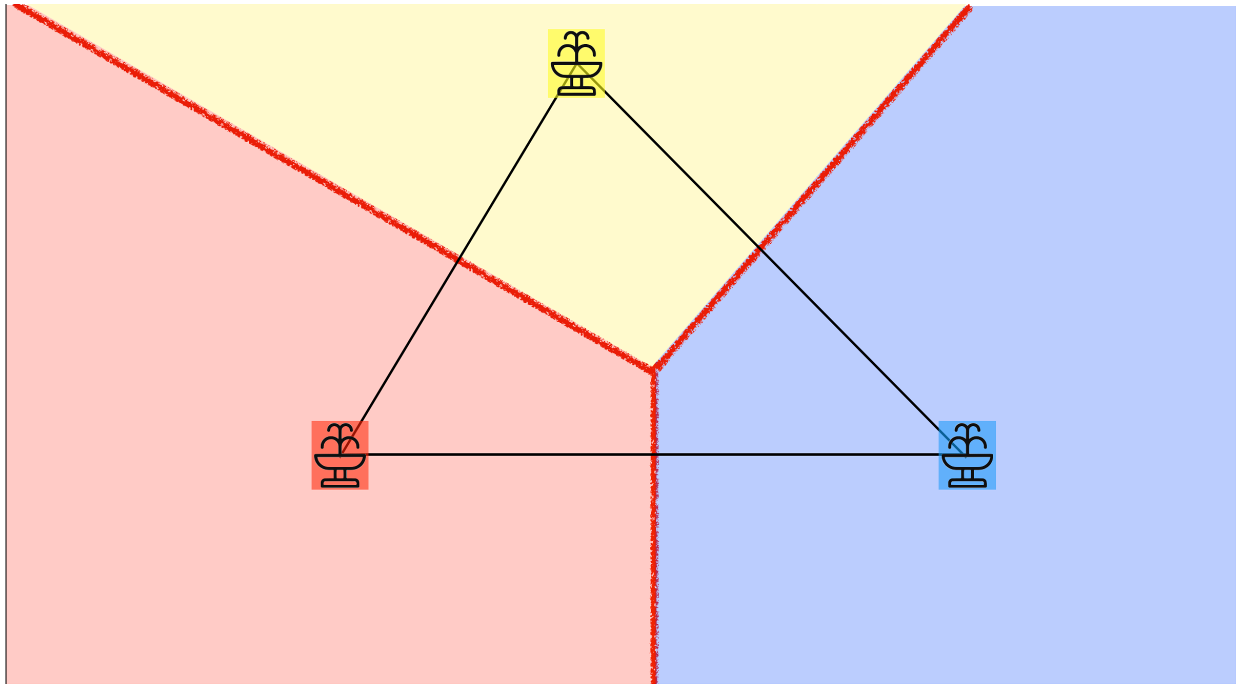
- i.e., the set of all points with equal distance from p and q.

Corollary 4.4

Voronoi region $V(p)$ of a point $p \in \mathcal{P}$:

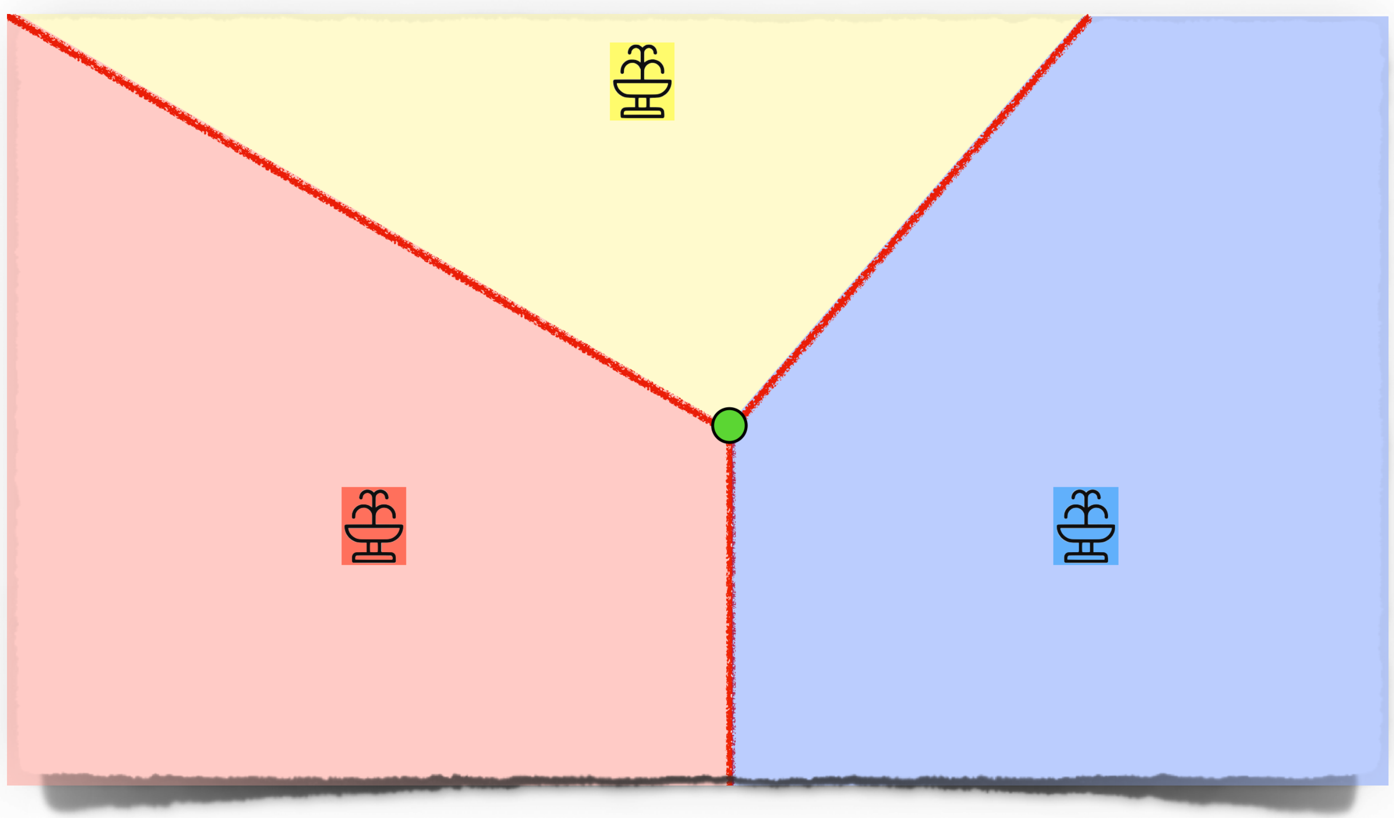
$$V(p) = \bigcap_{q \in \mathcal{P} \setminus \{p\}} H(p,q)$$



Lemma 4.5

$V(p_0), \ldots, V(p_{n-1})$ partition the plane into:

1. Convex set of points that are closest to precisely one site.
2. Sets of points (segments, rays or lines) that are closest to precisely two sites.
3. A finite number of points that are closest to at least three sites.



Proof:

Each $x \in \mathbb{R}^2$ has at least one closest site $\Rightarrow \mathbb{R}^2$ is completely partitioned.

Let $x \in \mathbb{R}^2$ closest to at least three sites $(q_1, q_2, q_3 \in \mathcal{P}$)

$$\Rightarrow d(x, q_1) = d(x, q_2) = d(x, q_3) = \min_{q \in \mathcal{P}} d(x, q)$$

$\Rightarrow x$ center of circumcircle $\bigcirc$ with $q_1, q_2, q_3 \in \bigcirc$

$\Rightarrow x$ is uniquely defined for each triple, of which there is a finite number.

Let $x \in \mathbb{R}^2$ be closest to precisely two sites ($q_1, q_2 \in \mathcal{P}$)

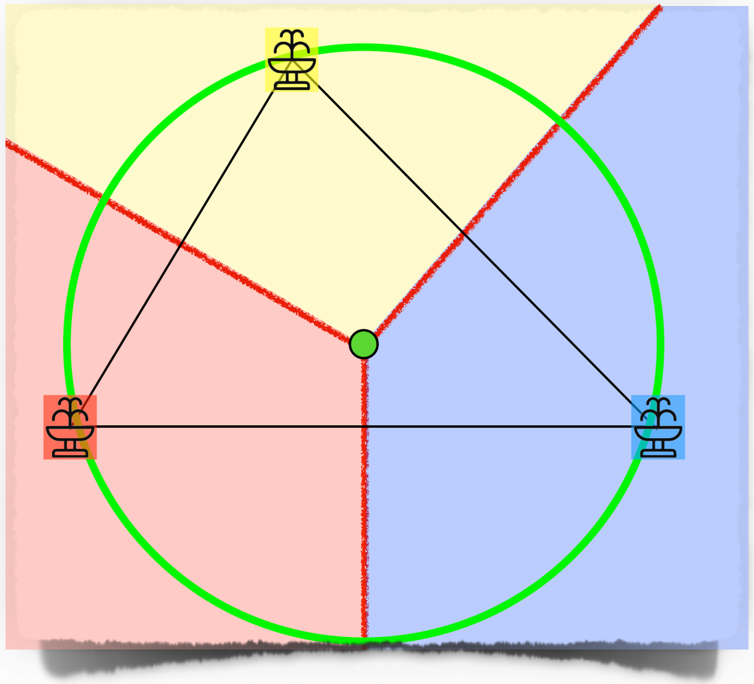
$\Rightarrow x$ belongs to bisector $B(q_1, q_2)$

Let $x \in \mathbb{R}^2$ be closest to precisely one site ($q_1 \in \mathcal{P}$)

$\Rightarrow x \in V(q_1)$ (in the interior)

And: Voronoi regions are separated by bisectors.

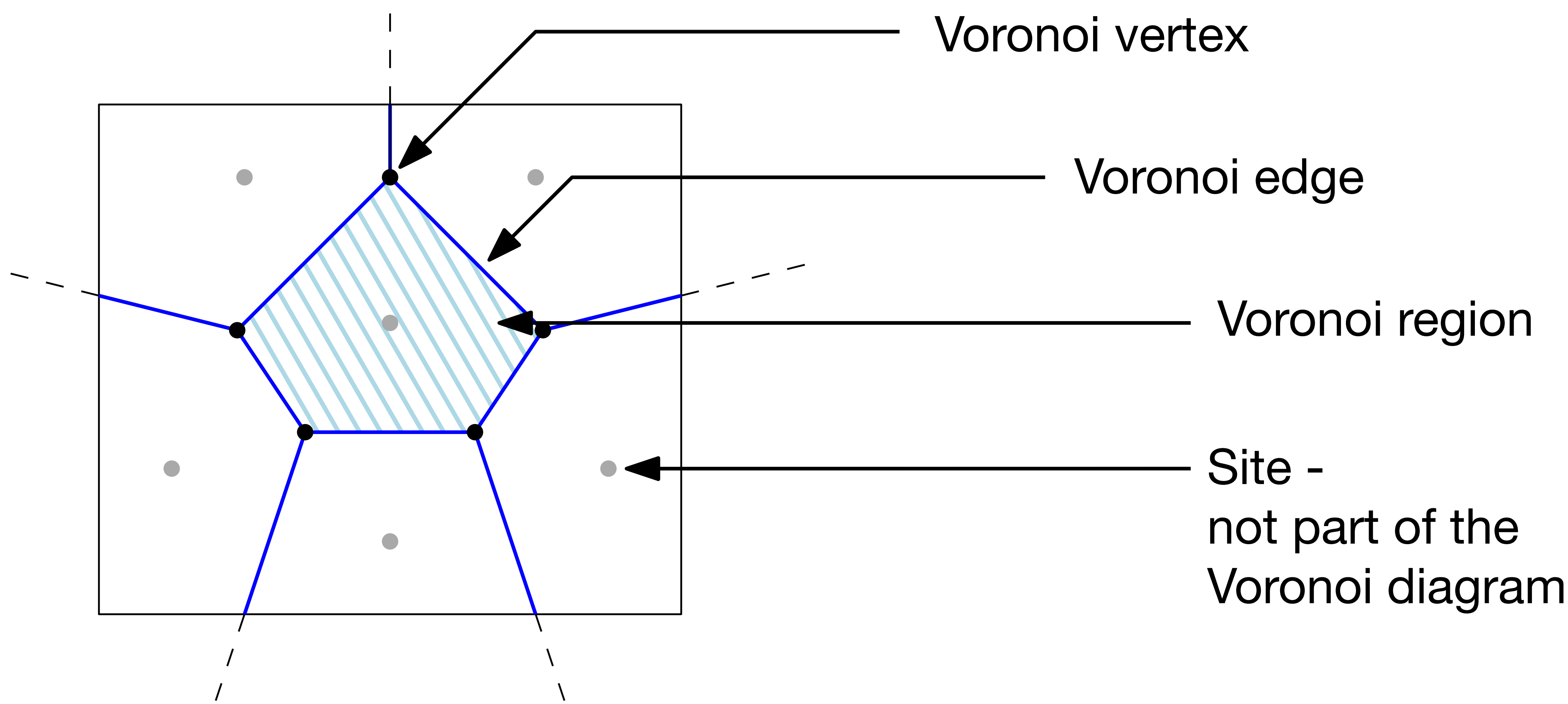
$\square$



Definition 4.6

The Voronoi diagram $Vor(\mathcal{P})$ is a partition of $\mathbb{R}^2$ into Voronoi regions with:

Voronoi vertices:	Points closest to at least three sites
Voronoi edges (or bisectors):	Points closest to precisely two sites
Voronoi regions:	Points closest to precisely one site

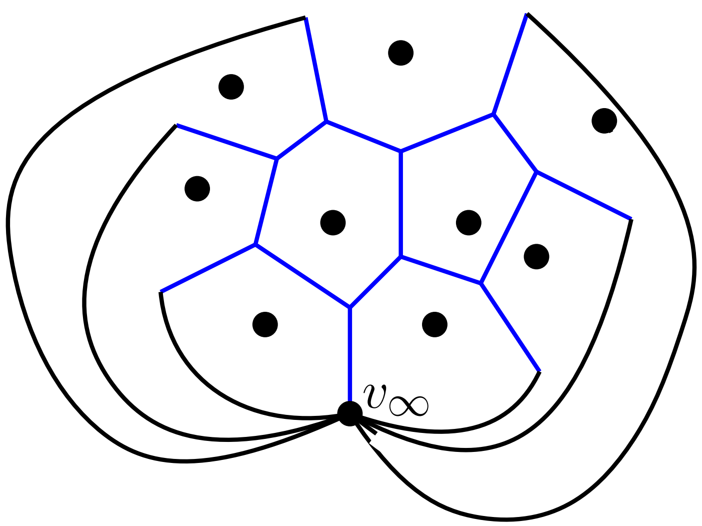


Theorem 4.7

$Vor(\mathcal{P})$ has precisely n Voronoi regions, at most $2n-5$ Voronoi vertices and at most $3n-6$ Voronoi edges.

Proof:

- Each $p \in \mathcal{P}$ induces a region.
- Embedding as a planar graph
  $\rightarrow$ Consider extra vertex v_∞



- Euler's formula: $v - e + f = 2$
- Number f of faces: Number n of Voronoi regions
- Number e of edges: Number n_e of Voronoi edges
- Number v of vertices: Number n_v of Voronoi vertices + 1
- Vertex degrees ≥ 3
- Edge increases sum of degrees by 2

$$2n_e \geq 3(n_v+1) \text{ \& } (n_v+1) - n_e + n \overset{(\dagger)}{=} 2 \quad \Leftrightarrow n_v \overset{(\star)}{=} n_e - n + 1$$

$$\overset{(\star)}{\Rightarrow} \text{1.: } 2n_e \geq 3(2+n_e-n) \Rightarrow 3n-6 \geq n_e$$

$$\overset{(\dagger)}{\Rightarrow} \text{2.: } 2(n_v+1+n-2) \geq 3(n_v+1) \Rightarrow 2n-5 \geq n_v$$

□

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Observation:

$Vor(\mathcal{P})$ can be considered an embedded planar graph.

Representing embedded graph:

- Algorithm for constructing $Vor(\mathcal{P})$

  $\rightarrow$ Efficient representation of $Vor(\mathcal{P})$ required

- Objects:
  - Vertices with coordinates
  - Edges (Pointers to end points)
  - Faces (CCW sequence of boundary edges)

Doubly-Connected Edge List [Muller und Preparata, 1978]

Theoretical Computer Science 7 (1978) 217–236.

FINDING THE INTERSECTION OF T POLYHEDRA*

D. E. MULLER[1] and F. I
Coordinated Science Laboratory, Ur

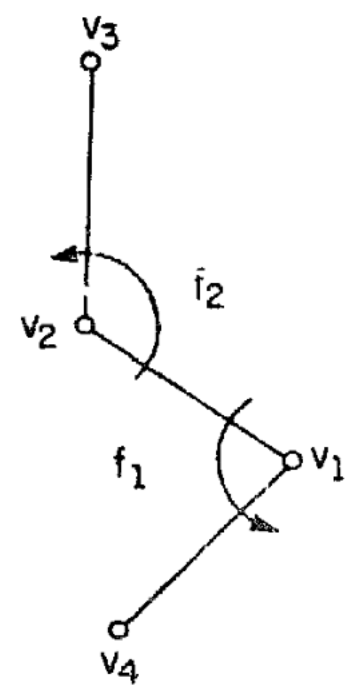
Communicated by M. Nivat
Received November 1977
Revised March 1978

Abstract. Given two convex polyhe test whether their intersection is em find a point in the intersection a algorithm runs in time O ($n \log n$) polyhedra. The part of the algorith upon the fact that if a point in the i from the convex hull of suitable ge point.

2. Derivation of a doubly connected edge list for a planar graph

Let $V=\{v_1,\ldots,v_n\}$ and $E=\{e_1,\ldots,e_m\}$ be the sets of vertices and edges respectively, of a planar graph embedded in the plane without crossing edges. We assume that (V, E) is represented as follows. To vertex $v_j \in V$ there corresponds cell $H[j]$ of an array $H[1:n]$, which contains a pointer to the first term of the cyclic list of the edges incident on v_j arranged in the order in which they appear as one proceeds counterclockwise around v_j. The latter lists are realized by means of two ... ERTEX[i], NEXT[i]) is the ... h (V, E) is precisely the one ... which constructs the convex ... surface of a convex poly- ... this collection of lists the

... commonly used represen- ... he dual graph, i.e., the graph ... ph, is not readily available.



	V1	V2	F1	F2	P1	P2
1						
2						
⋮						
a1	1	2	1	2	a2	a3
a2	4	1	1			
a3	2	3		2		

Fig. 1. Illustration of the DCEL.

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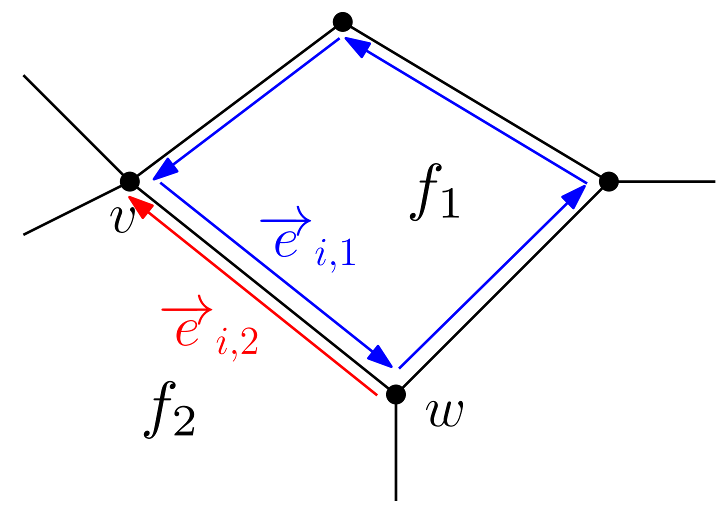
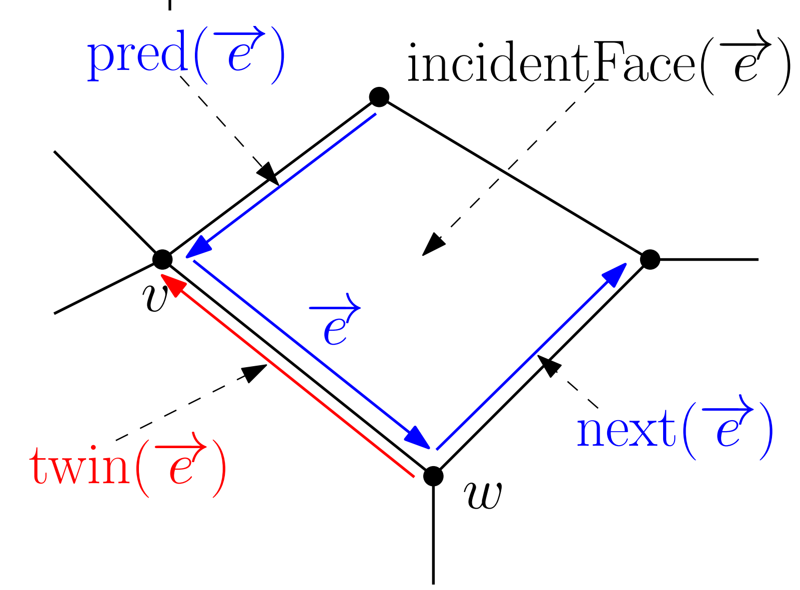
- Separate storage of vertices, edges and faces
- Subdividing edges into half-edges: $e_i = (v, w) \rightarrow \overrightarrow{e}_{i,1} = (v, w), \overrightarrow{e}_{i,2} = (w, v)$
- $e = (v, w)$ separates two regions $f_1, f_2 \Rightarrow$

$$\left[\begin{array}{c} \overrightarrow{e}_{i,1} = (v, w) \text{ is on boundary of } f_1 \\ \Leftrightarrow \\ w \text{ follows } v \text{ on boundary of } f_1 \text{(CCW)} \end{array} \right]$$



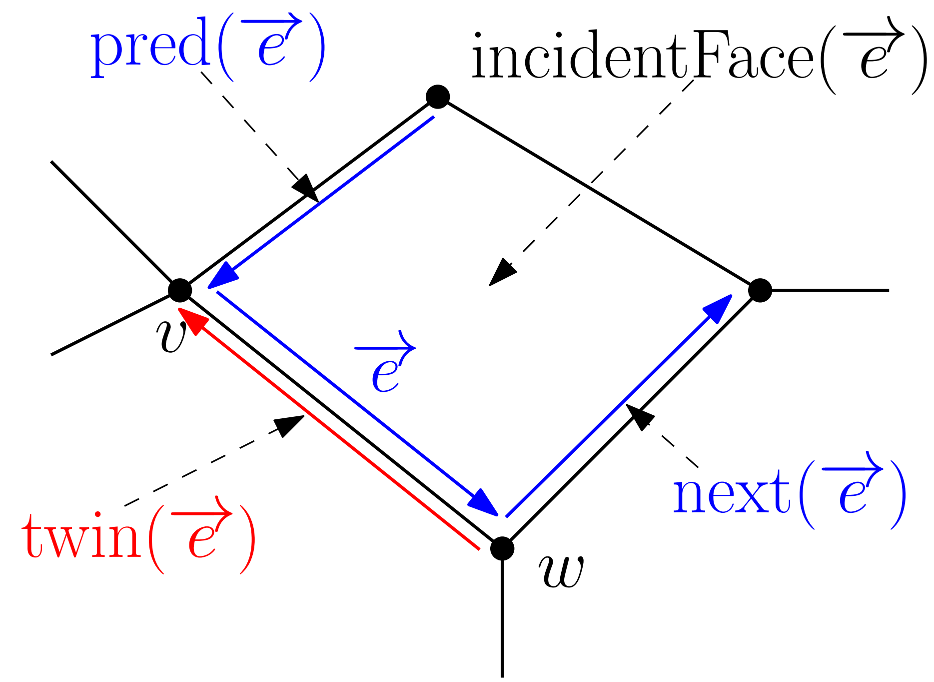
- Half-edge lies on boundary of unique face

  $\Rightarrow$ Half-edges have unique predecessor and successor



Representation:

- Half-edge $\overrightarrow{e}$ stores:
  - Pointer $\text{incidentFace}(\overrightarrow{e})$ to the face f bounded by edge $\overrightarrow{e}$
  - Pointer $\text{next}(\overrightarrow{e})$ to successor edge
  - Pointer $\text{pred}(\overrightarrow{e})$ to predecessor edge
  - Pointer $\text{origin}(\overrightarrow{e})$ to start vertex
  - Pointer $\text{twin}(\overrightarrow{e})$ to partner half-edge



- In essence: Storing boundary edges of a face f: Doubly linked list

- Storing vertices. Each vertex v stores:
  - Coordinates
  - Pointer to incident half-edge $\overrightarrow{e}$ with $\text{origin}(\overrightarrow{e}) = v$

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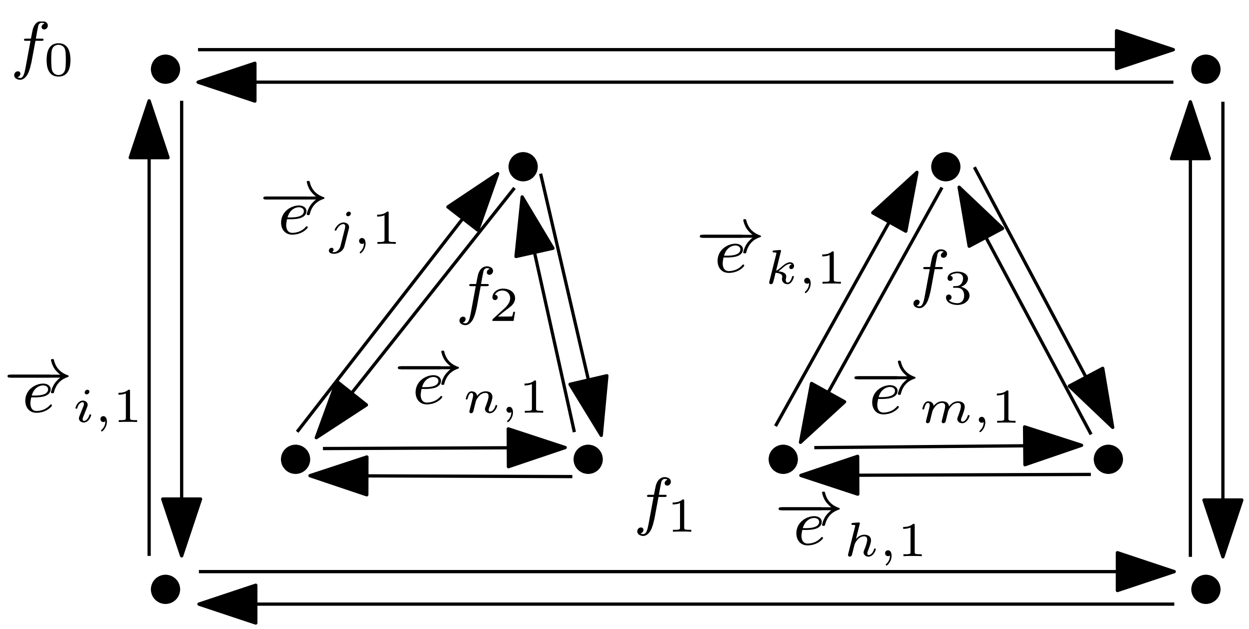
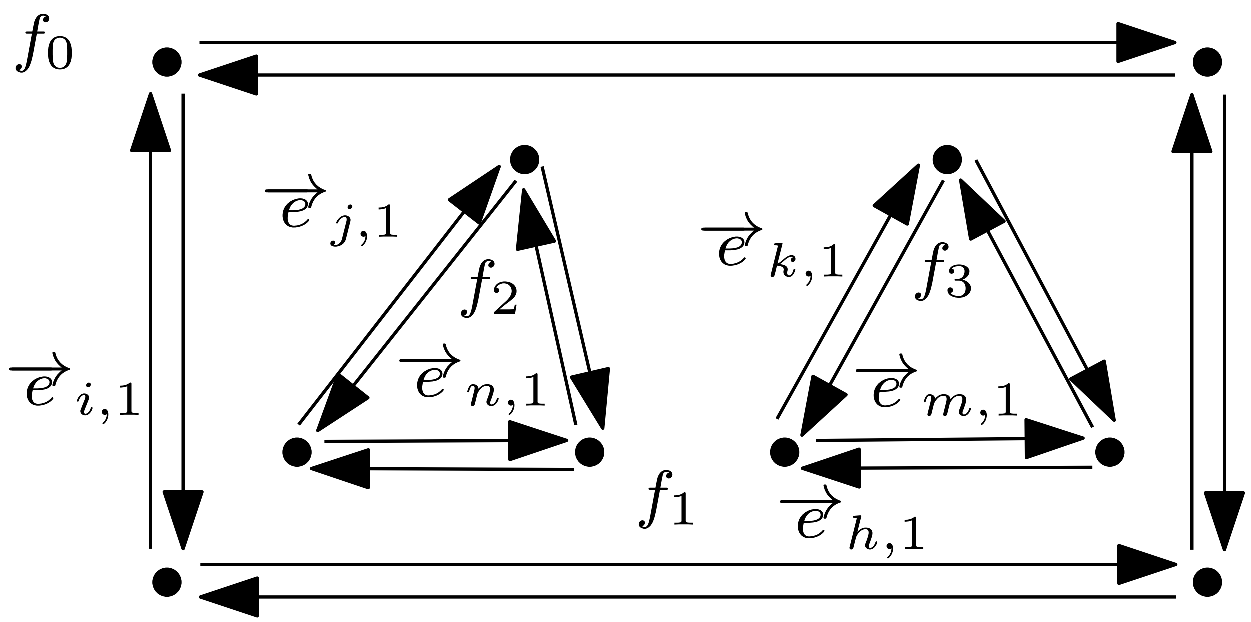
Storing faces:

- Exterior and interior boundaries (holes)
- Face f stores pointer $\text{outerComponent}(f)$ to some edge on outer boundary

  Exterior face: $\text{outerComponent}(f) = \text{null}$
- Face f stores list $\text{innerComponents}(f)$
  For each interior boundary one entry: pointer to some edge of component





Storing faces:

Face	outerComponent	innerComponents
f_0	null	$\vec{e}_{i,1}$
f_1	$\vec{e}_{h,1}$	$\{\vec{e}_{j,1}, \vec{e}_{k,1}\}$
f_2	$\vec{e}_{n,1}$	null
f_3	$\vec{e}_{m,1}$	null

Doubly-Connected Edge List (DCEL):

- Storing vertices, edges, faces in table
- Pointers to connect data; in particular: implicit storage of boundaries as doubly linked lists
- Constant memory per vertex and edge
- Total memory for faces: linear
- Total memory: linear
- Operations on DCEL $\rightarrow$ Exercise

Thank you for today!