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Bounding the tripartite-circle crossing number of complete tripartite graphs

Problem

In a *tripartite-circle drawing* of $K_{m,n,p}$, each part of the vertex partition is placed on one of three disjoint circles in the plane and the edges do not cross the circles. The *tripartite-circle crossing number* $\mathrm{cr}_{③}(K_{m,n,p})$ is the minimum number of crossings among all tripartite-circle drawings of $K_{m,n,p}$.

Results

General case:

$$\sum_{\substack{\{x,y\}\in\binom{\{m,n,p\}}{2}\\ z\in\{m,n,p\}\setminus\{x,y\}}}\left[\mathrm{cr}_{②}(K_{x,y}) + xy\left\lfloor\frac{z}{2}\right\rfloor\left\lfloor\frac{z-1}{2}\right\rfloor\right] \le \mathrm{cr}_{③}(K_{m,n,p}) \le \sum_{\substack{\{x,y\}\in\binom{\{m,n,p\}}{2}\\ z\in\{m,n,p\}\setminus\{x,y\}}}\left[\binom{x}{2}\binom{y}{2} + xy\left\lfloor\frac{z}{2}\right\rfloor\left\lfloor\frac{z-1}{2}\right\rfloor\right]$$

Balanced case:

$$3n\binom{n}{3} + 3n^2\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor \le \mathrm{cr}_{③}(K_{n,n,n}) \le 3\binom{n}{2}\binom{n}{2} + 3n^2\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$$

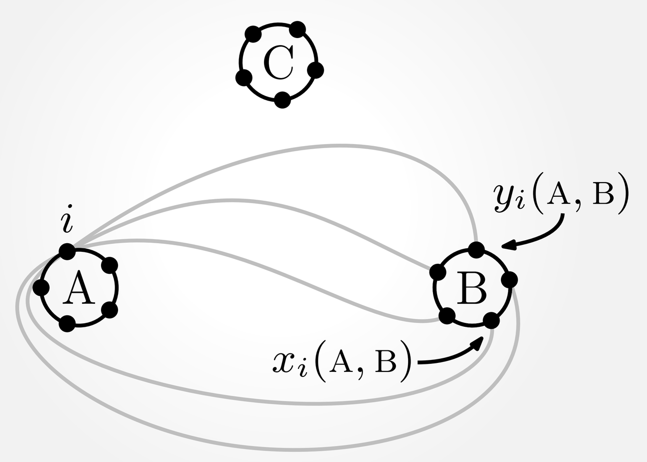
$$= \tfrac{5}{4}n^4 + O(n^3) \qquad\qquad = \tfrac{3}{2}n^4 + O(n^3)$$

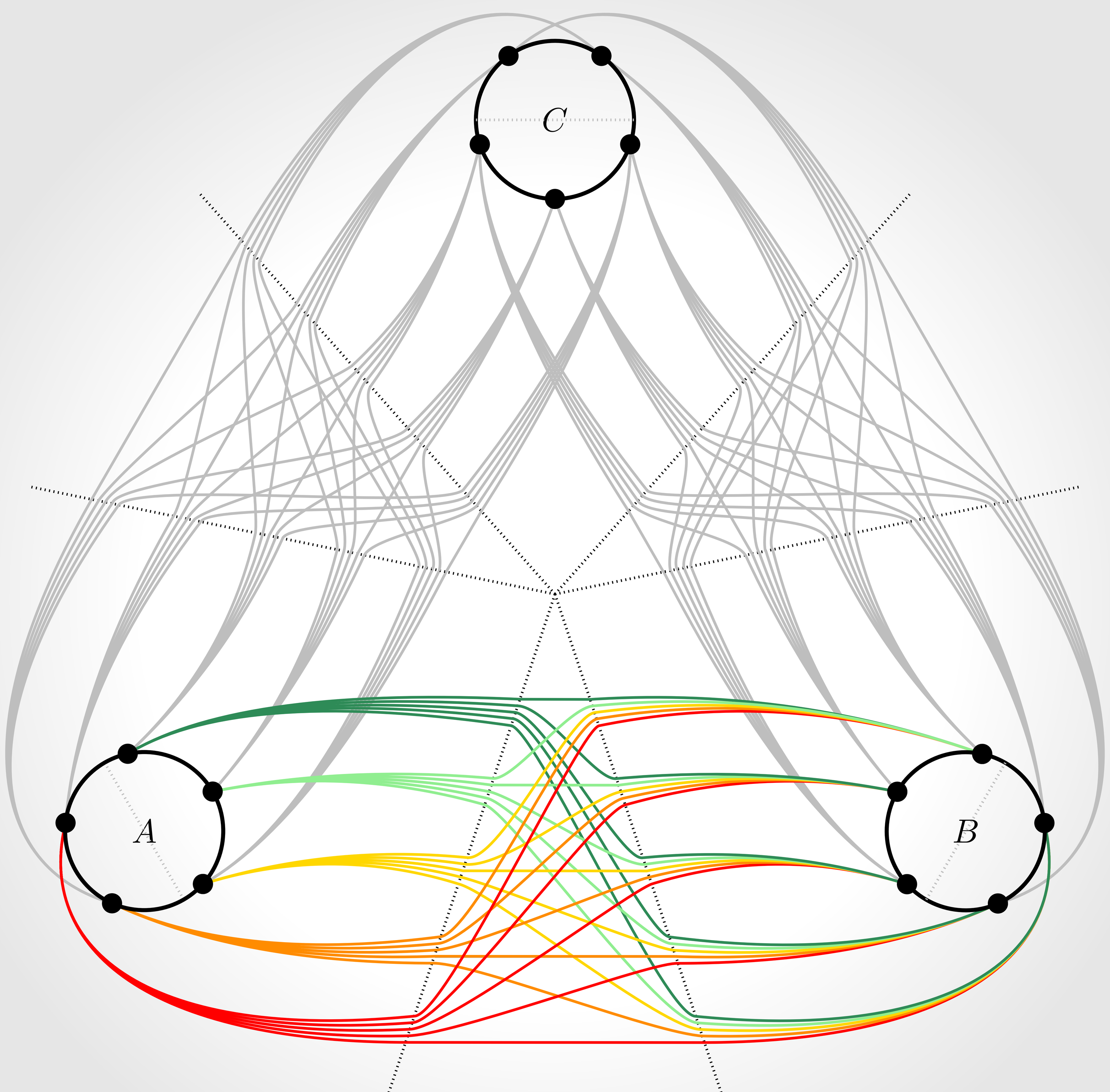
Crossing-minimal drawings are *good*:

no edge crosses itself — two edges share at most one point

Counting crossings I

For each vertex i on circle A, we identify two special neighbors on circle B: In a good drawing, the edges of i with B partition the exterior of B. The region containing A is enclosed by two edges of i and an arc of B; the clockwise first vertex on B is $x_i(\mathrm{A,B})$. Similarly, one region contains circle C and $y_i(\mathrm{A,B})$ is the cw first vertex on B.





Counting crossings II

In a good drawing where the circles A, B, C have a, b, c vertices, respectively. Richter and Thomassen [1] show that the number of crossings of type AB/AB is

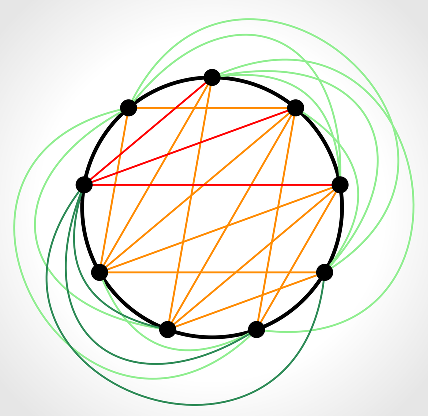
$$\sum_{1\le i<j\le a}\binom{d_{ij}}{2} + \binom{n-d_{ij}}{2}$$

for $d_{ij} := x_i(\mathrm{A,B}) - x_j(\mathrm{A,B}) \pmod b$.

We show that the number of crossings of type AB/BC can be expressed with $d_{ij} := y_i(\mathrm{A,B}) - y_j(\mathrm{B,C}) \pmod b$ as

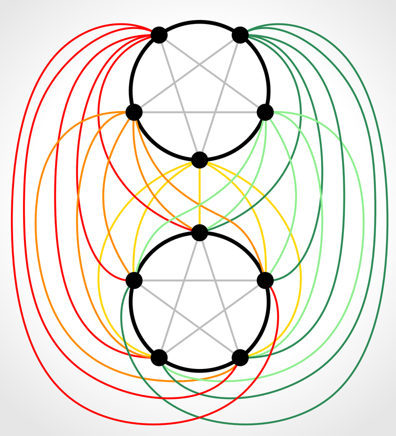
$$\sum_{\substack{1\le i\le a\\ 1\le j\le c}}\binom{d_{ij}}{2} + \binom{n-d_{ij}}{2}.$$

Motivation



1-circle drawing

The *Harary-Hill Conjecture* states that the crossing number of K_n is $\frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor =: H(n)$. In the 1950s, Harary and Hill showed that a crossing optimal 2-circle drawing of $K_{\frac{n}{2},\frac{n}{2}}$ together with all straight line segments joining the vertices on the same circle has $H(n)$ crossings.
In the 1960s, Blažek and Koman presented a 1-circle drawing of K_n with $H(n)$ crossings. Therefore, it has been asked whether a 3-circle drawing of $K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}$ together with all segments between vertices on the same circle can achieve $H(n)$ crossings. Our results prove that such a drawing does not exist.



2-circle drawing

Ref.: [1] R.B. Richter and C. Thomassen, Relations between crossing numbers of complete and complete bipartite graphs. The American Mathematical Monthly 104-2 (1997) 131–37.

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