Bounding the tripartite-circle crossing number of complete tripartite graphs

Charles Camacho[1], Silvia Fernández-Merchant[2], Marija Jelic[3], Rachel Kirsch[4], Linda Kleist[5], Elizabeth Bailey Matson[6], and Jennifer White[7]

[1] Oregon State University camachoc@math.oregonstate.edu
[2] California State University silvia.fernandez@csun.edu
[3] University of Belgrade marijaj@matf.bg.ac.rs
[4] London School of Economics r.kirsch1@lse.ac.uk
[5] Technische Universität Berlin kleist@math.tu-berlin.de
[6] Alfred University matson@alfred.edu
[7] Saint Vincent College jennifer.white@stvincent.edu

1 Introduction

The *crossing number* of a graph G, denoted by $\mathrm{cr}(G)$, is the minimum number of edge-crossings over all drawings of G on the plane. To date, even the crossing numbers of complete and complete bipartite graphs are open. For the crossing number of the complete bipartite graph Zarankiewicz [6] showed that

$$\mathrm{cr}(K_{m,n}) \le \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor,$$

and conjectured that equality holds. Harary and Hill [4] and independently Guy [3] conjectured that the crossing number of the complete graph K_n is

$$\mathrm{cr}(K_n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor =: H(n).$$

The construction of Harary and Hill is a so-called *cylindrical drawing*, in which the vertices lie on the circles of a cylinder, and edges of the graph cannot cross the circles. Towards the Zarankiewicz Conjecture, these drawings can be restricted to *bipartite cylindrical drawings*, in which each set of the vertex partition lies on its own circle. A *k-circle drawing* of a graph G is a drawing of G in the plane where the vertices are placed on k disjoint circles and the edges do not cross the circles. The *k-circle crossing number* of a graph G is the minimum number of crossings in a k-circle drawing of G. For the special case when G is a k-partite graph, we can further require that each of the k vertex classes is placed on one of the k circles. The corresponding crossing number is called the *k-partite-circle crossing number* and is denoted by $\mathrm{cr}_{\textcircled{k}}(G)$. Richter and Thomassen [5] showed that $\mathrm{cr}_{\textcircled{2}}(K_{n,n}) = n\binom{n}{3}$. Ábrego, Fernández-Merchant, and Sparks [1] generalized this result for $m \le n$ to

$$\mathrm{cr}_{\textcircled{2}}(K_{n,m}) = \binom{n}{2}\binom{m}{2} + \sum_{0 \le i < j \le m-1} \left(\left\lfloor \frac{n}{m} j \right\rfloor - \left\lfloor \frac{n}{m} i \right\rfloor \right) \left(\left\lfloor \frac{n}{m} j \right\rfloor - \left\lfloor \frac{n}{m} i \right\rfloor - n \right).$$

2 Our Results

We investigate the tripartite-circle crossing number of the complete tripartite graph. Drawings that minimize the number of crossings are *good*, i.e., no edge crosses itself and any two edges share at most one point. We develop methods to count the number of crossings in good drawings and provide concrete drawings to obtain upper bounds.

Theorem 1. *For any integers m, n, and p,*

$$\sum_{\substack{\{x,y\}\in\binom{\{m,n,p\}}{2}\\ z\in\{m,n,p\}\setminus\{x,y\}}} \left(cr_{\textcircled{2}}(K_{x,y}) + \ xy \left\lfloor\frac{z}{2}\right\rfloor \left\lfloor\frac{z-1}{2}\right\rfloor \right) \leq cr_{\textcircled{3}}(K_{m,n,p})$$

$$\leq \sum_{\substack{\{x,y\}\in\binom{\{m,n,p\}}{2}\\ z\in\{m,n,p\}\setminus\{x,y\}}} \left(\binom{x}{2}\binom{y}{2} + \ xy \left\lfloor\frac{z}{2}\right\rfloor \left\lfloor\frac{z-1}{2}\right\rfloor \right).$$

For the balanced case, Figure 1 illustrates the drawing, and the formulas simplify to

$$3n\binom{n}{3} + 3n^2\left\lfloor\frac{n}{2}\right\rfloor \left\lfloor\frac{n-1}{2}\right\rfloor \leq \mathrm{cr}_{\textcircled{3}}(K_{n,n,n}) \leq 3\binom{n}{2}^2 + 3n^2\left\lfloor\frac{n}{2}\right\rfloor \left\lfloor\frac{n-1}{2}\right\rfloor.$$

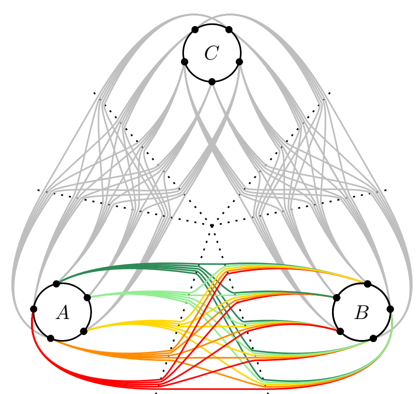

Fig. 1. A tripartite-circle drawing of $K_{n,n,n}$ proving the upper bound.

Connection to the Harary-Hill Conjecture The drawings of K_n presented by Harary and Hill [4] have $H(n)$ crossings and consist of a 2-circle drawing of $K_{n/2,n/2}$ together with all straight line segments joining vertices on the same circle. Moreover, Blažek and Koman [2] presented a 1-circle drawing of K_n with $H(n)$ crossings. Therefore it has been asked whether a 3-circle drawing of $K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}$ together with all straight line segments joining vertices on the same circle can achieve $H(n)$ crossings. Our result proves that such a drawing does not exist.

References

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