Unit Contact Representations of Grid Subgraphs with Regular Polytopes in 2D and 3D

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Abstract. We present a strategy to construct unit proper contact representations (UPCR) for subgraphs of certain highly symmetric grids. This strategy can be applied to obtain graphs admitting UPCRs with squares and cubes, whose recognition is NP-complete.

We show that subgraphs of the square grid allow for UPCR with squares which strengthens the previously known cube representation. Indeed, we give UPCR for subgraphs of a d-dimensional grid with d-cubes. Additionally, we show that subgraphs of the triangular grid admit a UPCR with cubes, implying that the same holds for each subgraph of an Archimedean grid. Considering further polygons, we construct UPCR with regular $3k$-gons of the hexagonal grid and UPCR with regular $4k$-gons of the square grid.

1 Introduction

In this paper, we study *unit contact representations* (UCR) which are *contact representations* (CR) with congruent objects. We are particularly interested in *proper contacts* (PCR), that is, contacts are realized by segments of non-zero length in 2D or polygons with non-zero area in 3D. Contacts not of this type are disregarded. Typical objects considered are regular polygons and cubes.

1.1 Related Work

Considering homothetic copies of disks, the celebrated circle packing theorem of Koebe [10] states that every planar graph has a CR with disks. Schramm [11] gives the following generalization: Assigning a convex set in 2D with smooth boundary to each vertex, every planar graph has a CR with non-degenerated homothetic copies of the prescribed sets. Allowing convex sets without smooth boundary, this results in CRs with possibly degenerated homothetic copies of the assigned sets [11, 12]. Gonçalves, Lévêque and Pinlou [8] observe that this result can be exploited for triangle CRs with non-degenerated homothetic triangles for 4-connected planar triangulations. Felsner and Francis [6] employ possibly degenerated homothetic triangle representations to show that all planar graphs have a CR with cubes, where contacts are not necessarily proper.

Studying PCRs with polygons, Gansner et al. [7] show that every planar graph has a PCR with hexagons, but not always with pentagons.

Considering UCRs with congruent objects, Breu and Kirkpatrick [4] prove that the recognition of unit disk graphs is NP-complete; indeed, this holds even for the recognition of bounded-ratio disk graphs. For a survey on recognition-complexity results with balls and disks we refer to [9].

Czyzowicz et al. [5] are interested in discrete versions of unit disk graphs and study UCRs of non-rotated copies of regular k-gons with two different contact types: vertex-to-vertex and whole edge contact. For even k, these graph classes coincide (for odd k the second class is empty). It turns out that these graphs are also unit disk graphs and that the recognition of graphs allowing for a representation with $4k$-gons is NP-complete.

In 3D, Bremner et al. [3], show that it is NP-complete to decide whether a graph admits a UPCR with cubes. In [1, 2] UPCR with cubes of subgraphs of 5 Archimedean grids are given. These are partly obtained by threshold colorings. However, threshold colorings cannot be used to find UPCR with cubes for all subgraphs of Archimedean grids; in particular, not for all subgraphs of the triangular grid [1].

1.2 Our Contributions

We develop a strategy to construct unit proper contact representations (UPCR) in 2D and 3D for subgraphs of certain highly symmetric grids. This strategy is used to show that subgraphs of the square grid allow for UPCR with squares. This is a strengthening of the unit cube representation of Alam et al. [2]. We generalize this result to $4k$-gons as well as to higher dimensions, namely, we prove that subgraphs of a d-dimensional grid have UPCR with d-cubes. Furthermore, we show that subgraphs of the triangular grid admit UPCRs with cubes, implying that the same holds for all subgraphs of Archimedean grids (grids originating from regular and semi-regular tilings of the plane). This solves an open problem posed by Alam et al. [2]. Considering other geometric objects, we construct UPCRs with regular $3k$-gons of the hexagonal grid.

Additionally, we observe that with the ideas of [3], we can show the NP-completeness of recognizing unit square proper contact graphs.

2 Definitions and Properties

Let $G = (V, E)$ be a graph. A function $\phi\colon V \to \mathcal{P}(\mathbb{R}^d), v \mapsto S_v$ is called a *proper contact representation* (PCR) in $\mathbb{R}^d$ of G if the sets in $\phi(V) := \bigcup_{v\in V} \phi(v)$ are pairwise interiorly disjoint and $(u, v) \in E \iff S_u \cap S_v$ is $(d-1)$-dimensional. Usually, the assigned sets are compact and path-connected and in this paper regular polytopes. A PCR $\mathcal{C}$ is called *unit* (UPCR), if all sets in $\mathcal{C}$ are congruent. This means for UPCR with regular n-gons, the polygons (in a component of G) can be transformed into one another by translations for even n, and by translations and rotation by π for odd n. In particular, we consider UPCRs of squares (USqPCR), cubes (UCuPCR) and triangles (UTriPCR). We define a *unit square* in $\mathbb{R}^2$ by its characteristic corner (x, y) as $S(x, y) := [x, x + 1] \times [y, y+1]$, and analogously, a *unit d-cube* in $\mathbb{R}^d$ as $Q(x) := [x_1, x_1+1]\times\ldots\times[x_d, x_d+1]$. Further, we define an upward and downward *unit triangle* with height $h := \sin(\frac{\pi}{3})$ as

$$\triangle(x,y) := \left\{ \binom{a}{b} \in \mathbb{R}^2 \mid \binom{a}{b} = \binom{x}{y} + r\binom{1}{0} + s\binom{1/2}{h};\ r,s \geq 0;\ r+s \leq 1 \right\},$$
$$\bigtriangledown(x,y) := \left\{ \binom{a}{b} \in \mathbb{R}^2 \mid \binom{a}{b} = \binom{x}{y} + r\binom{1}{0} + s\binom{1/2}{-h};\ r,s \geq 0;\ r+s \leq 1 \right\}.$$

For two touching polygons S_u, S_v in the plane, we define the *size of contact* $\mathrm{cs}(S_u, S_v)$ by the length of its realizing segment; and for two touching d-dimensional polytopes S_u, S_v, we define $\mathrm{cs}(S_u, S_v)$ by the shortest edge length of the $(d-1)$-dimensional polytope realizing the contact. The *contact size* of a PCR $\mathcal{C}$ is given by $\mathrm{cs}(\mathcal{C}) := \min\{\mathrm{cs}(S_u, S_v) \mid (u,v) \in E\}$. The *translation* of a set $S \in \mathbb{R}^n$ by a vector $t \in \mathbb{R}^n$ is defined by the addition $S + t := \{(x+t) \in \mathbb{R}^n \mid x \in S\}$. The *space* $\mathrm{sp}(S_u, S_v, d)$ of two objects S_u, S_v in direction d is the maximum $\delta \in \mathbb{R}$ such that either object can be translated by δd and they remain interiorly disjoint. Finally, $[n] := \{0, \ldots, n\}$.

2.1 Some Properties

We start with some basic properties of UPCRs. Clearly, PCRs in the plane may only exist for planar graphs. Due to the congruent objects, graphs with UPCR have bounded maximum degree and fulfill spatial constraints.

Observation 1 *Let G be a graph admitting a UPCR with regular n-gons. Then, the maximum degree is bounded by 6, i.e. $\Delta(G) \leq 6$.*

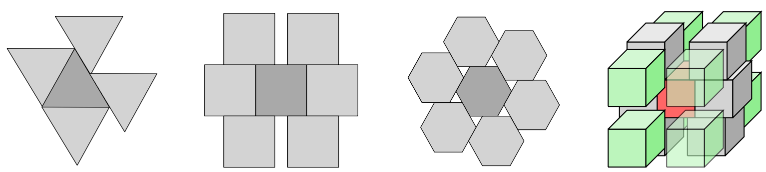


Fig. 1. Examples of polygons and a cube with maximum number of neighbors in a UPCR.

For triangles and squares, it is easy to see that the maximum degree is 4 and 6, respectively, see Figure 1. A general upper bound for regular polygons is given by their *kissing number*, the maximal number of congruent copies a single polygon can be touched by. Regular n-gons with $n \geq 5$ have a kissing number of ≤ 6, see [13, 14] for further explanation.

Together with $\Omega(\sqrt{n})$ polygons at the boundary of any (component of a) representation, this results in a low edge density: If a graph G on n vertices has a UPCR with k-gons, then the number of edges is $\leq 3n - \Omega(\sqrt{n})$. The analogous result for unit cube graphs has already been shown in [3]: For $G = (V, E)$ with UCuPCR it holds that $\Delta(G) \leq 14$ and $|E| \leq 7n - \Omega(n^{\frac{2}{3}})$.

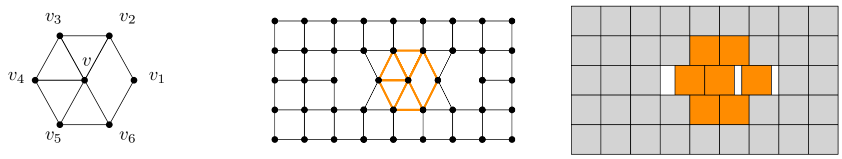
We formulate spatial restrictions, due to the congruent objects for *d-cubes*:

Observation 2 *Let G be a graph with a UPCR with d-cubes in $\mathbb{R}^d$. Then, every vertex has at most $(2r+1)^d$ vertices in distance $\leq r$.*

This follows easily by comparing the available space in distance $\leq r$ and the needed space to place all congruent d-cubes. Consequently, there exist binary trees with no UPCR with squares nor cubes.

For cubes, Bremner et al. [3], show that it is NP-complete to decide whether a graph admits a UCuPCR. With the same ideas, we can show the analogous result for squares:

Theorem 1 *The recognition of graphs with USqPCR or UCuPCR is NP-complete.*



(a) **(b)** **(c)**

Fig. 2. **(a)** Aligning graph **(b)** Box graph **(c)** USqPCR of box graph

Due to the similarity, we only want to provide the strategy of the proof. The logic engine of the proof by Bremner et al. [3] is composed of copies of the boxgraph, see fig. 2. Since in any USqPCR of the boxgraph the rectangular shape of the USqPCR remains, see fig. 2, the recognition of a USqPCR of the composed graph is equivalent to the recognition of a non-overlapping layout of the corresponding logic engine.

3 The Strategy

Let $G = (V, E)$ be a subgraph of some grid $\mathbb{G} = (V_\mathbb{G}, E_\mathbb{G})$. In order to find a UPCR of G, we set $V := V_\mathbb{G}$, since independent vertices can easily be removed from a CR. The idea of the method is to start with a UPCR $\hat{\phi} : V \to \mathcal{P}(\mathbb{R}^n)$ of $\mathbb{G}$, and then to modify $\hat{\phi}$ by removing unwanted contacts one by one, that are contacts corresponding to edges in $\overline{E} := E_\mathbb{G} \setminus E$. We partition the edge set $E_\mathbb{G} = \bigcup_i E_i$ and assign to $e \in E_i$ a direction $d_i \in \mathbb{R}^n$. For $e \in \overline{E}$ objects corresponding to vertices in a *moving set* $M(e) \subset V$ are translated by δd_i. The translation step $\delta > 0$ is chosen small enough, such that apart from e all contacts remain and no additional ones occur. This strategy yields a straight-forward construction of a final UPCR ϕ of G:

$$\phi(v) = \hat{\phi}(v) + \sum_i |\{e \in \overline{E} \cap E_i \mid v \in M(e)\}| \cdot \delta d_i.$$

4 Representations with Unit Squares

The square grid $\mathbb{S}_{m,n} = (V_\mathbb{S}, E_\mathbb{S})$ of size $(m \times n)$ is defined as follows:

$$\mathbb{S}_{m,n} := \Big(\{ v_{i,j} \mid i \in [m], j \in [n] \},\ \{ (v_{i,j}, v_{i',j'}) \mid \big\| \tbinom{i-i'}{j-j'} \big\|_1 = 1 \} \Big).$$

Theorem 2 *Every subgraph of the square grid $\mathbb{S}_{m,n}$ has a USqPCR.*

Proof. We consider subgraphs of type $G = (V_\mathbb{S}, E)$; for subgraphs with $V \subset V_\mathbb{S}$, the theorem follows by removing independent vertices. The proof applies the strategy described in Section 3. We fix an $\varepsilon \in (0, 1)$ and present a USqPCR $\hat{\phi}$ of $\mathbb{S}_{m,n}$, see also Figure 3:

$$\hat{\phi} \colon V \to \mathcal{P}(\mathbb{R}^2)$$
$$\hat{\phi}(v_{i,j}) = S(i - j\varepsilon, j + i\varepsilon).$$

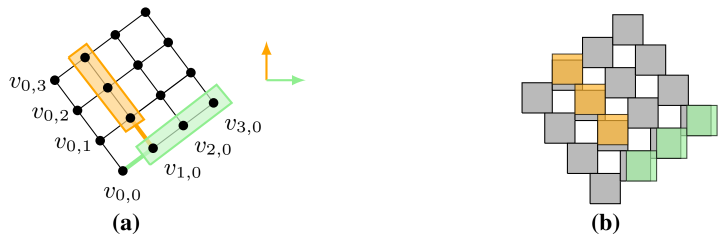


Fig. 3. (a) Square grid $\mathbb{S}_{3,3}$; for thick edges, the moving sets are indicated by the framed vertices **(b)** USqPCR $\hat{\phi}$ of $\mathbb{S}_{3,3}$ with illustration of modification step.

We claim that $\hat{\phi}$ is a USqPCR of $\mathbb{S}_{m,n}$

- $\text{cs}\big(\hat{\phi}(u), \hat{\phi}(v)\big) = 1 - \varepsilon$ for all $(u, v) \in E_{\mathbb{S}}$ and
- $\text{sp}\big(\hat{\phi}(u), \hat{\phi}(v), d\big) \geq \varepsilon$ for all $(u, v) \notin E_{\mathbb{S}}$ and the two directions d parallel to the square's sides: $d_1 := \binom{1}{0}$ and $d_2 := \binom{0}{1}$

Consider a vertex v and its four neighbors. Each contact is realized by one side of $\hat{\phi}(v)$. By definition of $\hat{\phi}$, the characteristic corners differ by $\pm\binom{1}{\varepsilon}, \pm\binom{-\varepsilon}{1}$. Hence, the squares are disjoint and $\text{cs}\big(\hat{\phi}(u), \hat{\phi}(v)\big) = (1 - \varepsilon)$, for all $(u, v) \in E_{\mathbb{S}}$. Consider a non-edge $(u, v) \notin E_{\mathbb{S}}$. The characteristic corners of $\hat{\phi}(u)$ and $\hat{\phi}(v)$ differ by $a\binom{1}{\varepsilon} + b\binom{-\varepsilon}{1}$ with $a, b \in \mathbb{Z}$ and $|a| + |b| \geq 2$. This implies that $\text{sp}\big(\hat{\phi}(u), \hat{\phi}(v), d_i\big) \geq \varepsilon$ for $i \in \{1, 2\}$.

We now define the moving sets and translation vectors: For $e = (v_{i,j}, v_{i+1,j})$, we set $d(e) := d_1 = \binom{1}{0}$ and $M(e) := \{v_{k,j} \in V \mid k > i\}$, otherwise $e = (v_{i,j}, v_{i,j+1})$, we set $d(e) := d_2 = \binom{0}{1}$ and $M(e) := \{v_{i,k} \in V \mid k > j\}$. For simplification, we define $E_i := \{e \in E \mid d(e) = d_i\}$ for $i \in \{1, 2\}$ and assume wlog $n \geq m$. The idea is to remove each contact of $e \in \overline{E}$ by translating $M(e)$ in direction $d(e)$. The integer $r_i(v)$ describes how often v is moved in direction d_i, that is $r_i(v) := |\{e \in \overline{E} \cap E_i \mid v \in M(e)\}|$. Observe that $r_i(v) \leq n$ for $i \in \{1, 2\}$ and $v \in V$. Choosing $\delta < \frac{1}{n}\min\{\varepsilon, 1 - \varepsilon\}$ the following mapping is a USqPCR of $G \subseteq \mathbb{S}_{m,n}$:

$$\phi\colon V \to \mathcal{P}(\mathbb{R}^2)$$
$$\phi(v) = \hat{\phi}(v) + r_1(v) \cdot \delta d_1 + r_2(v) \cdot \delta d_2.$$

We verify that ϕ is a USqPCR of G by showing three properties: Let $(u, v) \notin E_{\mathbb{S}}$. Recall that $\text{sp}\big(\hat{\phi}(u), \hat{\phi}(v), d_i\big)$ gives the maximum translation step λ such that $\hat{\phi}(u)$ and $\hat{\phi}(v)$ remain interiorly disjoint when either one of them is translated by λd_i. By construction $r_i(w)\delta \leq n\delta < \varepsilon \leq \text{sp}\big(\hat{\phi}(u), \hat{\phi}(v), d_i\big)$ for $w \in \{u, v\}$ and $i \in \{1, 2\}$. Consequently, $\phi(u)$ and $\phi(v)$ remain disjoint.

Let $(u, v) \in E_i$, then $\hat{\phi}(u)$ and $\hat{\phi}(v)$ have a contact segment parallel to d_{i+1} of length $\text{cs}(\hat{\phi}(u), \hat{\phi}(v)) \geq 1 - \varepsilon$. Hence, translations in direction d_{i+1} have no effect on the contact: $\text{cs}\big(\phi(u), \phi(v)\big) \geq \text{cs}\big(\hat{\phi}(u), \hat{\phi}(v)\big) - r_{i+1}(v)\delta \geq 1 - \varepsilon - n\delta > 0$. Wlog we suppose that $\hat{\phi}(u)$ is left or below of $\hat{\phi}(v)$, then by definition of the moving sets: $r_i(u) \leq r_i(v)$. This implies that $\phi(u)$ and $\phi(v)$ remain interiorly disjoint.

Since the translation vector d_i is not parallel to the contact segment, the contact remains iff $r_i(u) = r_i(v)$. By definition, this is the case iff $(u, v) \in E$. Note therefore, that if $(u,v) \in \overline{E}$, then $r_i(u) < r_i(v)$ since $u \notin M((u,v))$ and $v \in M((u,v))$. Consequently, $\phi(u)$ and $\phi(v)$ have proper contact iff $(u,v) \in E$. $\square$

Remark 1 *The construction has running time of $O(|V|^2)$: The parameters $r_i(v)$ can be determined in $O(n|V|)$ and the construction can be produced in $O(|V|)$. Since $n \leq |V|$, this gives an overall running time of $O(|V|^2)$.*

Remark 2 *Choosing specific values for ε and δ, further properties can be guaranteed: With $\varepsilon = \frac{1}{2}$ and $\delta = \frac{1}{2n+2}$ one obtains a USqPCR ϕ of G where the proper contacts and non-contacts are guaranteed to be of size δ. As ε and δ approach 0, the constact sizes are arbitrarily close to 1. This is exploited in Section 6 for USqPCR with $4k$-gons.*

5 Representations with Unit Cubes

In this section we investigate further subgraphs of grids for UPCR with cubes.

5.1 d-Dimensional Grid

Indeed, Theorem 2 can be generalized to all dimensions. As a generalization of the square grid, we define the d-dimensional grid $\mathbb{S}_n^d = (V_\mathbb{S}, E_\mathbb{S})$:

$$\mathbb{S}_n^d := \Big(\big\{ v_x | x \in [n]^d \big\}, \big\{ (v_x, v_y) \mid \|x - y\|_1 = 1 \big\} \Big)$$

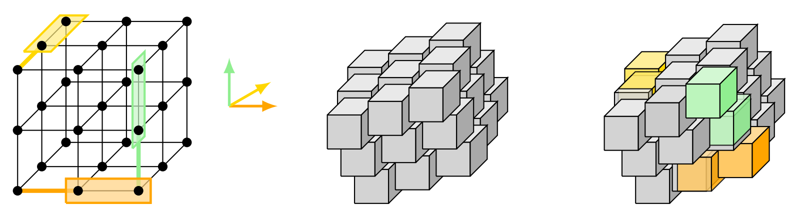


Fig. 4. (a) Cubic grid $\mathbb{S}_2^3$; for the thick edges, the moving sets are indicated by the framed vertices **(b)** UCuPCR $\hat{\phi}$ of $\mathbb{S}_2^3$ **(c)** Illustration of modification step.

Theorem 3 *Every subgraph of the grid $\mathbb{S}_n^d$ admits a UPCR with d-cubes.*

Proof. The proof applies the strategy presented in Section 3. First, we give a UPCR with d-cubes of $\mathbb{S}_n^d$, depicted in Figure 4. We choose $\varepsilon \in (0,1)$, $\delta < \frac{1}{n} \min\{\varepsilon, 1-\varepsilon\}$ and define the UPCR with the help of matrix $A \in \mathbb{R}^{d \times d}$:

$$\hat{\phi} \colon V \to \mathcal{P}(\mathbb{R}^d) \qquad\qquad A := \begin{pmatrix} 1 & \varepsilon & \dots & \varepsilon \\ -\varepsilon & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \varepsilon \\ -\varepsilon & \dots & -\varepsilon & 1 \end{pmatrix}$$
$$\hat{\phi}(v_x) = Q(A \cdot x)$$

In order to prove that $\hat{\phi}$ is a UPCR of $\mathbb{S}_n^d$ with d-cubes, we note: Two axis-aligned unit d-cubes with characteristic corners x and y, have proper $(d-1)$-dimensional contact iff there exist $k \in [d]$ such that $|x_k - y_k| = 1$ and $|x_i - y_i| < 1$ for all $i \in [d] \setminus \{k\}$. It remains to show that two cubes $\hat{\phi}(v_x)$ and $\hat{\phi}(v_y)$ have proper contact iff $(v_x, v_y) \in E_{\mathbb{S}_n^d}$: Suppose $(v_x, v_y) \in E_{\mathbb{S}}$. Then, it holds $\|x-y\|_1 = 1$; that is $\exists! k \in [d]$ with $|x_k - y_k| = 1$ and $|x_i - y_i| = 0$ for all $i \in [d] \setminus \{k\}$. Wlog let $x \geq_{\text{dom}} y$. Consider the characteristic corners $A \cdot x$ and $A \cdot y$ of the cubes $\hat{\phi}(v_x)$ and $\hat{\phi}(v_y)$. Let e_k denote the k^{th} standard basis vector of $\mathbb{R}^d$, then for the characteristic corners hold: $(A \cdot x - A \cdot y) = A \cdot (x-y) = A \cdot e_k$. This implies that the two cubes $\hat{\phi}(v_x)$ and $\hat{\phi}(v_y)$ have proper contact which is of size $cs(\hat{\phi}(v_x), \hat{\phi}(v_y)) = 1 - \varepsilon$.

Suppose $(v_x, v_y) \notin E_{\mathbb{S}}$, then $\|x - y\|_1 \geq 2$. Either there is a coordinate $k \in [d]$, such that $|x_k - y_k| \geq 2$ or there exist at least two coordinates $k, j \in [d]$ such that $|x_k - y_k| \geq 1$. It is easy to check, that in either case, there is a coordinate in $(A \cdot x - A \cdot y)$ which has an absolute value of $\geq 1 + \varepsilon$, and hence, the cubes $\hat{\phi}(v_x)$ and $\hat{\phi}(v_y)$ are disjoint. Thus for $(v_x, v_y) \notin E_{\mathbb{S}}$ the space is $\text{sp}(\hat{\phi}(v_x), \hat{\phi}(v_y), e_k) \geq \varepsilon$ for $k = 1, \ldots, d$.

We proceed with the translation vectors and moving sets: For $e = (v_x, v_y) \in E_{\mathbb{S}}$, there exists a unique $k \in [d]$ with $|x_k - y_k| = 1$ and $x_i = y_i$ for $i \neq k$. Wlog we assume $x \leq_{\text{dom}} y$. Then, the direction vector is defined as $d(e) := e_k$ and the moving set as $M(e) := \{v_z \mid z \in [n]^d, z_k > x_k, z_i = x_i \text{ for all } i \neq k\}$.

Observe that the translation vector $d(e)$ of edge e is parallel to each of the following crucial $(d-1)$-dimensional facets: These are facets realizing the contacts corresponding to edges of type $(u, v) \in E \backslash \{e\}$ with $u \in M(e)$ and $v \notin M(e)$. Additionally, it holds that $r_i(w)\delta \leq n\delta < \varepsilon \leq \text{sp}\big(\hat{\phi}(u), \hat{\phi}(v), d_i\big)$ for $w \in \{u, v\} \notin E_{\mathbb{S}}$ and $i \in \{1, \ldots, d\}$. Following the same lines as the proof of Theorem 2, this shows that the construction yields a UPCR with d-cubes for each subgraph of $\mathbb{S}_n^d$. $\square$

5.2 Triangular Grid

Alam et al. [2] asked whether subgraphs of the triangular grid have UCuPCRs. In this section, we answer this question affirmatively.

We introduce an unconventional definition of the triangular grid $\mathbb{T}_{m,n} = (V_{\mathbb{T}}, E_{\mathbb{T}})$ of size $m \times n$. For $i \in [m]$, and $j \in [n]$, the vertex set is $V := \{b_{i,j}\} \cup \{t_{i,j}\}$. The set of edges is $E := \bigcup_{i=1}^{5} E_i$, where $E_1 := (x_{i,j}, x_{i,j+1})$ with $x \in \{b, t\}$, $E_2 := (b_{i,j}, t_{i,j})$, $E_3 := (b_{i,j}, t_{i,j-1})$, $E_4 := (b_{i,j}, t_{i-1,j-1})$, and $E_5 := (b_{i,j}, t_{i-1,j})$, see also Figure 5. For fixed j, we call the vertex set of type $\{x_{i,j}\}_{i \in [m]}$ a *line* and two neighboring lines a *level*. Edges between two lines are called *level edges*. Note that we have $2n$ lines.

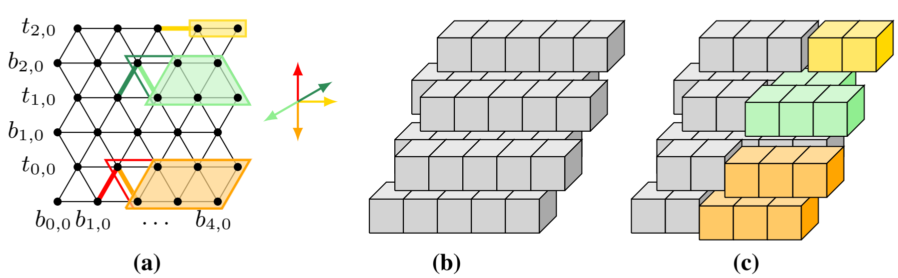


Fig. 5. **(a)** Triangular grid $\mathbb{T}_{4,2}$; for thick edges, the moving sets are indicated by framed vertices **(b)** UCuPCR $\hat{\phi}$ of $\mathbb{T}_{4,2}$ **(c)** Illustration of modification step.

Theorem 4 *Every subgraph of the triangular grid $\mathbb{T}_{m,n}$ has a UCuPCR.*

Proof. The proof uses the observation that subgraphs of $\mathbb{T}_{1,n}$ allow for USqPCR. Therefore, contacts corresponding to level edges are realized alternating in planes parallel to the xy and xz plane. Choosing $\varepsilon \in (0,1)$ and $\delta < \frac{1}{n+1}\min\{\varepsilon, 1-\varepsilon\}$, we define a UCuPCR of $\mathbb{T}_{m,n}$, see also Figure 5:

$$\hat{\phi}\colon V \to \mathcal{P}(\mathbb{R}^3)$$
$$\hat{\phi}(v) = \begin{cases} Q\big(i, \quad j(1+\varepsilon), \quad j\big) & \text{if } v = b_{i,j}, \\ Q\big(i+\varepsilon,\ j(1+\varepsilon)+\varepsilon,\ j+1\big) & \text{if } v = t_{i,j}. \end{cases}$$

It is easy to verify that $\hat{\phi}$ is a UCuPCR of $\mathbb{T}_{m,n}$ with contact size $cs(\hat{\phi}) \geq (1-\varepsilon)$. Additionally, it holds that $sp(\hat{\phi}(u), \hat{\phi}(v), d_i) \geq \varepsilon$ for the following direction vectors:

For an edge $e \in E_i$ we set the direction vector d_i to $d_1 := (1,0,0)$, $d_2 := (0,0,1)$, $d_3 := (0,0,-1)$, $d_4 := (0,1,0)$, $d_5 := (0,-1,0)$. The moving set belonging to an edge is slightly more involved. For a level edge e, $M(e)$ roughly consists of the vertices in the level with larger index; for a line edge, it consists of the line vertices with larger index: for $e = (b_{i,j}, t_{k,l}) \in E_2 \cup E_4$, we define $M(e) := \{b_{i,r} \mid r > j\} \cup \{t_{k,r} \mid r \geq l\}$, and for $e = (b_{i,j}, t_{k,l}) \in E_3 \cup E_5$, we define $M(e) := \{b_{i,r} \mid r \geq j\} \cup \{t_{k,r} \mid r > l\}$, and for $e = (x_{i,j}, x_{i,j+1}) \in E_1$, we define $M(e) := \{x_{i,r} \mid r > j\}$. Figure 5 depicts the moving sets, direction vectors and modifications.

As the proof is analogous to the proof of Theorem 2, we give some useful observations and leave the rest to the reader: The translation vector $d(e)$ of an edge e is parallel to each of the crucial cube facets, realizing contacts corresponding to edges of type $(u,v) \in E \backslash \{e\}$ with $u \in M(e)$ and $v \notin M(e)$. Also note that $r_i(v) \leq n+1$ and hence $r_i(u) \cdot \delta \leq (n+1)\delta \leq \varepsilon \leq \mathrm{sp}(\hat{\phi}(u), \hat{\phi}(v), d_i)$ for all $(u,v) \notin E_{\mathbb{S}}$ and all d_i. Combining this, the construction yields a UCuPCR for each subgraph of $\mathbb{T}_{m,n}$. □

5.3 Archimedean Grids

There exist eleven grids originating from regular and semi-regular tilings of the plane, so called *Archimedean grids*, which are depicted in Table 1. As mentioned before, UCuPCR for subgraphs of five Archimedean grids are known [1, 2]. With the results of the previous section, UCuPCR for subgraphs of all Archimedean grids follow directly:

Corollary 1 *Every subgraph of an Archimedean grid has a UCuPCR.*

Proof. Observe that Archimedean grids are subgraphs of the triangular grid. For proving this fact, we provide convincing pictures in Table 1. With this observation, the claim follows directly from Theorem 4. □

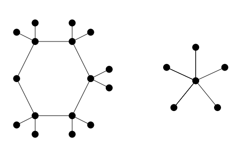
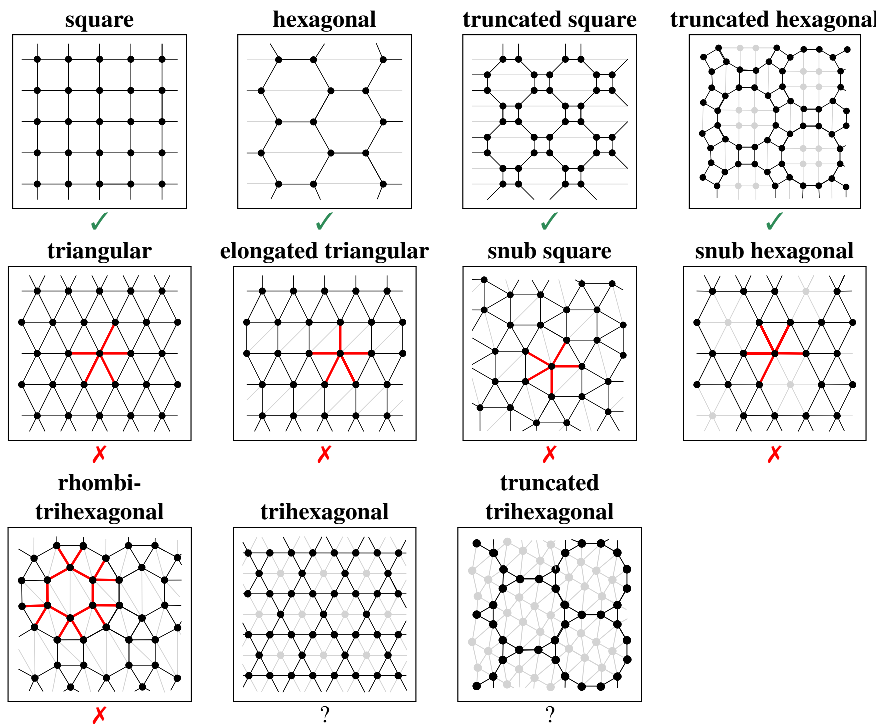


Fig. 6. The pufferfish graph and the star $K_{1,5}$

The remaining question is, whether subgraphs of Archimedean grids admit a representation with unit squares. In general, this is not the case since we find two forbidden subgraphs: $K_{1,5}$ and the *pufferfish graph*, which is a C_6 together with two private neighbors for all but one vertex, see Figure 6. Table 1 summarizes for which Archimedean grids, all subgraphs allow for a USqPCR.

Table 1. The table gives an overview for which Archimedean grids each subgraph allows for a USqPCR. By Corollary 1, each subgraph has a UCuPCR.



Lemma 1 *The pufferfish graph and $K_{1,5}$ have no USqPCR.*

Proof. We first note that for two touching squares, at least one corner of each square is involved in the segment of contact; two corners are involved iff the squares have full side contact. In a USqPCR of $K_{1,5}$, there must be a corner involved in two contacts which leads to full side contacts. Since a square has only four sides, it follows that $K_{1,5}$ has no USqPCR.

Assume the pufferfish graph has a USqPCR. We first observe that no two squares have full side contact: Suppose there are two such squares, then one of them represents a vertex v of the C_6 which has 4 independent neighbors. Due to its 4 independent neighbors, it follows that all of its contacts are full side contacts. It is easy to see that repeating the argument for a neighbor of v with degree 4 leads to a contradiction. Hence, no two squares have full side contact and each corner of the squares in the C_6 is involved in a contact segment.

Consequently, the inner face bounded by the squares of the C_6 is an orthogonal 8-gon (of T- or Z-shape) with side lengths ≤ 1. This implies that private neighbors cannot be placed in the inner face and the two convex corners belonging to different squares do not account for a contact: a contradiction. □

Corollary 2 *Not all subgraphs of the triangular, elongated triangular, snub square, snub hexagonal, and rhombitrihexagonal grid have a USqPCR.*

Proof. $K_{1,5}$ (and the pufferfish graph) are induced subgraphs of the triangular, elongated triangular, snub square, and snub hexagonal grid. The pufferfish graph is an induced subgraph of rhombitrihexagonal grid. □

6 Representations with Regular Polygons

In this section, we consider UPCR with further regular polygons and for ease, refer to them as polygons. To do so, we introduce the notion of *pseudo polygons*. A *pseudo n-gon* (with side length s) is a subset of a regular n-gon, which includes a central segment (of length $\geq s$) of each boundary edge. A segment of a boundary edge is called *central* if their midpoints coincide. It can be understood as a n-gon with cut-off corners, consider Figure 7.

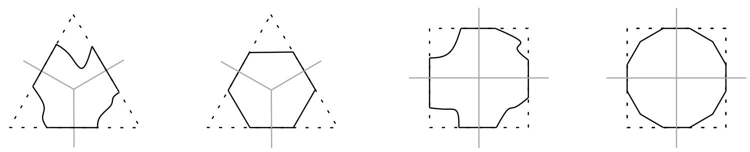


Fig. 7. Examples of pseudo-triangles and pseudo-squares

Lemma 1. *Let G be a graph with a UPCR ϕ with regular k-gons and $cs(\phi) > 1 - s$. Then, G has a UPCR with pseudo k-gons with side length at least s.*

Proof. This UPCR is obtained from ϕ by inscribing a pseudo k-gon into each k-gon: Consider two touching k-gons and the sides realizing the proper contact. The midpoints of these sides differ by $< s$ since $\mathrm{cs}(\phi) > 1-s$. Hence, every contact can be certified by two intersecting central segments of size s. Since the pseudo k-gons (with side length s) contain these segments, the contacts remain. Additionally, no new contacts occur, because each pseudo k-gon is a subsets of a k-gon. Hence, ϕ still serves as a UPCR.

□

6.1 Representations with Regular $4k$-gons

Corollary 3 *Every subgraph of the square grid $\mathbb{S}_{m,n}$ has a UPCR with regular $4k$-gons, $k \geq 1$.*

Proof. Note that $4k$-gons are pseudo-squares with side length $s_k := \sin(\frac{\pi}{4k})$. Choosing $\varepsilon < \frac{s_k}{2}$ in the proof of Theorem 2, we obtain a UPCR ϕ for every subgraph G of $\mathbb{S}_{m,n}$ with $cs(\phi) > 1 - s_k$. With this, the claim follows directly from Theorem 2 and Lemma 1. □

6.2 Representations with Regular $3k$-gons

We define the hexagonal grid $\mathbb{H}_{m,n} = (V_{\mathbb{H}}, E_{\mathbb{H}})$ as a subgraph of the square grid $\mathbb{S}_{m,n}$:

$$\mathbb{H}_{m,n} := \Big(V_{\mathbb{S}}, E_{\mathbb{S}} \backslash \big\{(v_{i,j}, v_{i+1,j}) \in E_{\mathbb{S}} \mid (i+j) \text{ odd}\big\}\Big).$$

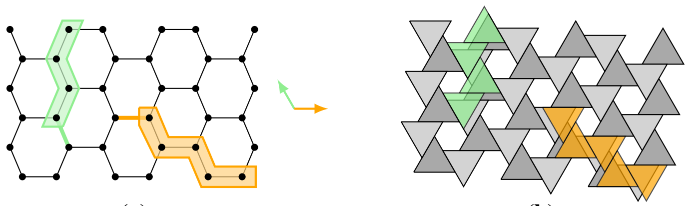


Fig. 8. (a) Hexagonal Grid $\mathbb{H}_{5,5}$; for thick edges, the moving sets are indicated by framed vertices **(b)** UTriPCR $\hat{\phi}$ of $\mathbb{H}_{5,5}$ with illustration of modification step.

Theorem 5 *Every subgraph of the hexagonal grid $\mathbb{H}_{m,n}$ has a UTriPCR.*

Proof. For the proof, we apply the already known technique. With $\varepsilon \in (0,1)$, a UTriPCR of $\mathbb{H}_{m,n}$ is given by the following mapping and depicted in Figure 8:

$$\phi \colon V \to \mathcal{P}(\mathbb{R}^3)$$

$$\phi(v_{i,j}) = \begin{cases} \triangle\Big(\frac{i}{4}\big[\binom{3}{2h} + \varepsilon\binom{3}{-6h}\big] + \frac{j}{4}\big[\binom{-1}{2h} + \varepsilon\binom{3}{2h}\big]\Big) & \text{if } i+j \text{ even,} \\ \triangledown\Big(\frac{i}{4}\big[\binom{3}{2h} + \varepsilon\binom{3}{-6h}\big] + \frac{j}{4}\big[\binom{-1}{2h} + \varepsilon\binom{3}{2h}\big] + \frac{1}{4}\big[\binom{-1}{h} + \varepsilon\binom{-1}{h}\big]\Big) & \text{else.} \end{cases}$$

Analyzing the shifts of the characteristic corner, it is not hard to verify that $\hat{\phi}$ is a UTriPCR of the triangular grid with contact size $cs(\hat{\phi}) \geq (1-\varepsilon)$ and has moving space $sp\big(\phi(u), \phi(v), d\big) \geq \varepsilon$ for $(u,v) \in E_{\mathbb{H}}$ and direction vectors d parallel to the sides of the triangles.

Indeed, two types of edges and direction vectors suffice: For $e = (v_{i,j}, v_{i,j+1})$, we define direction $d(e) := (-\frac{1}{2}, h)$ and moving set $M(e) := \{v_{i,k} \in V \mid k > j\}$; for $e = (v_{i,j}, v_{i+1,j})$, we define direction $d(e) := (1,0)$ and moving set $M(e) := \{v_{i+1+k,j-k} \in V \mid k \in [m-i]\} \cup \{v_{i+1+k,j-1-k} \in V \mid k \in [m-i]\}$. A crucial property is that, again, the translation vector $d(e)$ of an edge e is parallel to each of the segments realizing contacts corresponding to edges of type $(u,v) \in E \backslash \{e\}$ with $u \in M(e)$ and $v \notin M(e)$. Moreover, $d(e)$ is not parallel to the segment realizing this contact in $\hat{\phi}$. Note also that each edge belongs to at most n moving sets with the same translation direction. Therefore, choosing $\delta < \frac{1}{n}\min\{\varepsilon, 1-\varepsilon\}$ the construction analogous to Theorem 2 yields a UTriPCR for each subgraph G of $\mathbb{H}_{m,n}$. $\square$

Corollary 4 *Every subgraph of the hexagonal grid $\mathbb{H}_{m,n}$ has a UPCR with regular $3k$-gons, $k \geq 1$.*

Proof. Note that $3k$-gons are pseudo-triangles with side length $s_k := \tan(\frac{\pi}{3k})h$. Choosing $\varepsilon < \frac{s_k}{2}$ in the proof of Theorem 5, we obtain a UPCR ϕ for every subgraph G of $\mathbb{H}_{m,n}$ with $cs(\phi) > 1 - s_k$. With this, the claim follows directly from Theorem 5 and Lemma 1. $\square$

7 Open Questions

We want to conclude with a list of open questions:

- What is the complexity of recognizing graphs admitting UPCRs with regular polygons other than squares?
- Can we characterize the graphs with USqPCRs? Or with other polygons?
- Do the trihexagonal and truncated trihexagonal grid admit a USqPCR?
- Do subgraphs of duals of Archimedean grids not containing $K_{1,9}$ have UCuPCR? What about USqPCR for duals not containing $K_{1,5}$, namely the snubsquare grid?

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