On the Area-Universality of Triangulations

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Abstract. We study straight-line drawings of planar graphs with pre-scribed face areas. A plane graph is *area-universal* if for every area as-signment on the inner faces, there exists a straight-line drawing realizing the prescribed areas.

For triangulations with a special vertex order, we present a sufficient criterion for area-universality that only requires the investigation of one area assignment. Moreover, if the sufficient criterion applies to one plane triangulation, then all embeddings of the underlying planar graph are also area-universal. To date, it is open whether area-universality is a property of a plane or planar graph.

We use the developed machinery to present area-universal families of tri-angulations. Among them we characterize area-universality of accordion graphs showing that area-universal and non-area-universal graphs may be structural very similar.

Keywords: area-universality · triangulation · planar graph · face area

1 Introduction

By Fary's theorem [11,20,22], every plane graph has a straight-line drawing. We are interested in straight-line drawings with the additional property that the face areas correspond to prescribed values. Particularly, we study *area-universal* graphs for which all prescribed face areas can be realized by a straight-line drawing. Usually, in a *planar* drawing, no two edges intersect except in common vertices. It is worthwhile to be slightly more generous and allow *crossing-free* drawings, i.e., drawings that can be obtained as the limit of a sequence of planar straight-line drawings. Note that a crossing-free drawing of a triangulation is not planar (*degenerate*) if and only if the area of at least one face vanishes. Moreover, we consider two crossing-free drawings of a plane graph as *equivalent* if the cyclic order of the incident edges at each vertex and the outer face coincide.

For a plane graph G, we denote the set of faces by F, and the set of inner faces by F'. An *area assignment* is a function $\mathcal{A}\colon F' \to \mathbb{R}_{\geq 0}$. We say G is *area-universal* if for every area assignment $\mathcal{A}$ there exists an equivalent crossing-free drawing where every inner face $f \in F'$ has area $\mathcal{A}(f)$. We call such a drawing $\mathcal{A}$-*realizing* and the area assignment $\mathcal{A}$ *realizable*.

Related Work. Biedl and Ruiz Velázquez [6] showed that planar partial 3-trees, also known as subgraphs of *stacked triangulations* or *Apollonian networks*, are area-universal. In fact, every subgraph of a plane area-universal graph is area-universal. Ringel [18] gave two examples of graphs that have drawings where all face areas are of equal size, namely the octahedron graph and the icosahedron graph. Thomassen [21] proved that plane 3-regular graphs are area-universal. Moreover, Ringel [18] showed that the octahedron graph is not area-universal. Kleist [15] generalized this result by introducing a simple counting argument which shows that no Eulerian triangulation, different from K_3, is area-universal. Moreover, it is shown in [15] that every 1-subdivision of a plane graphs is area-universal; that is, every area assignment of a plane graph has a realizing polyline drawing where each edge has at most one bend. Evans et al. [10,16] present classes of area-universal plane quadrangulations. In particular, they verify the conjecture that plane bipartite graphs are area-universal for quadrangulations with up to 13 vertices. Particular graphs have also been studied: It is known that the square grid [9] and the unique triangulation on seven vertices [4] are area-universal. Moreover, non-area-universal triangulations on up to ten vertices have been investigated in [13].

The computational complexity of the decision problem of area-universality for a given graph was studied by Dobbins et al. [7]. The authors show that this decision problem belongs to Universal Existential Theory of the Reals ($\forall\exists\mathbb{R}$), a natural generalization of the class Existential Theory of the Reals ($\exists\mathbb{R}$), and conjecture that this problem is also $\forall\exists\mathbb{R}$-complete. They show hardness of several variants, e.g., the analogue problem of volume universality of simplicial complexes in three dimensions.

In a broader sense, drawings of planar graphs with prescribed face areas can be understood as *cartograms*. Cartograms have been intensely studied for duals of triangulations [1,3,5,14] and in the context of rectangular layouts, dissections of a rectangle into rectangles [8,12,24]. For a detailed survey of the cartogram literature, we refer to [17].

Our contribution. In this work we present three characterizations of area-universal triangulations. We use these characterizations for proving area-universality of certain triangulations. Specifically, we consider triangulations with a vertex order, where (most) vertices have at least three neighbors with smaller index, called *predecessors*. We call such an order a *p-order*. For triangulations with a p-order, the realizability of an area assignment reduces to finding a real root of a univariate polynomial. If the polynomial is surjective, we can guarantee area-universality. In fact, this is the only known method to prove the area-universality of a triangulation besides the simple argument for plane 3-trees relying on K_4.

We discover several interesting facts. First, to guarantee area-universality it is enough to investigate one area assignment. Second, if the polynomial is surjective for one plane graph, then it is for every embedding of the underlying planar graph. Consequently, the properties of one area assignment can imply the area-universality of all embeddings of a planar graph. This may indicate that area-universality is a property of planar graphs.

We use the method to prove area-universality for several graph families including accordion graphs. To obtain an *accordion graph* from the plane octahedron graph, we introduce new vertices of degree 4 by subdividing an edge of the central triangle. Figure 1 presents four examples of accordion graphs. Surprisingly, the insertion of an even number of vertices yields a non-area-universal graph while the insertion of an odd number of vertices yields an area-universal graph. Accordions with an even number of vertices are Eulerian and thus not area-universal [15]. Consequently, area-universal and non-area-universal graphs may have a very similar structure. (In [16], we use the method to classify small triangulations with p-orders on up to ten vertices.)
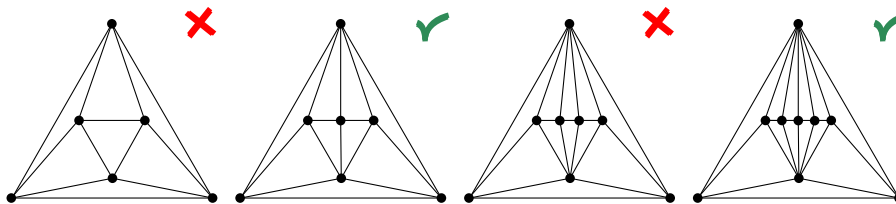

Fig. 1: Examples of accordion graphs. A checkmark indicates area-universality and a cross non-area-universality.

Organization. We start by presenting three characterizations of area-universality of triangulations in Section 2. In Section 3, we turn our attention to triangulations with p-orders and show how the analysis of one area assignment can be sufficient to prove area-universality of all embeddings of the given triangulation. Then, in Section 4, we apply the developed method to prove area-universality for certain graph families; among them we characterize the area-universality of accordion graphs. We end with a discussion and a list of open problems in Section 5. In appendices A to C we present omitted proofs of Sections 2 to 4.

2 Characterizations of Area-Universal Triangulations

Throughout this section, let T be a plane triangulation on n vertices. A straight-line drawing of T can be encoded by the $2n$ vertex coordinates, and hence, by a point in the Euclidean space $\mathbb{R}^{2n}$. We call such a vector of coordinates a *vertex placement* and denote the set of all vertex placements encoding crossing-free drawings by $\mathcal{D}(T)$; we also write $\mathcal{D}$ if T is clear from the context.

It is easy to see that an $\mathcal{A}$-realizing drawing of a triangulation can be transformed by an affine linear map into an $\mathcal{A}$-realizing drawing where the outer face corresponds to any given triangle of correct total area $\Sigma\mathcal{A} := \sum_{f \in F'} \mathcal{A}(f)$, where F' denotes the set of inner faces as before.

Lemma 1. [15, Obs. 2] *A plane triangulation T with a realizable area assignment $\mathcal{A}$, has an $\mathcal{A}$-realizing drawing within every given outer face of area $\Sigma\mathcal{A}$.*

Likewise, affine linear maps can be used to scale realizing drawings by any factor. For any positive real number $\alpha \in \mathbb{R}$ and area assignment $\mathcal{A}$, let $\alpha\mathcal{A}$ denote the *scaled area assignment* of $\mathcal{A}$ where $\alpha\mathcal{A}(f) := \alpha \cdot \mathcal{A}(f)$ for all $f \in F'$.

Lemma 2. *Let $\mathcal{A}$ be an area assignment of a plane graph and $\alpha > 0$. The scaled area assignment $\alpha\mathcal{A}$ is realizable if and only if $\mathcal{A}$ is realizable.*

For a plane graph and $c > 0$, let $\mathbb{A}^c$ denote the set of area assignments with a total area of c. Lemma 2 directly implies the following property.

Lemma 3. *Let $c > 0$. A plane graph is area universal if all area assignments in $\mathbb{A}^c$ are realizable.*

2.1 Closedness of Realizable Area Assignments

In [15, Lemma 4], it is shown for triangulations that $\mathcal{A} \in \mathbb{A}^c$ is realizable if and only if in every open neighborhood of $\mathcal{A}$ in $\mathbb{A}^c$ there exists a realizable area assignment. For our purposes, we need a stronger version. Let $\mathbb{A}^{\leq c}$ denote the set of area assignments of T with a total area of at most c. For a fixed face f of T, $\mathbb{A}^{\leq c}|_{f \to a}$ denotes the subset of $\mathbb{A}^{\leq c}$ where f is assigned to a fixed $a > 0$.

Proposition 1. *Let T be a plane triangulation and $c > 0$. Then $\mathcal{A} \in \mathbb{A}^c$ is realizable if and only if for some face f with $\mathcal{A}(f) > 0$ every open neighborhood of $\mathcal{A}$ in $\mathbb{A}^{\leq 2c}|_{f \to \mathcal{A}(f)}$ contains a realizable area assignment.*

Intuitively, Proposition 1 enables us not to worry about area assignments with bad but unlikely properties. In particular, area-universality is guaranteed by the realizability of a dense subset of $\mathbb{A}^c$. Moreover, this stronger version allows to certify the realizability of an area assignment by realizable area assignments with slightly different total areas. The proof of Proposition 1 goes along the same lines as in [15, Lemma 4]; it is based on the fact that the set of drawings of T with a fixed face f and a total area of at most $2c$ is compact.

2.2 Characterization by 4-Connected Components

For a plane triangulation T, a *4-connected component* is a maximal 4-connected subgraph of T. Moreover, we call a triangle t of T *separating* if at least one vertex of T lies inside t and at least one vertex lies outside t; in other words, t is not a face of T.

Proposition 2. *A plane triangulation T is area-universal if and only if every 4-connected component of T is area-universal.*

Proof (Sketch). The proof is based on the fact that a plane graph G with a separating triangle t is area-universal if and only if G_{E}, the induced graph by t and its exterior, and G_{I}, the induced graph by t and its interior, are area-universal. In particular, Lemma 1 allows us to combine realizing drawings of G_{E} and G_{I} to a drawing of G.

Remark. Note that a plane 3-tree has no 4-connected component. (Recall that K_4 is 3-connected and a graph on $n > 4$ vertices is 4-connected if and only if it has no separating triangle.) This is another way to see their area-universality.

2.3   Characterization by Polynomial Equation System

Dobbins et al. [7, Proposition 1] show a close connection of area-universality and equation systems: For every plane graph G with area assignment $\mathcal{A}$ there exists a polynomial equation system $\mathcal{E}$ such that $\mathcal{A}$ is realizable if and only if $\mathcal{E}$ has a real solution. Here we strengthen the statement for triangulations, namely it suffices to guarantee the face areas; these imply all further properties such as planarity and the equivalent embedding. To do so, we introduce some notation.

A plane graph G induces an orientation of the vertices of each face. For a face f given by the vertices $v_1, \ldots, v_k$, we say f is *counter clockwise (ccw)* if the vertices $v_1, \ldots, v_k$ appear in ccw direction on a walk on the boundary of f; otherwise f is *clockwise (cw)*. Moreover, the function $\mathrm{AREA}(f, D)$ measures the area of a face f in a drawing D. For a ccw triangle t with vertices v_1, v_2, v_3, we denote the coordinates of v_i by (x_i, y_i). Its area in D is given by the determinant

$$\mathrm{Det}(v_1, v_2, v_3) := \det\big(c(v_1), c(v_2), c(v_3)\big) = 2 \cdot \mathrm{AREA}(t, D), \tag{1}$$

where $c(v_i) := (x_i, y_i, 1)$. Since the (complement of the) outer face f_o has area $\Sigma\mathcal{A}$ in an $\mathcal{A}$-realizing drawing, we define $\mathcal{A}(f_o) := \Sigma\mathcal{A}$. For a set of faces $\tilde{F} \subset F$, we define the *area equation system of $\tilde{F}$* as

$$\mathrm{AEQ}(T, \mathcal{A}, \tilde{F}) := \{\mathrm{Det}(v_i, v_j, v_k) = \mathcal{A}(f) \mid f \in \tilde{F}, f =: (v_i, v_j, v_k) \text{ ccw}\}.$$

For convenience, we omit the factor of 2 in each *area equation*. Therefore, without mentioning it any further, we usually certify the realizability of $\mathcal{A}$ by a $^1/_2\mathcal{A}$-realizing drawing. That is, if we say a triangle has area a, it may have area $^1/_2a$. Recall that, by Lemma 2, consistent scaling has no further implications.

Proposition 3. *Let T be a triangulation, $\mathcal{A}$ an area assignment, and f a face of T. Then $\mathcal{A}$ is realizable if and only if $\mathrm{AEQ}(T, \mathcal{A}, F \setminus \{f\})$ has a real solution.*

The key idea is that a (scaled) vertex placement of an $\mathcal{A}$-realizing drawing is a real solution of $\mathrm{AEQ}(T, \mathcal{A}, F \setminus \{f\})$ and vice versa. The main task is to guarantee crossing-freeness of the induced drawing; it follows from the following neat fact.

Lemma 4. *Let D be a vertex placement of a triangulation T where the orientation of each inner face in D coincides with the orientation in T. Then D represents a crossing-free straight-line drawing of T.*

A proof of Lemma 4 can be found in [2, in the end of the proof of Lemma 4.2]. An alternative proof relies on the properties of the determinant, in particular, on the fact that for any vertex placement D the area of the triangle formed by its outer vertices evaluates to

$$\mathrm{AREA}(f_o, D) = \sum_{f \in F'} \mathrm{AREA}(f, D). \tag{2}$$

Equation (2) shows that for every face $f \in F'$, the equation systems $\mathrm{AEQ}(T, \mathcal{A}, F')$ and $\mathrm{AEQ}(T, \mathcal{A}, F \setminus \{f\})$ are equivalent. This fact is also used for Proposition 3.

Remark 1. In fact, Lemma 4 and Proposition 3 generalize to *inner triangulations*, i.e., 2-connected plane graphs where every inner face is a triangle.

3   Area-Universality of Triangulations with p-orders

We consider planar triangulations with the following property: An order of the vertices $(v_1, v_2, \ldots, v_n)$, together with a set of *predecessors* $\mathrm{pred}(v_i) \subset N(v_i)$ for each vertex v_i, is a *p-order* if the following conditions are satisfied:

– $\mathrm{pred}(v_i) \subseteq \{v_1, v_2, \ldots, v_{i-1}\}$, i.e., the predecessors of v_i have an index $< i$,
– $\mathrm{pred}(v_1) = \emptyset$, $\mathrm{pred}(v_2) = \{v_1\}$, $\mathrm{pred}(v_3) = \mathrm{pred}(v_4) = \{v_1, v_2\}$, and
– for all $i > 4$: $|\mathrm{pred}(v_i)| = 3$, i.e., v_i has exactly three predecessors.

Note that $\mathrm{pred}(v_i)$ specifies a subset of preceding neighbors. Moreover, a p-order is defined for a planar graph independent of a drawing. We usually denote a p-order by $\mathcal{P}$ and state the order of the vertices; the predecessors are then implicitly given by $\mathrm{pred}(v_i)$. Figure 2 illustrates a p-order.

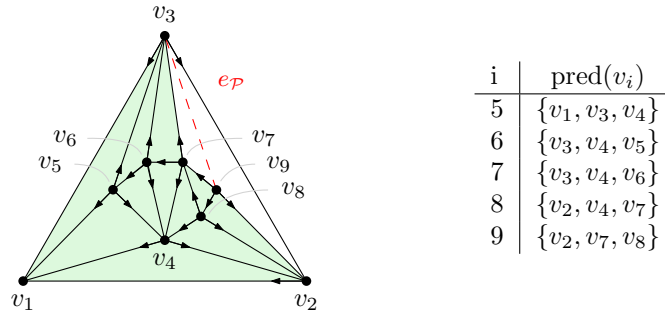| i	$\mathrm{pred}(v_i)$
5	$\{v_1, v_3, v_4\}$
6	$\{v_3, v_4, v_5\}$
7	$\{v_3, v_4, v_6\}$
8	$\{v_2, v_4, v_7\}$
9	$\{v_2, v_7, v_8\}$

Fig. 2: A plane 4-connected triangulation with a p-order $\mathcal{P}$. In an almost realizing vertex placement constructed with $\mathcal{P}$, all face areas are realized except for the two faces incident to the unoriented (dashed) edge $e_{\mathcal{P}}$ of $\mathcal{O}_{\mathcal{P}}$ (Lemma 8).

We pursue the following one-degree-of-freedom mechanism to construct realizing drawings for a plane triangulation T with a p-order $(v_1, v_2, \ldots, v_n)$ and an area assignment $\mathcal{A}$:
– Place the vertices v_1, v_2, v_3 at positions realizing the area equation of the face $v_1 v_2 v_3$. Without loss of generality, we set $v_1 = (0,0)$ and $v_2 = (1,0)$.
– Insert v_4 such that the area equation of face $v_1 v_2 v_4$ is realized; this is fulfilled if y_4 equals $\mathcal{A}(v_1 v_2 v_4)$ while $x_4 \in \mathbb{R}$ is arbitrary. The value x_4 is our variable.
– Place each remaining vertex v_i with respect to its predecessors $\mathrm{pred}(v_i)$ such that the area equations of the two incident face areas are respected; the coordinates of v_i are *rational* functions of x_4.
– Finally, all area equations are realized except for two special faces f_a and f_b. Moreover, the face area of f_a is a *rational* function $\mathfrak{f}$ of x_4.
– If $\mathfrak{f}$ is *almost surjective*, then there is a vertex placement D respecting all face areas and orientations, i.e., D is a real solution of $\mathrm{AEQ}(T, \mathcal{A}, F)$.
– By Proposition 3, D guarantees the realizability of $\mathcal{A}$.
– If this holds for *enough* area assignments, then T is area-universal.

3.1 Properties of p-orders

A p-order $\mathcal{P}$ of a plane triangulation T induces an orientation $\mathcal{O}_\mathcal{P}$ of the edges: For $w \in \mathrm{pred}(v_i)$, we orient the edge from v_i to w, see also Figure 2. By Proposition 2, we may restrict our attention to 4-connected triangulations. We note that 4-connectedness is not essential for our method but yields a cleaner picture.

Lemma 5. *Let T be a planar 4-connected triangulation with a p-order $\mathcal{P}$. Then $\mathcal{O}_\mathcal{P}$ is acyclic, $\mathcal{O}_\mathcal{P}$ has a unique unoriented edge $e_\mathcal{P}$, and $e_\mathcal{P}$ is incident to v_n.*

It follows that the p-order encodes all but one edge which is easy to recover. Therefore, the p-order of a planar triangulation T encodes T. In fact, T has a p-order if and only if there exists an edge e such that $T - e$ is 3-degenerate.

Convention. Recall that a drawing induces an orientation of each face. We follow the convention of stating the vertices of inner faces ccw and of the outer face in cw direction. This convention enables us to switch between different plane graphs of the same planar graph without changing the order of the vertices. To account for our convention, we redefine $\mathcal{A}(f_o) := -\Sigma \mathcal{A}$ for the outer face f_o. Then, for different embeddings, only the right sides of the AEQs change.

The next properties can be proved by induction and are shown in Figure 3.

Lemma 6. *Let T be a plane 4-connected triangulation with a p-order $\mathcal{P}$ specified by $(v_1, v_2, \ldots, v_n)$ and let T_i denote the subgraph of T induced by $\{v_1, v_2, \ldots, v_i\}$. For $i \geq 4$,*
  - *T_i has one 4-face and otherwise only triangles,*
  - *T_{i+1} can be constructed from T_i by inserting v_{i+1} in the 4-face of T_i, and*
  - *the three predecessors of v_i can be named $(p_\mathrm{F}, p_\mathrm{M}, p_\mathrm{L})$ such that $p_\mathrm{F} p_\mathrm{M} v_i$ and $p_\mathrm{M} p_\mathrm{L} v_i$ are (ccw inner and cw outer) faces of T_i.*
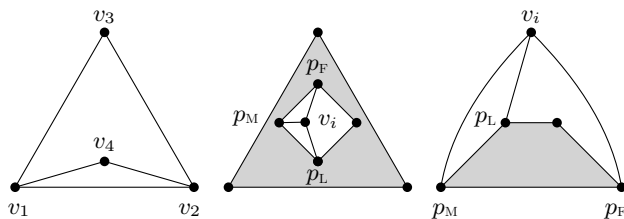

Fig. 3: Illustration of Lemma 6: (a) T_4, (b) v_i is inserted in an inner 4-face, (c) v_i is inserted in outer 4-face.
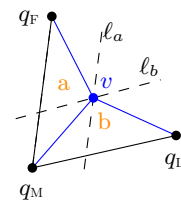
Fig. 4: Illustration of Lemma 7.

Remark 2. For every (non-equivalent) plane graph T' of T, the three predecessors $(p_\mathrm{F}, p_\mathrm{M}, p_\mathrm{L})$ of v_i in T' and T coincide.

Remark 3. Lemma 6 can be used to show that the number of 4-connected planar triangulations on n vertices with a p-order is $\Omega(2^n/n)$.

3.2   Constructing Almost Realizing Vertex Placements

Let T be a plane triangulation with an area assignment $\mathcal{A}$. We call a vertex placement D of T *almost $\mathcal{A}$-realizing* if there exist two faces f_a and f_b such that D is a real solution of the equation system $\text{AEQ}(T, \mathcal{A}, \tilde{F})$ with $\tilde{F} := F \setminus \{f_a, f_b\}$. In particular, we insist that the orientation and area of each face, except for f_a and f_b be correct, i.e., the area equations are fulfilled. Note that an almost realizing vertex placement does not necessarily correspond to a crossing-free drawing.

Observation. An almost $\mathcal{A}$-realizing vertex placement D fulfilling the area equations of all faces except for f_a and f_b, certifies the realizability of $\mathcal{A}$ if additionally the area equation of f_a is satisfied.

We construct almost realizing vertex placements with the following lemma.

Lemma 7. *Let $a, b \geq 0$ and let $q_{\text{F}}, q_{\text{M}}, q_{\text{L}}$ be three vertices with a non-collinear placement in the plane. Then there exists a unique placement for vertex v such the ccw triangles $q_{\text{F}} q_{\text{M}} v$ and $q_{\text{M}} q_{\text{L}} v$ fulfill the area equations for a and b, respectively.*

Proof. Consider Figure 4. To realize the areas, v must be placed on a specific line ℓ_a and ℓ_b, respectively. Note that ℓ_a is parallel to the segment $q_{\text{F}}, q_{\text{M}}$ and ℓ_b is parallel to the segment $q_{\text{M}}, q_{\text{L}}$. Consequently, ℓ_a and ℓ_b are not parallel and their intersection point yields the unique position for vertex v. The coordinates of v are specified by the two equations $\text{Det}(q_{\text{F}}, q_{\text{M}}, v) \stackrel{!}{=} a$ and $\text{Det}(q_{\text{M}}, q_{\text{L}}, v) \stackrel{!}{=} b$.

Note that if ℓ_a and ℓ_b are parallel and do not coincide, then there is no position for v realizing the area equations of the two triangles. Based on Lemma 7, we obtain our key lemma.

Lemma 8. *Let T be a plane 4-connected triangulation with a p-order $\mathcal{P}$ specified by $(v_1, v_2, \ldots, v_n)$. Let f_a, f_b be the faces incident to $e_{\mathcal{P}}$ and $f_0 := v_1 v_2 v_3$. Then there exists a constant $c > 0$ such that for a dense subset $\mathbb{A}_D$ of $\mathbb{A}^c$, every $\mathcal{A} \in \mathbb{A}_D$ has a finite set $\mathcal{B}(\mathcal{A}) \subset \mathbb{R}$, rational functions $x_i(\cdot, \mathcal{A})$, $y_i(\cdot, \mathcal{A})$, $\mathfrak{f}(\cdot, \mathcal{A})$ and a triangle $\triangle$, such that for all $x_4 \in \mathbb{R} \setminus \mathcal{B}(\mathcal{A})$, there exists a vertex placement $D(x_4)$ with the following properties:*

(i) f_0 coincides with the triangle $\triangle$,
(ii) $D(x_4)$ is almost realizing, i.e., a real solution of $\text{AEQ}(T, \mathcal{A}, F \setminus \{f_a, f_b\})$,
(iii) every vertex v_i is placed at the point $\big(x_i(x_4, \mathcal{A}), y_i(x_4, \mathcal{A})\big)$, and
(iv) the area of face f_a in $D(x_4)$ is given by $\mathfrak{f}(x_4, \mathcal{A})$.

The idea of the proof is to use Lemma 7 in order to construct $D(x_4)$ inductively. Therefore, given a vertex placement $v_1, \ldots, v_{i-1}$, we have to ensure that the vertices of $\text{pred}(v_i)$ are not collinear. To do so, we consider algebraically independent area assignments. We say an area assignment $\mathcal{A}$ of T is *algebraically independent* if the set $\{\mathcal{A}(f) | f \in F'\}$ is algebraically independent over $\mathbb{Q}$. In fact, the subset of algebraically independent area assignments $\mathbb{A}_I$ of $\mathbb{A}^c$ is dense when c is transcendental.

We call the function $\mathfrak{f}$, constructed in the proof of Lemma 8, the *last face function* of T and interpret it as a function in x_4 whose coefficients depend on $\mathcal{A}$.

3.3 Almost Surjectivity and Area-Universality

In the following, we show that almost surjectivity of the last face function implies area-universality. Let A and B be sets. A function $f\colon A \to B$ is *almost surjective* if f attains all but finitely many values of B, i.e., $B \setminus f(A)$ is finite.

Theorem 1. *Let T be a 4-connected plane triangulation with a p-order $\mathcal{P}$ and let $\mathbb{A}_D, \mathbb{A}^c, \mathfrak{f}$ be obtained by Lemma 8. If the last face function $\mathfrak{f}$ is almost surjective for all area assignments in $\mathbb{A}_D$, then T is area-universal.*

Proof. By Lemma 3, it suffices to show that every $\mathcal{A} \in \mathbb{A}_D$ is realizable. Let f_0 be the triangle formed by v_1, v_2, v_3 and $\mathbb{A}^+ := \mathbb{A}^{\leq 2c}|_{f_0 \to \mathcal{A}(f_0)}$. By Proposition 1, $\mathcal{A}$ is realizable if every open neighborhood of $\mathcal{A}$ in $\mathbb{A}^+$ contains a realizable area assignment. Let f_a and f_b denote the faces incident to $e_{\mathcal{P}}$ and $a := \mathcal{A}(f_a)$. Lemma 8 guarantees the existence of a finite set $\mathcal{B}$ such that for all $x_4 \in \mathbb{R} \setminus \mathcal{B}$, there exists an almost $\mathcal{A}$-realizing vertex placement $D(x_4)$. Since $\mathcal{B}$ is finite and $\mathfrak{f}$ is almost surjective, for every ε with $0 < \varepsilon < c$, there exists $\tilde{x} \in \mathbb{R} \setminus \mathcal{B}$ such that $a \leq \mathfrak{f}(\tilde{x}) \leq a + \varepsilon$, i.e., the area of face f_a in $D(\tilde{x})$ is between a and $a + \varepsilon$. (If f_a and f_b are both inner faces, then the face f_b has an area between $b - \varepsilon$ and b, where $b := \mathcal{A}(f_b)$. Otherwise, if f_a or f_b is the outer face, then the total area changes and face f_b has area between b and $b + \varepsilon$.) Consequently, for some $\mathcal{A}'$ in the ε-neighborhood of $\mathcal{A}$ in $\mathbb{A}^+$, $D(\tilde{x})$ is a real solution of $\text{AEQ}(T, \mathcal{A}', F \setminus \{f_b\})$ and Proposition 3 ensures that $\mathcal{A}'$ is realizable. By Proposition 1, $\mathcal{A}$ is realizable. Thus, T is area-universal.

To prove area-universality, we use the following sufficient condition for almost surjectivity. We say two real polynomials p and q are *crr-free* if they do not have common real roots. For a rational function $f := \frac{p}{q}$, we define the *max-degree* of f as $\max\{|p|, |q|\}$, where $|p|$ denotes the degree of p. Moreover, we say f is crr-free if p and q are. The following property follows from the fact that polynomials of odd degree are surjective.

Lemma 9. *Let $p, q\colon \mathbb{R} \to \mathbb{R}$ be polynomials and let Q be the set of the real roots of q. If the polynomials p and q are crr-free and have odd max-degree, then the function $f\colon \mathbb{R} \backslash Q \to \mathbb{R}$, $f(x) = \frac{p(x)}{q(x)}$ is almost surjective.*

For the final result, we make use of several convenient properties of algebraically independent area assignments. For $\mathcal{A}$, let $\mathfrak{f}_{\mathcal{A}}$ denote the last face function and $d_1(\mathfrak{f}_{\mathcal{A}})$ and $d_2(\mathfrak{f}_{\mathcal{A}})$ the degree of the numerator and denominator polynomial of $\mathfrak{f}_{\mathcal{A}}$ in x_4, respectively. Since $\mathfrak{f}_{\mathcal{A}}$ is a function in x_4 whose coefficients depend on $\mathcal{A}$, algebraic independence directly yields the following property.

Claim 1. *For two algebraically independent area assignments $\mathcal{A}, \mathcal{A}' \in \mathbb{A}_I$ of a 4-connected triangulation with a $p-$order $\mathcal{P}$, the degrees of the last face functions $\mathfrak{f}_{\mathcal{A}}$ and $\mathfrak{f}_{\mathcal{A}'}$ with respect to $\mathcal{P}$ coincide, i.e., $d_i(\mathfrak{f}_{\mathcal{A}}) = d_i(\mathfrak{f}_{\mathcal{A}'})$ for $i \in [2]$.*

In fact, the degrees do not only coincide for all algebraically independent area assignments, but also for different embeddings of the plane graph. For a plane triangulation T, let T^* denote the corresponding planar graph and $[T]$ the set (of equivalence classes) of all plane graphs of T^*.

Claim 2. *Let T be a plane 4-connected triangulation with a p-order $\mathcal{P}$. Then for every plane graph $T' \in [T]$, and algebraically independent area assignments $\mathcal{A}$ of T and $\mathcal{A}'$ of T', the last face functions $\mathfrak{f}_\mathcal{A}$ and $\mathfrak{f}'_{\mathcal{A}'}$ with respect to $\mathcal{P}$ have the same degrees, i.e., $d_i(\mathfrak{f}_\mathcal{A}) = d_i(\mathfrak{f}'_{\mathcal{A}'})$ for $i \in [2]$.*

This implies our final result:

Corollary 1. *Let T be a plane triangulation with a p-order $\mathcal{P}$. If the last face function $\mathfrak{f}$ of T is crr-free and has odd max-degree for one algebraically independent area assignment, then every plane graph in $[T]$ is area-universal.*

4   Applications

We now use Theorem 1 and Corollary 1 to prove area-universality of some classes of triangulations. The considered graphs rely on an operation that we call *diamond addition*. Consider the left image of Figure 5. Let G be a plane graph and let e be an inner edge incident to two triangular faces that consist of e and the vertices u_1 and u_2, respectively. Applying a *diamond addition of order k* on e results in the graph G' which is obtained from G by subdividing edge e with k vertices, $v_1, \ldots, v_k$, and inserting the edges $v_i u_j$ for all pairs $i \in [k]$ and $j \in [2]$. Figure 5 illustrates a diamond addition on e of order 3.
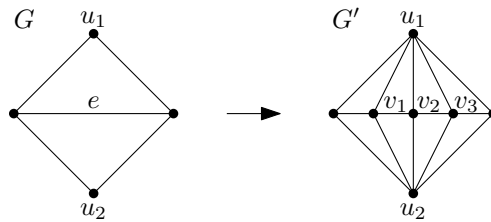

Fig. 5: Obtaining G' from G by a diamond addition of order 3 on edge e.

4.1   Accordion graphs

An accordion graph can be obtained from the plane octahedron graph $\mathcal{G}$ by a diamond addition: Choose one edge of the central triangle of $\mathcal{G}$ as the *special edge*. The *accordion* graph $\mathcal{K}_\ell$ is the plane graph obtained by a diamond addition of order ℓ on the special edge of $\mathcal{G}$. Consequently, $\mathcal{K}_\ell$ has $\ell + 6$ vertices. We speak of an *even accordion* if ℓ is even and of an *odd accordion* if ℓ is odd. Figure 1 illustrates the accordion graphs $\mathcal{K}_i$ for $i \leq 3$. Note that $\mathcal{K}_0$ is $\mathcal{G}$ itself and $\mathcal{K}_1$ is the unique 4-connected plane triangulation on seven vertices. Due to its symmetry, it holds that $[\mathcal{K}_\ell] = \{\mathcal{K}_\ell\}$.

Theorem 2. *The accordion graph $\mathcal{K}_\ell$ is area-universal if and only if ℓ is odd.*

Proof (Sketch). Performing a diamond addition of order ℓ on some plane graph changes the degree of exactly two vertices by ℓ while all other vertex degrees remain the same. Consequently, if ℓ is even, all vertices of $\mathcal{K}_\ell$ have even degree, and hence, $\mathcal{K}_\ell$ as an Eulerian triangulation is not area-universal as shown by the author in [15, Theorem 1].

It remains to prove the area-universality of odd accordion graphs with the help of Theorem 1. Consider an arbitrary but fixed algebraically independent area assignment $\mathcal{A}$. We use the p-order depicted in Figure 6 to construct an almost realizing vertex placement. We place the vertices v_1 at $(0,0)$, v_2 at $(1,0)$, v_3 at $(1, \Sigma\mathcal{A})$, and v_4 at (x_4, a) with $a := \mathcal{A}(v_1 v_2 v_4)$. Consider also Figure 6.
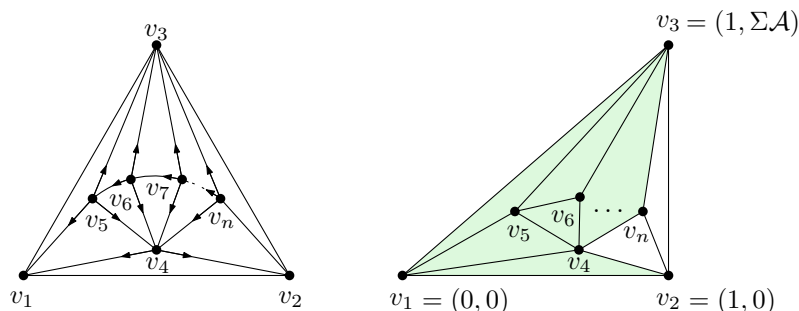

Fig. 6: A p-order of an accordion graph (left) and an almost realizing vertex placement (right), where the shaded faces are realized.

We use Lemma 8 to construct an almost realizing vertex placement. Note that for all vertices v_i with $i > 5$, the three predecessors of v_i are $p_{\mathrm{F}} = v_3$, $p_{\mathrm{M}} = v_{i-1}$ and $p_{\mathrm{L}} = v_4$. One can show that the vertex coordinates of v_i can be expressed as $x_i = \mathcal{N}_i^x / \mathcal{D}_i$ and $y_i = \mathcal{N}_i^y / \mathcal{D}_i$, where $\mathcal{N}_i^x, \mathcal{N}_i^y, \mathcal{D}_i$ are polynomials in x_4. Moreover, the polynomials fulfill the following crucial properties.

Lemma 10. *For all $i \geq 5$, it holds that $|\mathcal{D}_5| = 1$ and*

$$|\mathcal{N}_{i+1}^x| = |\mathcal{D}_{i+1}| = |\mathcal{N}_{i+1}^y| + 1 = |\mathcal{D}_i| + 1.$$

Consequently, $|\mathcal{N}_i^x| = |\mathcal{D}_i|$ is odd if and only if i is odd. In particular, for odd ℓ, $|\mathcal{N}_n^x| = |\mathcal{D}_n|$ is odd since the number of vertices $n = \ell + 6$ is odd.

Lemma 11. *For all $i \geq 5$ and $\circ \in \{x, y\}$, it holds that $\mathcal{N}_i^\circ$ and $\mathcal{D}_i$ are crr-free.*

Consequently, the area of the ccw triangle $v_2 v_3 v_n$ in $D(x_4)$ is given by the crr-free last face function

$$\mathfrak{f}(x) := \mathrm{Det}(v_2, v_3, v_n) = \Sigma\mathcal{A}(1 - x_n) = \Sigma\mathcal{A}\left(1 - \frac{\mathcal{N}_n^x}{\mathcal{D}_n}\right).$$

Since $|\mathcal{N}_n^x|$ and $|\mathcal{D}_n|$ are odd, the max-degree of $\mathfrak{f}$ is odd. Thus, Lemma 9 ensures that $\mathfrak{f}$ is almost surjective. By Theorem 1, $\mathcal{K}_\ell$ is area-universal for odd ℓ.

This result can be generalized to double stacking graphs.
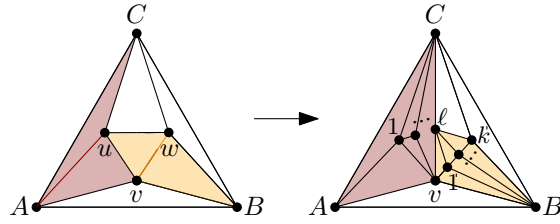
Fig. 7: A double stacking graph $\mathcal{H}_{\ell,k}$.

4.2 Double Stacking Graphs

Denote the vertices of the plane octahedron $\mathcal{G}$ by ABC and uvw as depicted in Figure 7. The *double stacking graph* $\mathcal{H}_{\ell,k}$ is the plane graph obtained from $\mathcal{G}$ by applying a diamond addition of order $\ell - 1$ on Au and a diamond addition of order $k - 1$ on vw. Note that $\mathcal{H}_{\ell,k}$ has $(\ell + k + 4)$ vertices. Moreover, $\mathcal{H}_{\ell,1}$ is isomorphic to $\mathcal{K}_{\ell-1}$; in particular, $\mathcal{H}_{1,1}$ equals $\mathcal{G}$. Note that $[\mathcal{H}_{\ell,k}]$ usually contains several (equivalence classes of) plane graphs.

Theorem 3. *A plane graph in $[\mathcal{H}_{\ell,k}]$ is area-universal if and only if $\ell \cdot k$ is even.*

If $\ell \cdot k$ is odd, every plane graph in $[\mathcal{H}_{\ell,k}]$ is Eulerian and hence not area-universal by [15, Theorem 1]. If $\ell \cdot k$ is even, we consider an algebraically independent area assignment of $\mathcal{H}_{\ell,k}$, show that its last face function is crr-free and has odd max-degree. Then we apply Corollary 1.

Theorem 3 implies that

Corollary 2. *For every $n \geq 7$, there exists a 4-connected triangulation on n vertices that is area-universal.*

5 Discussion and Open Problems

For triangulations with p-orders, we introduced a sufficient criterion to prove area-universality of all embeddings of a planar graph which relies on checking properties of one area assignments of one plane graph. We used the criterion to present two families of area-universal triangulations. Since area-universality is maintained by taking subgraphs, area-universal triangulations are of special interest. For instance, the area-universal double stacking graphs are used in [10,16] to show that all plane quadrangulations with at most 13 vertices are area-universal. The analysis of accordion graphs showns that area-universal and non-area-universal graphs can be structural very similar. The class of accordion graphs gives a hint why understanding area-universality seems to be a difficult problem. In conclusion, we pose the following open questions:

- – Is area-universality a property of plane or planar graphs?
- – What is the complexity of deciding the area-universality of triangulations?
- – Can area-universal graphs be characterized by *local* properties?

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A   Proofs of Section 2

A.1   Proof of Proposition 1

Proposition 1 follows from the fact that set of vertex placements $\mathcal{D}^{\leq c}|_{f \to \triangle}$ of T is compact, where $\mathcal{D}^{\leq c}|_{f \to \triangle}$ denotes the set of crossing-free vertex placements where f coincides with a fixed triangle $\triangle$ of positive area and additionally the total area in each drawing does not exceed c.

Lemma 12. *Let T be a plane triangulation T, let f be some face of T, and $c \in \mathbb{R}_{>0}$. Then the set of vertex placements $\mathcal{D}^{\leq c}|_{f \to \triangle}$ of T is compact.*

Proof. First note that closedness follows from the fact that we allow for degenerate drawings. Second, we show that $\mathcal{D}^{\leq c}|_{f \to \triangle}$ is bounded. If f is the outer face, then clearly all inner vertex coordinates are bounded by the coordinates of $\triangle$. Hence, it remains to consider the case that f is an inner face. By assumption, $\triangle$ has positive area and hence three sides which are pairwise not parallel.

We show that all vertices lie inside a bounded region which consists of the intersection of three half spaces. For each side s of f, we consider the line ℓ_s such that s and any point of ℓ_s form a triangle of area c that intersects f. Let H_s denote the half space defined by ℓ_s that contains f. We claim that any drawing D in $\mathcal{D}^{\leq c}|_{f \to \triangle}$ lies in H_s. Suppose a vertex v of T lies outside H_s, then the triangle t' formed by s and v is contained in D since the outer face is triangular and thus convex. However, the area of t' exceeds c. Therefore all vertices lie within the intersection of the three half spaces H_s.

Now we are ready to prove the proposition.

Proposition 1. *Let T be a plane triangulation and $c > 0$. Then $\mathcal{A} \in \mathbb{A}^c$ is realizable if and only if for some face f with $\mathcal{A}(f) > 0$ every open neighborhood of $\mathcal{A}$ in $\mathbb{A}^{\leq 2c}|_{f \to \mathcal{A}(f)}$ contains a realizable area assignment.*

Proof. Suppose $\mathcal{A} \in \mathbb{A}$ is realizable, then clearly, $\mathcal{A}$ itself lies in every of its neighborhoods and serves as a certificate of a realizing drawing.

Suppose every open neighborhood of $\mathcal{A}$ in $\mathbb{A}^{\leq 2c}|_{f \to \mathcal{A}(f)}$ contains a realizable area assignment. Hence, we may construct a sequence of realizable assignments $(\mathcal{A}_i)_{i \in \mathbb{N}}$ converging to $\mathcal{A}$. Since $\mathcal{A}_i(f) = \mathcal{A}(f)$ for all i, Lemma 1 allows to pick an A_i-realizing drawing D_i such that the placement of f coincides with a fixed triangle $\triangle$ of area $\mathcal{A}(f)$, i.e., $D_i \in \mathcal{D}^{\leq 2c}|_{f \to \triangle}$. By Lemma 12, the sequence $(D_i)_{i \in \mathbb{N}}$ is bounded. Therefore, by the Bolzano-Weierstrass theorem, $(D_i)_{i \in \mathbb{N}}$ contains a converging subsequence with limit D. By the compactness, D is contained in $\mathcal{D}^{\leq 2c}|_{f \to \triangle}$ and thus yields a crossing-free drawing of G. Note that for every inner face $f' \in F'$ it holds that:

$$\text{AREA}(f', D) = \lim_{i \to \infty} \text{AREA}(f', D_i) = \lim_{i \to \infty} \mathcal{A}_i(f') = \mathcal{A}(f').$$

Consequently, D guarantees that $\mathcal{A}$ is realizable.

A.2   Proof of Proposition 2

Proposition 2. *A plane triangulation T is area-universal if and only if every 4-connected component of T is area-universal.*

Proof. If T is area-universal, then also every subgraph and consequently all its 4-connected components are area-universal.

We prove the other direction by induction. For the induction base, note that on $n = 3$ and $n = 4$ vertices there exist unique triangulations, namely the complete graphs on three and four vertices, which are area-universal as stacked triangulations and have no 4-connected components.

For the induction step consider a triangulation T on $n > 4$ vertices. Now we use the fact that a triangulation on $n > 4$ vertices is 4-connected if and only if it has no separating triangle. If T is 4-connected, then the statement is vacuous. So suppose T has a separating triangle t. Let T_i denote the triangulation consisting of t and its interior, and let T_o denote the triangulation consisting of t and its exterior, see also Figure 8.
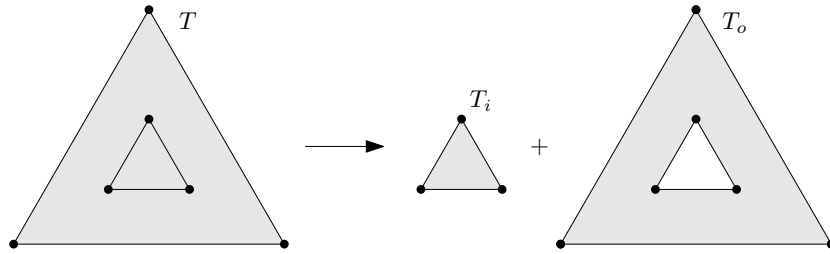
Fig. 8: Decomposing a triangulation along separating triangles.

By assumption all 4-connected components of T, and thus also of T_i and T_o, are area-universal. By induction it follows that T_i and T_o are area-universal. We show how to obtain a realizing drawing of T for any area assignment $\mathcal{A}$ of T. For T_o, consider a realizing drawing D_o of $\mathcal{A}_o$ where $\mathcal{A}_o(t)$ equals the sum of all faces in T_i and $\mathcal{A}_o(f) = \mathcal{A}(f)$ for all other faces of T_o. Now, for T_i consider a realizing drawing D_i of $\mathcal{A}_i$, the restriction of $\mathcal{A}$ to T_i. By Lemma 1, we may assume that the outer face of D_i coincides with t in D_o. By construction, $A_o(t) = \Sigma A_i$ and hence the area of t in D_o coincides with the area of the outer triangle of T_i. Hence, the union of D_o and D_i yields an $\mathcal{A}$-realizing drawing of T.

A.3   Proof of Proposition 3

Proposition 3. *Let T be a triangulation, $\mathcal{A}$ an area assignment, and f a face of T. Then $\mathcal{A}$ is realizable if and only if $\textsc{aeq}(T, \mathcal{A}, F \setminus \{f\})$ has a real solution.*

Proof. The proof consists of two directions. If $\mathcal{A}$ is realizable, then the vertex placement of an $1/2\mathcal{A}$-realizing drawing is a real solution of $\textsc{aeq}(T, \mathcal{A}, F \setminus \{f\})$. Recall that by Lemma 2, $\mathcal{A}$ is realizable if $1/2\mathcal{A}$ is.

   If $\textsc{aeq}(T, \mathcal{A}, F \setminus \{f\})$ has a real solution S, S yields a vertex placement D satisfying $\mathcal{A}$ and preserving the orientation of all but one face f. It remains to show that D corresponds to a crossing-free drawing. If f is the outer face, then Lemma 4 implies that D is an equivalent straight-line drawing of T. Equation (2) shows that for every face $f' \in F'$, the equation systems $\textsc{aeq}(T, \mathcal{A}, F')$ and $\textsc{aeq}(T, \mathcal{A}, F \setminus \{f'\})$ are equivalent. Consequently, we may assume that f is the outer face f_o. Thus, it remains to prove Equation (2). Let $v_1, v_2, \ldots, v_k$ denote the vertices of the outer face f_o (of an inner triangulation) in counter clockwise orientation. Recall that $\mathrm{Det}(u, v, w)$ denotes the determinant of the homogeneous coordinates of u, v, w as defined in Equation (1). Moreover, $\det(u, v)$ denotes the determinant of the 2-dimensional coordinates of u and v. Then, by the properties of the determinant, for any vertex placement D it holds that

$$
\begin{aligned}
2 \cdot \textsc{area}(f_o, D) = \sum_{i=2}^{k-1} \mathrm{Det}(v_1, v_i, v_{i+1}) &= \sum_{i=1}^{k-1} \det(v_i, v_{i+1}) \\
&= \sum_{i=1}^{k-1} \det(v_i, v_{i+1}) + \sum_{e=(u,v)\text{inner}} \big(\det(u, v) + \det(u, v) \big) \\
&= \sum_{f=(u,v,w)\in F'} \mathrm{Det}(u, v, w) = 2 \cdot \sum_{f\in F'} \textsc{area}(f, D).
\end{aligned}
$$

In the last line, we use the fact that every inner edge appears in both direction and every outer edge in one direction. Thus we can traverse each inner face in ccw direction. This fact is illustrated in Figure 9.
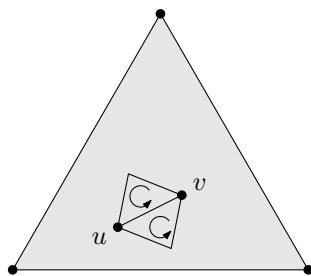

Fig. 9: Illustration of the proof of Proposition 3.

B Proofs of Section 3

Here we present the omitted proofs of Section 3. We start with the properties of p-orders.

Lemma 5. *Let T be a planar 4-connected triangulation with a p-order $\mathcal{P}$. Then $\mathcal{O}_\mathcal{P}$ is acyclic, $\mathcal{O}_\mathcal{P}$ has a unique unoriented edge $e_\mathcal{P}$, and $e_\mathcal{P}$ is incident to v_n.*

Proof. By definition of $\mathcal{O}_\mathcal{P}$ if an edge (v_{i+1}, v_j) is oriented from v_{i+1} to v_j, then $i > j$. Hence the orientation is acyclic. In particular, no edge is oriented in two directions. The number of unoriented edges follows by double counting the edges of T. On the one hand, by Euler's formula, the number of edges in a triangulation is $|E| = 3n - 6$. On the other hand, the number of oriented edges $E_\uparrow$ is given by the sum of the outdegrees.

$$|E_\uparrow| = \sum_{i=1}^{n} \mathrm{outdeg}(v_{i+1}) = 0 + 1 + 2 + 2 + 3(n-4) = 3n - 7.$$

Hence, $|E| - |E_\uparrow| = 1$ and thus there is exactly one edge e without orientation. Observe that the last vertex v_n in the p-order has indegree 0. If T has minimum degree 4, then e is incident to v_n; otherwise v_n would be a vertex of degree 3. This is a contradiction.

Lemma 6. *Let T be a plane 4-connected triangulation with a p-order $\mathcal{P}$ specified by $(v_1, v_2, \ldots, v_n)$ and let T_i denote the subgraph of T induced by $\{v_1, v_2, \ldots, v_i\}$. For $i \geq 4$,*
  – T_i has one 4-face and otherwise only triangles,
  – T_{i+1} can be constructed from T_i by inserting v_{i+1} in the 4-face of T_i, and
  – the three predecessors of v_i can be named $(p_\mathrm{F}, p_\mathrm{M}, p_\mathrm{L})$ such that $p_\mathrm{F} p_\mathrm{M} v_i$ and $p_\mathrm{M} p_\mathrm{L} v_i$ are (ccw inner and cw outer) faces of T_i.

Proof. We prove this statement by induction. For the induction base, note that T_4 has three faces: the triangle $v_3 v_2 v_1$, the triangle $v_1 v_2 v_4$, and the 4-face $v_1 v_4 v_2 v_3$. Figure 3 (a) depicts T_4 for the case that $v_1 v_2 v_3$ is the outer face. By 4-connectedness, v_3 and v_4 cannot share an edge. Therefore, with this notation, the inner faces are ccw and the outer face is cw oriented – independent of the choice of the outer face.

Now we consider the induction step and insert v_{i+1} in T_i. Since T is 4-connected, v_{i+1} can only be placed in the unique 4-face f of T_i. Clearly, any three vertices of f are consecutive on the boundary cycle of f. Hence, the predecessors of v_{i+1} form a path of length three along f. We define p_M as the middle vertex of this path. Naming the remaining predecessors by p_F and p_L, $p_\mathrm{F} p_\mathrm{M} v_{i+1}$ and $p_\mathrm{M} p_\mathrm{L} v_{i+1}$ are (not necessarily correctly oriented) triangles in T_{i+1}. Since T is 4-connected, these triangles of T_{i+1} are faces in T and thus also in T_{i+1}. Furthermore, $p_\mathrm{F} v_{i+1} p_\mathrm{L} w$ forms a 4-face of T_{i+1} where w is the vertex of f which is not in $\mathrm{pred}(v_{i+1})$.

For the correct orientation we distinguish two cases: If f is an inner face, we define p_{F} as the ccw first vertex (of the path of predecessors in f) and p_{L} as the ccw last vertex. Figure 3 (b) illustrates this definition for the case that f is an inner face. Otherwise, f is the outer face and we define p_{F} as the cw first vertex and p_{L} the cw last vertex. This case is displayed in Figure 3 (c). Then, $p_{\text{F}}p_{\text{M}}v_{i+1}$ and $p_{\text{M}}p_{\text{L}}v_{i+1}$ are ccw faces in T_{i+1} if and only if they are inner faces of T.

As mentioned in remark 3, Lemma 6 can be used to obtain a lower bound on the number of 4-connected planar triangulations on n vertices with a p-order.

Proposition 4. *The number of 4-connected planar triangulations on n vertices with a p-order is $\Omega(1/n \cdot 2^n)$.*

Proof. Firstly, we show that a 4-connected triangulation T on n vertices has at most $9n \cdot 2^n$ different p-orders. We consider every p-order in the reverse order $v_n, ..., v_1$. T has at most $3n$ edges which may serve as the unique unoriented edge. Its deletion yields a 4-face. By Lemma 6, in every p-order for $i = n, ..., 5$ the vertex v_i is a vertex incident to a 4-face in T_i. Removing v_i from T_i yields a graph T_{i-1} that has again a unique 4-face. It follows from Lemma 5 that v_i is a vertex of degree 3 in T_i. Consequently, all neighbors of v_i in T_i are the predecessors of v_i.

Consider Figure 10 and observe that two adjacent vertices of degree 3 on a 4-face $x_1y_1y_2x_2$ certify a separating triangle, unless $n \leq 5$: Since all other faces of T_i are triangles, every pair of adjacent vertices has a common neighbor outside of the 4-face. If x_i has degree 3, the common neighbor of x_iy_i and x_1x_2 coincides; we call it z. Thus, zy_1y_2 is a separating triangle unless T_i contains only these five vertices. By 4-connectivity, each 4-face has at most two vertices of degree 3 for $i \geq 5$ and there are at most two choices for the vertex v_i. For $i = 5$, the number of choices is upper bounded by the four vertices of the 4-cycle and for $i \leq 4$, by another four; two for v_1, v_2 and two for v_3, v_4.
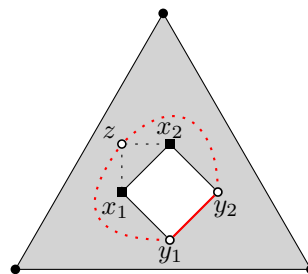
Fig. 10: Two adjacent vertices of degree three in a 4-face certify a separating triangle.

Consequently, for a specific unoriented edge, the number of vertex orderings is at most $2^{n-5} \cdot 4 \cdot 4 = 2^{n-1}$. This makes a total of at most $3n \cdot 2^{n-1}$ different p-orders for a fixed triangulation.

In order to build a 4-connected triangulation with a p-order $v_1, v_2, \ldots, v_n$, we specify the middle predecessor v_{M} of v_i for $5 \leq i \leq n$ from the 4-face of T_{i-1}. By Lemma 6, the remaining two predecessors of v_i are the two neighbors of v_{M} in the 4-face. Thus, we have four choices for v_{M} in each step $i > 5$. For $i = 5$, neither v_3 nor v_4 can serve as the middle predecessor since this results in a separating triangle. Thus, we obtain at least $2 \cdot 4^{n-5}$ different p-orders. By the above observation at most $3n \cdot 2^{n-1}$ belong to the same triangulation. Hence there exist $\Omega(1/n \cdot 2^n)$ 4-connected planar triangulation on n vertices.

B.1   Proof of Lemma 8

For the proof of Lemma 8, we introduce the concept of algebraically independent area assignments. A set of real numbers $\{a_1, a_2, \ldots, a_k\}$ is *algebraically independent* over $\mathbb{Q}$ if for each polynomial $p(x_1, x_2, \ldots, x_k)$ with coefficients from $\mathbb{Q}$, different from the 0-polynomial, it holds that $p(a_1, a_2, \ldots, a_k) \neq 0$. We say a face area assignment $\mathcal{A}$ of a plane graph G is *algebraically independent* if the set $\{\mathcal{A}(f) | f \in F'\}$ is algebraically independent over $\mathbb{Q}$. Note that then for all $f_0 \in F'$ it holds that $\{\mathcal{A}(f) | f \in F' \setminus f_0\} \cup \{\Sigma \mathcal{A}\}$ is algebraically independent. For a transcendental c, we denote the subset of $\mathbb{A}^c$ consisting of all algebraically independent area assignments by $\mathbb{A}_I$ and show that $\mathbb{A}_I$ is a dense subset.

Claim 3. *If c is transcendental c, then $\mathbb{A}_I$ is dense in $\mathbb{A}^c$.*

Proof. We show by induction that the set of algebraically independent k-tuples is dense. Our proof is built upon the fact that the algebraic closure of a countable field is countable [19, p. 343, Cor. B-2.41].

For the induction base, we consider $k = 1$. Since the algebraic closure of $\mathbb{Q}$ is countable, its complement $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Now we consider the induction step from $k - 1$ to k. In order to show that the algebraically independent k-tuple are dense, it suffices to show that for each a in $\mathbb{R}^n$ and each ε-ball B of a, there exists an algebraically independent b in B. By induction hypothesis, we find a $(b_2, \ldots, b_k)$ with algebraically independent entries, that is arbitrarily close to $(a_2, \ldots, a_k)$. Let K denote the algebraic closure of $\mathbb{Q}(b_2, ..., b_n)$, the smallest field containing $\mathbb{Q}$ and $\{b_2, \ldots, b_k\}$. As a rational function field over $\mathbb{Q}$, the field $\mathbb{Q}(b_2, ..., b_n)$ is countable. Thus, since the algebraic closure of a countable field is countable, K is countable. Thus in each open neighborhood of a_1, there exists a b_1 in the complement of K. Therefore, in each ε-ball of $a = (a_1, \ldots, a_n)$ there exists an algebraically independent $b = (b_1, \ldots, b_n)$.

Now, we are ready to prove the main lemma.

Lemma 8. *Let T be a plane 4-connected triangulation with a p-order $\mathcal{P}$ specified by $(v_1, v_2, \ldots, v_n)$. Let f_a, f_b be the faces incident to $e_\mathcal{P}$ and $f_0 := v_1 v_2 v_3$. Then there exists a constant $c > 0$ such that for a dense subset $\mathbb{A}_D$ of $\mathbb{A}^c$, every $\mathcal{A} \in \mathbb{A}_D$ has a finite set $\mathcal{B}(\mathcal{A}) \subset \mathbb{R}$, rational functions $x_i(\cdot, \mathcal{A})$, $y_i(\cdot, \mathcal{A})$, $\mathfrak{f}(\cdot, \mathcal{A})$ and a triangle $\triangle$, such that for all $x_4 \in \mathbb{R} \setminus \mathcal{B}(\mathcal{A})$, there exists a vertex placement $D(x_4)$ with the following properties:*

(i) f_0 coincides with the triangle $\triangle$,
(ii) $D(x_4)$ is almost realizing, i.e., a real solution of $\mathrm{AEQ}(T, \mathcal{A}, F \setminus \{f_a, f_b\})$,
(iii) every vertex v_i is placed at the point $\big(x_i(x_4, \mathcal{A}), y_i(x_4, \mathcal{A})\big)$, and
(iv) the area of face f_a in $D(x_4)$ is given by $\mathfrak{f}(x_4, \mathcal{A})$.

Proof. Let c be transcendental. We show that the claim holds for $\mathbb{A}_D := \mathbb{A}_I$. Thus, we consider an arbitrary $\mathcal{A} \in \mathbb{A}_I$ and think of $\mathcal{A}$ as an *abstract* area assignment, where the prescribed areas are still variables. The idea of the proof

is simple. Given a placement of $v_1, \ldots, v_{i-1}$, we want to insert v_i by Lemma 7. Thus, we need to guarantee that the predecessors are not collinear.

We rename the vertices v_1, v_2, v_3, v_4 such that the triangle $v_1 v_3 v_2$ and $v_1 v_2 v_4$ are ccw inner and cw outer faces. For simplicity, we can think of f_0 as being the outer face. However, the construction works in all settings.

We place v_1 at $(0,0)$, v_2 at $(0,1)$ and set $y_3 := -\mathcal{A}(f_0)$; recall that for the outer face f_o it holds that $\mathcal{A}(f_o) := -\Sigma \mathcal{A}$. We use the freedom to specify x_3 at a later state. This guarantees property (i). Furthermore, for $a := \mathcal{A}(v_1 v_2 v_4)$ we place v_4 at (x_4, a). Consequently, the face area of the triangle $v_1 v_2 v_4$ is realized for all choices of x_4.

For property (ii), we show that for all but finitely many values of x_4, we obtain an almost $\mathcal{A}$-realizing vertex placement $D(x_4)$. By Lemma 6, the three predecessors of v_i can be named $(p_{\text{F}}, p_{\text{M}}, p_{\text{L}})$ such that $p_{\text{F}} p_{\text{M}} v_i$ and $p_{\text{M}} p_{\text{L}} v_i$ are ccw inner or cw outer faces of T. By Lemma 7, the vertex coordinates of v_i follow directly from its three predecessors $p_{\text{F}}, p_{\text{M}}, p_{\text{L}}$ – unless these are collinear. Denoting the coordinates of p_i by (x_i, y_i) and solving the area equations yields the following coordinates of v_i:

$$x_i = x_{\text{M}} + \frac{a(x_{\text{L}} - x_{\text{M}}) - b(x_{\text{M}} - x_{\text{F}})}{x_{\text{F}}(y_{\text{M}} - y_{\text{L}}) + x_{\text{M}}(y_{\text{L}} - y_{\text{F}}) + x_{\text{L}}(y_{\text{F}} - y_{\text{M}})} \tag{3}$$

$$y_i = y_{\text{M}} + \frac{a(y_{\text{L}} - y_{\text{M}}) - b(y_{\text{M}} - y_{\text{F}})}{x_{\text{F}}(y_{\text{M}} - y_{\text{L}}) + x_{\text{M}}(y_{\text{L}} - y_{\text{F}}) + x_{\text{L}}(y_{\text{F}} - y_{\text{M}})} \tag{4}$$

Note that the predecessors are collinear if and only if the denominators of Equations (3) and (4) vanish.

Assume for now, that we are considering a position for x_4 such that no triple of predecessors becomes collinear. For $i = 5, \ldots, n$, we place v_i according to Lemma 7 and satisfy two new area equations. Together with the realized face area of the triangles $v_1 v_2 v_4$ and $v_1 v_2 v_3$, the number of realized face areas is

$$2(n - 4) + 2 = 2n - 6.$$

Consequently, all but two face areas, namely f_a and f_b, are realized and $D(x_4)$ is an almost realizing vertex placement. Let $\mathcal{B}(\mathcal{A})$ denote the set of all x_4 where a triple of predecessors becomes collinear. We postpone to show that $\mathcal{B}(\mathcal{A})$ is finite. It is sufficient to show that the denominator of each vertex is not the 0-polynomial. We prove this simultaneously with property (iii).

Now, we show (ii). For each vertex v_i, we wish to represent its coordinates (x_i, y_i) in $D(x_4)$ by rational functions with a common denominator. Specifically, we aim for polynomials $\mathcal{N}_i^x, \mathcal{N}_i^y, \mathcal{D}_i$ in x_4, which are different from the 0-polynomial, such that

$$x_i = \frac{\mathcal{N}_i^x}{\mathcal{D}_i} \qquad \text{and} \qquad y_i = \frac{\mathcal{N}_i^y}{\mathcal{D}_i}.$$

Moreover, we assume that the leading coefficient of $\mathcal{D}_i$ is 1. We show the existence of such a representation by induction. Hence assume that we have such a

representation of $v_1, \ldots, v_{i-1}$. By the placement of the initial vertices, it holds that $\mathcal{D}_i := 1$ for all $i \in [4]$.

Now, we consider the vertex v_i with $i > 4$. By Lemma 6, we denote the three predecessors of v_i by p_F, p_M, p_L such that the triangles $p_F p_M v_i$ and $p_M p_L v_i$ are ccw inner or cw outer faces of T; we call the prescribed face areas of the two triangles a_i and b_i, respectively. Equations (3) and (4) yield the coordinates of vertex v_i in $D(x_4)$. Since we will aim for the fact, that the representation is *crr-free*, that is the polynomials share no common real root, we are already here more careful.

For the later argument, it is convenient to consider v_5 explicitly. By the 4-connectedness of T, neither v_3 nor v_4 is the middle predecessor of v_5. Thus, by symmetry, we may assume that $\mathrm{pred}(v_5) = \{v_1, v_4, v_3\}$ and Equations (3) and (4) simplify to

$$x_5 = \frac{a_5 x_4 + b_5 x_3}{y_3 x_4 - a x_3} \qquad \text{and} \qquad y_5 = \frac{a a_5 + b_5 y_3}{y_3 x_4 - a x_3}. \tag{5}$$

Note that for $x_3 = 0$, the denominators of x_5 and y_5 would vary in a crr-free representation. Thus, for $x_3 \neq 0$, we define $\mathcal{N}_5^x := x_4 a_5 + b_5 x_3$, $\mathcal{N}_5^y := a a_5 + b_5$, $\mathcal{D}_5 := y_3 x_4 - a x_3$. Clearly, none of them is the 0-polynomial.

Now, we consider the induction step for vertex v_i with $i > 5$. Note that, due to the 4-connectedness, v_1 is not a predecessor of v_i. Equations (3) and (4) yield the coordinates of vertex v_i in the almost realizing vertex placement. By assumption, $\mathcal{D}_F \cdot \mathcal{D}_M \cdot \mathcal{D}_L$ is not the 0-polynomial, since none of its factors is the 0-polynomial. Therefore, we may expand the right term by $\mathcal{D}_F \cdot \mathcal{D}_M \cdot \mathcal{D}_L$. Using the representations $x_j = \mathcal{N}_j^x / \mathcal{D}_j$ and $y_j = \mathcal{N}_j^y / \mathcal{D}_j$ for $j \in \{F, M, L\}$ yields the following identities:

$$x_i = \frac{\mathcal{N}_M^x}{\mathcal{D}_M} + \frac{\mathcal{D}_M(a_i \mathcal{N}_L^x \mathcal{D}_F + b_i \mathcal{N}_F^x \mathcal{D}_L) - (a_i + b_i)\mathcal{N}_M^x \mathcal{D}_F \mathcal{D}_L}{\tilde{\mathcal{D}}_i} \tag{6}$$

$$y_i = \frac{\mathcal{N}_M^y}{\mathcal{D}_M} + \frac{\mathcal{D}_M(a_i \mathcal{N}_L^y \mathcal{D}_F + b_i \mathcal{N}_F^y \mathcal{D}_L) - (a_i + b_i)\mathcal{N}_M^y \mathcal{D}_F \mathcal{D}_L}{\tilde{\mathcal{D}}_i} \tag{7}$$

with $\tilde{\mathcal{D}}_i := \mathcal{N}_F^x(\mathcal{N}_M^y \mathcal{D}_L - \mathcal{N}_L^y \mathcal{D}_M) + \mathcal{N}_M^x(\mathcal{N}_L^y \mathcal{D}_F - \mathcal{N}_F^y \mathcal{D}_L) + \mathcal{N}_L^x(\mathcal{N}_F^y \mathcal{D}_M - \mathcal{N}_M^y \mathcal{D}_F).$ (8)

Note that the denominators of x_i and y_i are identical and the numerators are symmetric in the x- and y-coordinates of their predecessors, respectively. Hence, for $\circ \in \{x, y\}$, we define $\mathcal{N}_i^\circ$ to unify the notation.

We wish to argue that $\tilde{\mathcal{D}}_i$ is not the 0-polynomial. The existence of distinct $j, k \in \{F, M, L\}$ such that neither $\mathcal{N}_j^x$ nor $\mathcal{N}_k^y$ are the 0-polynomial guarantees that one summand of $\tilde{\mathcal{D}}_i$ does not vanish. Note that such a pair does always exist since there are only two polynomials which might be the 0-polynomial, namely $\mathcal{N}_2^y$ and $\mathcal{N}_3^x$; here we use the fact that v_1 is no predecessor of any v_i with $i > 5$.

We now expand to find the desired representations of x_i and y_i. The denominator is the least common multiple of $\mathcal{D}_M$ and $\tilde{\mathcal{D}}_i$, none of which is the 0-polynomial. Thus, we define E_i and F_i to be crr-free polynomials such that

$$\mathcal{D}_M E_i = \tilde{\mathcal{D}}_i F_i. \tag{9}$$

In Equations (6) and (7), we expand the left summand by E_i and the right summand by F_i. Then the coordinates of vertex v_i can be expressed by

$$\mathcal{N}_i^\circ := F_i \mathcal{D}_\mathrm{M}(a_i \mathcal{N}_\mathrm{L}^\circ \mathcal{D}_\mathrm{F} + b_i \mathcal{N}_\mathrm{F}^\circ \mathcal{D}_\mathrm{L}) + \mathcal{N}_\mathrm{M}^\circ(E_i - (a_i + b_i)\mathcal{D}_\mathrm{F}\mathcal{D}_\mathrm{L}F_i) \qquad (10)$$

$$\mathcal{D}_i := \mathcal{D}_\mathrm{M}E_i = \tilde{\mathcal{D}}_i F_i \qquad (11)$$

Thus, each coordinate of v_i is a rational function in x_4, where the coefficients are polynomials in $\mathcal{A}$. Due to the algebraically independence of $\mathcal{A}$, the coefficients cannot vanish and $\mathcal{D}_i$ is not the 0-polynomial. Using the fact that $\mathcal{D}_\mathrm{F} \cdot \mathcal{D}_\mathrm{M} \cdot \mathcal{D}_\mathrm{L}$ is not the 0-polynomial, any $\mathcal{N}_j^\circ$ with $j \in \{\mathrm{F,M,L}\}$ which is not the 0-polynomial, certifies that $\mathcal{N}_i^\circ$ is not the zero polynomial. Recall that we already guaranteed the existence of distinct $j, k \in \{\mathrm{F,M,L}\}$ such that neither $\mathcal{N}_j^x$ nor $\mathcal{N}_k^y$ are the 0-polynomial. Thus, such a pair also implies that $\mathcal{N}_i^\circ$ is not the zero polynomial for all choices of $\circ \in \{x, y\}$. Consequently, we have proved property (ii) and (iii).

Moreover, (iii) immediately implies (iv): The area of face f_a can be expressed as the determinant of its three vertex coordinates. Thus, if the vertex coordinates are rational functions in x_4, so is the area of face f_a.

We interpret $\mathfrak{f}$ as a rational function in x_4 whose coefficients depend on $\mathcal{A}$.

B.2 Almost surjectivity and area-universality

We start by proving Lemma 9.

Lemma 9. *Let $p, q \colon \mathbb{R} \to \mathbb{R}$ be polynomials and let Q be the set of the real roots of q. If the polynomials p and q are crr-free and have odd max-degree, then the function $f \colon \mathbb{R}\backslash Q \to \mathbb{R}$, $f(x) = \frac{p(x)}{q(x)}$ is almost surjective.*

Proof. Let $c \in \mathbb{R}\backslash\{0\}$ and consider $g \colon \mathbb{R} \to \mathbb{R}$, $g(x) := p(x) - cq(x)$. The leading coefficients of the polynomials p and $c \cdot q$ cancel for at most one choice of c. For all other values, the degree of g is $\deg g = \max\{\deg p, \deg q\}$ and by assumption odd. Consequently, as a real polynomial of odd degree, g has a real root $\tilde{x}$. If $q(\tilde{x}) \neq 0$, then

$$g(\tilde{x}) = 0 \iff f(\tilde{x}) = c.$$

Suppose $q(\tilde{x}) = 0$. Then, $q(\tilde{x}) = 0 = g(\tilde{x}) = p(\tilde{x})$. Hence $\tilde{x}$ is a zero of both p and q. A contradiction to the assumption that p and q are crr-free.

Now, we aim to prove Corollary 1 which relies on several interesting properties of algebraically independent area assignments. In particular, it remains to show Claim 2.

Claim 2. *Let T be a plane 4-connected triangulation with a p-order $\mathcal{P}$. Then for every plane graph $T' \in [T]$, and algebraically independent area assignments $\mathcal{A}$ of T and $\mathcal{A}'$ of T', the last face functions $\mathfrak{f}_\mathcal{A}$ and $\mathfrak{f}'_{\mathcal{A}'}$ with respect to $\mathcal{P}$ have the same degrees, i.e., $d_i(\mathfrak{f}_\mathcal{A}) = d_i(\mathfrak{f}'_{\mathcal{A}'})$ for $i \in [2]$.*

Proof. We assume that $v_1v_2v_3$ is the cw outer face of T and $w_1w_2w_3$ is the cw outer face of T'. Then $v_1v_2v_3$ is a ccw inner face of T' and $w_1w_2w_3$ is a ccw inner face of T. Compared to T, the orientation of the faces $v_1v_2v_3$ and $w_1w_2w_3$ in T' changes; while the orientation of all other faces remains: This is easily seen when considering the drawings on the sphere; which are obtained by one-point compactification of some point in the respective outer faces. Due to the 3-connectedness and Whitney's uniqueness theorem, the drawings T and T' on the sphere are equivalent [23]. Moreover, in the drawing of the sphere $v_1v_2v_3$ and $w_1w_2w_3$ are ccw. Then choosing one face as the outer face and applying a stereographic projection of the punctured sphere where a point of the outer face is deleted, results in a drawing in the plane where all faces remain ccw while the outer face becomes cw.

With respect to the area assignment $\mathcal{A}$ of T, $v_1v_2v_3$ is assigned to the total area $-\Sigma\mathcal{A}$ and $w_1w_2w_3$ to some value c. As an intermediate step we consider the area assignment $\mathcal{A}''$ of T' where all area assignments remain but $w_1w_2w_3$ obtains the total area $-\Sigma\mathcal{A}$ and $v_1v_2v_3$ some value c. Clearly, $\mathcal{A}''$ is algebraically independent since $\mathcal{A}$ is. Fortunately, the negative sign accounts for the fact that the orientation changes; constructing realizing drawings by Lemma 8 T and T' are treated by the very same procedure. Thus, $\mathfrak{f}'_{\mathcal{A}''}$ can be obtained from $\mathfrak{f}_{\mathcal{A}}$ by swapping all occurrences of c and $-\Sigma\mathcal{A}$. Consequently, the degrees of the denominator and numerator polynomials of the last face functions $\mathfrak{f}_{\mathcal{A}}$ and $\mathfrak{f}'_{\mathcal{A}''}$ coincide. Moreover, by Claim 1, the degrees of $\mathfrak{f}'_{\mathcal{A}''}$ and $\mathfrak{f}'_{\mathcal{A}'}$ coincide.

Consequently, Lemma 9, Claim 2, and Theorem 1 imply Corollary 1.

Corollary 1. *Let T be a plane triangulation with a p-order $\mathcal{P}$. If the last face function $\mathfrak{f}$ of T is crr-free and has odd max-degree for one algebraically independent area assignment, then every plane graph in $[T]$ is area-universal.*

Proof. If the last face function $\mathfrak{f}$ of T has odd max-degree for some $\mathcal{A} \in \mathbb{A}_I$, then this holds true for all area assignments in $\mathbb{A}_I$ by Claim 1. Consequently, Lemma 9 guarantees that the last face function $\mathfrak{f}(\cdot, \mathcal{A})$ is almost surjective for all $\mathcal{A} \in \mathbb{A}_I$. Since $\mathbb{A}_I$ is dense in $\mathbb{A}$ as proved in Claim 3, Theorem 1 implies the area-universality of T.

For every other plane graph $T' \in [T]$, the last face function $\mathfrak{f}'$ has also odd maximum degree by Claim 2. Here we used the fact that $\mathfrak{f}'$ can be obtained from $\mathfrak{f}$ by exchanging two algebraically independent numbers. By the same reasoning, $\mathfrak{f}'$ is also crr-free. Thus, the above argument shows that T' is area-universal.

C   Proofs of Section 4

Here we present the omitted proofs of Section 4. We start by helpful lemmas to analyze the coordinate functions and their degrees.

C.1   Analyzing the Coordinates and their Degrees

Throughout this section, let T be a plane 4-connected triangulation with a p-order and $\mathcal{A}$ an algebraically independent area assignment. We use Lemma 8 to obtain an almost realizing drawing $D(x_4)$ and want to use Lemma 9 to guarantee almost surjectivity of the last face function $\mathfrak{f}$. Thus, we are interested in the max-degree of $\mathfrak{f}$.

As shown in Lemma 8, we can represent the coordinates (x_i, y_i) of each vertex v_i and the last face function by rational functions. Specifically, we have a representation of x_i and y_i by polynomials $\mathcal{N}_i^x, \mathcal{N}_i^y, \mathcal{D}_i$ in x_4 such that

$$x_i = \frac{\mathcal{N}_i^x}{\mathcal{D}_i} \quad \text{and} \quad y_i = \frac{\mathcal{N}_i^y}{\mathcal{D}_i}.$$

Due to Lemma 9, we aim for the fact that $\mathcal{N}_i^x$ and $\mathcal{N}_i^y$ are crr-free with $\mathcal{D}_i$ and are interested in their degrees. As before, we denote the degree of a polynomial p by $|p|$. Moreover, we say that a polynomial $p(x_1, \ldots, x_k)$ *depends* on x_j if and only if $p(x_1, \ldots, x_j, \ldots, x_k) \neq p(x_1, \ldots, 0, \ldots, x_k)$. Equations (10) and (11) show:

Observation. For $i \in \{4, \ldots, n\}$, $\mathcal{N}_i^\circ$ depends on a_i and b_i, while $\mathcal{D}_i$ does not.

In order to study their degree, we define $d_i^\circ := |\mathcal{N}_i^\circ| - |\mathcal{D}_i|$.

Lemma 13. *Let v_i be a vertex with the three predecessors $p_\text{F}, p_\text{M}, p_\text{L}$ in $\mathcal{P}$. For the vertex coordinates of v_i in $D(x_4)$, it holds that the (not necessarily crr-free) polynomials $\mathcal{N}_i^\circ, \mathcal{D}_i, \tilde{\mathcal{D}}_i$, defined in Equations (8), (10) and (11), have the following degrees:*

$$
\begin{aligned}
|\mathcal{N}_i^\circ| =&|\mathcal{D}_\text{M}| + |\mathcal{D}_\text{F}| + |\mathcal{D}_\text{L}| + |F_i| \\
&+ \max\left\{d_\text{L}^\circ, d_\text{F}^\circ, d_\text{M}^\circ + \max\{|E_i| - |\mathcal{D}_\text{F}| - |\mathcal{D}_\text{L}| - |F_i|, 0\}\right\} \\
|\mathcal{D}_i| =&|\mathcal{D}_\text{M}| + |E_i| = |\tilde{\mathcal{D}}_i| + |F_i| \\
|\tilde{\mathcal{D}}_i| =&|\mathcal{D}_\text{F}| + |\mathcal{D}_\text{M}| + |\mathcal{D}_\text{L}| \\
&+ \max\left\{d_\text{F}^x + \max\{d_\text{M}^y, d_\text{L}^y\}, d_\text{M}^x + \max\{d_\text{L}^y, d_\text{F}^y\}, d_\text{L}^x + \max\{d_\text{F}^y, d_\text{M}^y\}\right\}
\end{aligned}
$$

Proof. We need to determine the degrees of the polynomials in Equations (8), (10) and (11). Here we use the fact that for all polynomials p, q which are not the 0-polynomial it holds $|p \cdot q| = |p| + |q|$. With the convention that $|0| = -\infty$, the above identity also holds for the 0-polynomial. Moreover, unless $|p| = |q|$ and the leading coefficients are canceling, it holds that $|p + q| = \max\{|p|, |q|\}$. By algebraic independence, cancellation of leading coefficients does not occur.

However, in order to apply Lemma 9, we also need that $\mathcal{N}_i^x$ and $\mathcal{N}_i^y$ are crr-free with $\mathcal{D}_i$. Therefore, we are interested in sufficient conditions.

Lemma 14. *Let $\circ \in \{x, y\}$. Suppose $\mathcal{N}_j^\circ$ and $\mathcal{D}_j$ are crr-free for all $j < i$. Then, $\mathcal{N}_i^\circ$ and $\mathcal{D}_i$ have a common zero z if and only if the following properties hold*
- *z is a zero of E_i*
- *z is independent of a_i and b_i*

and additionally

(i) z is a zero of at least two of $\{\mathcal{D}_F, \mathcal{D}_M, \mathcal{D}_L\}$ or

(ii) z is a zero of both of $\{\mathcal{N}_L^\circ \mathcal{D}_M - \mathcal{N}_M^\circ \mathcal{D}_L, \mathcal{N}_F^\circ \mathcal{D}_M - \mathcal{N}_M^\circ \mathcal{D}_F\}$.

Due to their technicality, we have moved the proofs of Lemma 14 and of the two following lemmas to appendix C.2. Now, we study a more specific situation which occurs for accordion and double stacking graphs.

Stacking on same angle Recall that, by Lemma 6, vertex v_i is inserted in a 4-face of T_{i-1}. In this section, we analyze the situation that in the p-order several vertices are repeatedly inserted in the same angle. In particular, we say vertices v_{i+1} and v_{i+2} are *stacked on the same angle* if their first and last predecessors are identical and the middle predecessor of v_{i+2} is v_{i+1}. Specifically, v_{i+1} and v_{i+2} have predecessors (p_F, v_i, p_L) and (p_F, v_{i+1}, p_L), respectively. Figure 11 illustrates two vertices which are stacked on the same angle.
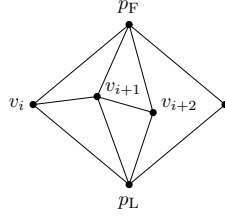

Fig. 11: Vertices v_{i+1} and v_{i+2} are stacked on the same angle.

Lemma 15. *If v_{i+1} and v_{i+2} are stacked on the same angle in the p-order and $\mathcal{N}_{i+1}^\circ$ and $\mathcal{D}_{i+1}$ are crr-free, then it holds that*

$$E_{i+2} = E_{i+1} - (a_{i+1} + b_{i+1})\mathcal{D}_F \mathcal{D}_L F_{i+1}$$
$$F_{i+2} = F_{i+1}.$$

For the proof of Lemma 15, we refer to appendix C.2. For the degrees, we obtain the following expressions.

Lemma 16. *If v_{i+1} and v_{i+2} are stacked on the same angle in the p-order, and if $\mathcal{N}_{i+1}^\circ, \mathcal{D}_{i+1}$, and $\mathcal{N}_{i+2}^\circ, \mathcal{D}_{i+2}$ are crr-free. Then, for $M := \max\{|E_{i+1}|, |\mathcal{D}_F| + |\mathcal{D}_L| + |F_{i+1}|)\}$ and $\circ \in \{x, y\}$ it holds that*

$$|\mathcal{N}_{i+2}^\circ| = |\mathcal{D}_{i+1}| + M + d_{i+1}^\circ$$
$$|\mathcal{D}_{i+2}| = |\mathcal{D}_{i+1}| + M.$$

In particular, it holds that $d_{i+2}^\circ = d_{i+1}^\circ$.

For the proof of Lemma 16, we refer to appendix C.2.

C.2   Proofs of the Degree-Lemmas

In this section, we present the pending proofs of the previous section.

Lemma 14. *Let $\circ \in \{x, y\}$. Suppose $\mathcal{N}_j^\circ$ and $\mathcal{D}_j$ are crr-free for all $j < i$. Then, $\mathcal{N}_i^\circ$ and $\mathcal{D}_i$ have a common zero z if and only if the following properties hold*
   - z is a zero of E_i
   - z is independent of a_i and b_i
and additionally
(i) z is a zero of at least two of $\{\mathcal{D}_\mathrm{F}, \mathcal{D}_\mathrm{M}, \mathcal{D}_\mathrm{L}\}$ or
(ii) z is a zero of both of $\{\mathcal{N}_\mathrm{L}^\circ \mathcal{D}_\mathrm{M} - \mathcal{N}_\mathrm{M}^\circ \mathcal{D}_\mathrm{L}, \mathcal{N}_\mathrm{F}^\circ \mathcal{D}_\mathrm{M} - \mathcal{N}_\mathrm{M}^\circ \mathcal{D}_\mathrm{F}\}$.

Proof. The proof consists of two directions. For both recall the formulas for $\mathcal{N}_i^\circ$ and $\mathcal{D}_i$ given in Equations (10) and (11). Suppose that $\mathcal{N}_i^\circ$ and $\mathcal{D}_i$ have a common zero z. We think of z as an algebraic function of $(a, a_5, b_5, \ldots, a_i, b_i)$. By appendix C.1, the variables a_i and b_i do not occur in D_i. Consequently, z does not depend on a_i and b_i, and is thus algebraically independent of a_i and b_i. Since $\mathcal{D}_i = \mathcal{D}_\mathrm{M} E_i$, by Equation (11), z is a zero of at least one of $\mathcal{D}_\mathrm{M}$ or E_i. We distinguish three cases.

Case 1: If z is a zero of both, $\mathcal{D}_\mathrm{M}$ and E_i, then Equation (10) simplifies to

$$\mathcal{N}_i^\circ[z] = -(a_i + b_i)(\mathcal{N}_\mathrm{M}^\circ \mathcal{D}_\mathrm{F} \mathcal{D}_\mathrm{L} F_i)[z] = 0.$$

By the assumption of being crr-free, z is neither a zero of $\mathcal{N}_\mathrm{M}^\circ$ nor F_i. Hence, z is a zero of $\mathcal{D}_\mathrm{F} \mathcal{D}_\mathrm{L}$. In conclusion, z is a zero of E_i, $\mathcal{D}_\mathrm{M}$ as well as $\mathcal{D}_\mathrm{F}$ or $\mathcal{D}_\mathrm{L}$ (or both). In other words, condition (i) is fulfilled.

Case 2: If z is a zero of $\mathcal{D}_\mathrm{M}$ and not of E_i, then Equation (10) reads as

$$\mathcal{N}_i^\circ[z] = (\mathcal{N}_\mathrm{M}^\circ(E_i - (a_i + b_i))\mathcal{D}_\mathrm{F} \mathcal{D}_\mathrm{L} F_i)[z] = 0.$$

Since z, a_i, b_i are algebraically independent, $\mathcal{N}_i^\circ[z]$ vanishes on each summand. However, z is not a zero of E_i by assumption of Case 2 and z is not a zero of $\mathcal{N}_\mathrm{M}^\circ$ since $\mathcal{D}_\mathrm{M}$ and $\mathcal{N}_\mathrm{M}^\circ$ are crr-free. Thus, we arrive at a contradiction and this case does not occur.

Case 3: If z is a zero of E_i and not of $\mathcal{D}_\mathrm{M}$, z is not a zero of F_i; since E_i and F_i are crr-free. Consequently, Equation (10) implies $(\mathcal{D}_\mathrm{M}(a_i \mathcal{N}_\mathrm{L}^\circ \mathcal{D}_\mathrm{F} + b_i \mathcal{N}_\mathrm{F}^\circ \mathcal{D}_\mathrm{L}) - (a_i + b_i)\mathcal{N}_\mathrm{M}^\circ \mathcal{D}_\mathrm{F} \mathcal{D}_\mathrm{L})[z] = 0$. Reordering for a_i and b_i results in

$$\left(a_i \mathcal{D}_\mathrm{F}(\mathcal{D}_\mathrm{M} \mathcal{N}_\mathrm{L}^\circ - \mathcal{N}_\mathrm{M}^\circ \mathcal{D}_\mathrm{L}) + b_i \mathcal{D}_\mathrm{L}(\mathcal{D}_\mathrm{M} \mathcal{N}_\mathrm{F}^\circ - \mathcal{N}_\mathrm{M}^\circ \mathcal{D}_\mathrm{F})\right)[z] = 0.$$

As argued above, z, a_i, and b_i are algebraically independent, and thus z is a zero of both summands. If z is a zero of $\mathcal{D}_\mathrm{F}$, then it is also a zero of $\mathcal{D}_\mathrm{L} \mathcal{D}_\mathrm{M} \mathcal{N}_\mathrm{F}^\circ$. However, by assumption, it is not a zero of $\mathcal{D}_\mathrm{M} \mathcal{N}_\mathrm{F}^\circ$, and thus it is a zero of $\mathcal{D}_\mathrm{L}$. Likewise, if z is a zero of $\mathcal{D}_\mathrm{L}$ then it follows that z is also a zero of $\mathcal{D}_\mathrm{F}$. Hence, (i) is satisfied.

Thus, in the following we may assume that z is not a zero of $\mathcal{D}_\mathrm{F} \mathcal{D}_\mathrm{L}$, but of both polynomials $(\mathcal{N}_\mathrm{L}^\circ \mathcal{D}_\mathrm{M} - \mathcal{N}_\mathrm{M}^\circ \mathcal{D}_\mathrm{L})$ and $(\mathcal{N}_\mathrm{F}^\circ \mathcal{D}_\mathrm{M} - \mathcal{N}_\mathrm{M}^\circ \mathcal{D}_\mathrm{F})$. This is condition (ii).

It remains to show the reverse direction. Since z is a zero of E_i, it follows that z is a zero of $\mathcal{D}_i$. By construction of our cases, z was a zero of $\mathcal{N}_i^\circ$. Alternatively, it is easy to check that z is also a zero of $\mathcal{N}_i^\circ$ as given in Equation (10) in all cases. Consequently, z is a zero of both $\mathcal{N}_i^\circ$ and $\mathcal{D}_i$.

Lemma 15. *If v_{i+1} and v_{i+2} are stacked on the same angle in the p-order and $\mathcal{N}_{i+1}^\circ$ and $\mathcal{D}_{i+1}$ are crr-free, then it holds that*

$$E_{i+2} = E_{i+1} - (a_{i+1} + b_{i+1})\mathcal{D}_F\mathcal{D}_L F_{i+1}$$
$$F_{i+2} = F_{i+1}.$$

Proof. Note that the irreducibility of E_{i+1} and F_{i+1} directly implies the irreducibility of E_{i+2} and F_{i+2}: Since $F_{i+2} = F_{i+1}$, every zero of F_{i+2} is a zero of F_{i+1}. Therefore, z is a zero of E_{i+2} and F_{i+2} if and only if z is a zero of E_{i+1} and F_{i+1}. It remains to show that

$$E_{i+2}\mathcal{D}_{i+1} = F_{i+2}\tilde{\mathcal{D}}_{i+2}.$$

To do so, we will show that $\tilde{\mathcal{D}}_{i+1}$ is a factor of $\tilde{\mathcal{D}}_{i+2}$. By Equation (8), we obtain the following formula for $\tilde{\mathcal{D}}_{i+2}$:

$$\tilde{\mathcal{D}}_{i+2} = \mathcal{N}_{i+1}^x(\mathcal{N}_L^y\mathcal{D}_F - \mathcal{N}_F^y\mathcal{D}_L) + \mathcal{N}_{i+1}^y(\mathcal{N}_F^x\mathcal{D}_L - \mathcal{N}_L^x\mathcal{D}_F) + \mathcal{D}_{i+1}(\mathcal{N}_L^x\mathcal{N}_F^y - \mathcal{N}_F^x\mathcal{N}_L^y)$$

The last summand is clearly divisible by $\mathcal{D}_{i+1}$ and, hence by $\tilde{\mathcal{D}}_{i+1}$. Therefore we focus on the first two summands, in which we replace $\mathcal{N}_{i+1}^\circ$ with the help of Equation (10) by

$$\mathcal{N}_{i+1}^\circ = F_{i+1}\mathcal{D}_i(a_{i+1}\mathcal{N}_L^\circ\mathcal{D}_F + b_{i+1}\mathcal{N}_F^\circ\mathcal{D}_L) + \mathcal{N}_i^\circ E_{i+2}.$$

Replacing $\mathcal{N}_{i+1}^\circ$ by the above expressions and using Equation (8) for $\tilde{\mathcal{D}}_{i+1}$, we obtain:

$$\mathcal{N}_{i+1}^x(\mathcal{N}_L^y\mathcal{D}_F - \mathcal{N}_F^y\mathcal{D}_L) + \mathcal{N}_{i+1}^y(\mathcal{N}_F^x\mathcal{D}_L - \mathcal{N}_L^x\mathcal{D}_F)$$
$$= (F_{i+1}\mathcal{D}_i[a_{i+1}\mathcal{N}_L^x\mathcal{D}_F + b_{i+1}\mathcal{N}_F^x\mathcal{D}_L] + \mathcal{N}_i^x E_{i+2})(\mathcal{N}_L^y\mathcal{D}_F - \mathcal{N}_F^y\mathcal{D}_L)$$
$$+ (F_{i+1}\mathcal{D}_i[a_{i+1}\mathcal{N}_L^y\mathcal{D}_F + b_{i+1}\mathcal{N}_F^y\mathcal{D}_L] + \mathcal{N}_i^y E_{i+2})(\mathcal{N}_F^x\mathcal{D}_L - \mathcal{N}_L^x\mathcal{D}_F)$$
$$= \mathcal{D}_F\mathcal{D}_L\mathcal{D}_i F_{i+1}(\mathcal{N}_L^y\mathcal{N}_F^x - \mathcal{N}_F^y\mathcal{N}_L^x)(a_{i+1} + b_{i+1})$$
$$\quad + E_{i+2}(\mathcal{N}_i^x(\mathcal{N}_L^y\mathcal{D}_F - \mathcal{N}_F^y\mathcal{D}_L) + \mathcal{N}_i^y(\mathcal{N}_F^x\mathcal{D}_L - \mathcal{N}_L^x\mathcal{D}_F))$$
$$= \mathcal{D}_F\mathcal{D}_L\mathcal{D}_i F_{i+1}(\mathcal{N}_L^y\mathcal{N}_F^x - \mathcal{N}_F^y\mathcal{N}_L^x)(a_{i+1} + b_{i+1})$$
$$\quad + E_{i+2}(\tilde{\mathcal{D}}_{i+1} - \mathcal{D}_i(\mathcal{N}_L^x\mathcal{N}_F^y - \mathcal{N}_F^x\mathcal{N}_L^y))$$

We remove the term $E_{i+2}\tilde{\mathcal{D}}_{i+1}$ which clearly divisible by $\tilde{\mathcal{D}}_{i+1}$. In the remainder we factor out $(\mathcal{N}_F^x\mathcal{N}_L^y - \mathcal{N}_L^x\mathcal{N}_F^y)\mathcal{D}_i$ and recall the definitions of E_{i+2} and D_{i+1}. It remains

$$\mathcal{D}_F\mathcal{D}_L\mathcal{D}_i F_{i+1}(\mathcal{N}_F^x\mathcal{N}_L^y - \mathcal{N}_L^x\mathcal{N}_F^y)(a_{i+1} + b_{i+1}) + E_{i+2}(-\mathcal{D}_i(\mathcal{N}_L^x\mathcal{N}_F^y - \mathcal{N}_F^x\mathcal{N}_L^y))$$
$$= (\mathcal{N}_F^x\mathcal{N}_L^y - \mathcal{N}_L^x\mathcal{N}_F^y)\mathcal{D}_i(\mathcal{D}_F\mathcal{D}_L F_{i+1}(a_{i+1} + b_{i+1}) + E_{i+2})$$
$$= (\mathcal{N}_F^x\mathcal{N}_L^y - \mathcal{N}_L^x\mathcal{N}_F^y)\mathcal{D}_i E_{i+1} = (\mathcal{N}_F^x\mathcal{N}_L^y - \mathcal{N}_L^x\mathcal{N}_F^y)\mathcal{D}_{i+1}$$

By definition, $\mathcal{D}_{i+1}$ divisible by $\tilde{\mathcal{D}}_{i+1}$. The remainder consists of three summands, namely $(\mathcal{N}_{\mathrm{L}}^x \mathcal{N}_{\mathrm{F}}^y - \mathcal{N}_{\mathrm{F}}^x \mathcal{N}_{\mathrm{L}}^y) + E_{i+2} + (\mathcal{N}_{\mathrm{F}}^x \mathcal{N}_{\mathrm{L}}^y - \mathcal{N}_{\mathrm{L}}^x \mathcal{N}_{\mathrm{F}}^y)$. Note that the first and last summand of the remainder are canceling. Consequently, the remainder is E_{i+2}, i.e., $\hat{\mathcal{D}}_{i+2} = \hat{\mathcal{D}}_{i+1} E_{i+2}$. This directly implies that $F_{i+1} \hat{\mathcal{D}}_{i+2} = F_{i+1} \tilde{\mathcal{D}}_{i+1} E_{i+2} = \mathcal{D}_{i+1} E_{i+2}$ and therefore finishes the proof.

Lemma 16. *If v_{i+1} and v_{i+2} are stacked on the same angle in the p-order, and if $\mathcal{N}_{i+1}^\circ, \mathcal{D}_{i+1}$, and $\mathcal{N}_{i+2}^\circ$, $\mathcal{D}_{i+2}$ are crr-free. Then, for $M := \max\{|E_{i+1}|, |\mathcal{D}_{\mathrm{F}}| + |\mathcal{D}_{\mathrm{L}}| + |F_{i+1}|)\}$ and $\circ \in \{x, y\}$ it holds that*

$$|\mathcal{N}_{i+2}^\circ| = |\mathcal{D}_{i+1}| + M + d_{i+1}^\circ$$
$$|\mathcal{D}_{i+2}| = |\mathcal{D}_{i+1}| + M.$$

In particular, it holds that $d_{i+2}^\circ = d_{i+1}^\circ$.

Proof. Recall that, by definition $d_{i+2}^\circ = |\mathcal{N}_{i+2}^\circ| - |\mathcal{D}_{i+2}|$. Consequently, if the degree of $\mathcal{N}_{i+2}^\circ$ and $\mathcal{D}_{i+2}$ are as claimed, it follows directly that $d_{i+2}^\circ = d_{i+1}^\circ$. For $\circ \in \{x, y\}$, we define $m^\circ := |\mathcal{D}_{\mathrm{F}}| + |\mathcal{D}_{\mathrm{L}}| + |F_{i+1}| + \max\{d_{\mathrm{L}}^\circ, d_{\mathrm{F}}^\circ\}$. Together with Lemma 13, the degrees can be expressed as follows:

$$|\mathcal{N}_{j+1}^\circ| = |\mathcal{D}_j| + \max\left\{m^\circ, d_j^\circ + M\right\}$$
$$|\mathcal{D}_{j+1}| = |\mathcal{D}_j| + |E_{j+1}|$$

This implies the following formula for d_{j+1}°:

$$d_{j+1}^\circ = -|E_{j+1}| + \max\left\{m^\circ, d_i^\circ + M\right\}$$

Recall that by Lemma 15, $E_{i+2} = E_{i+1} - (a_{i+1} + b_{i+1})\mathcal{D}_{\mathrm{F}}\mathcal{D}_{\mathrm{L}}F_j$ and $F_{i+2} = F_{i+1}$. Consequently, it holds that $|E_{i+2}| = M$ and with the above formula it follows that $|\mathcal{D}_{i+2}| = |\mathcal{D}_{i+1}| + |E_{i+2}| = |\mathcal{D}_{i+1}| + M$, as claimed. For the numerator, we replace d_{i+1}° in $\mathcal{N}_{i+2}^\circ$:

$$\begin{aligned}
|\mathcal{N}_{i+2}^\circ| &= |\mathcal{D}_{i+1}| + \max\left\{m^\circ, d_{i+1}^\circ + M\right\} \\
&= |\mathcal{D}_{i+1}| + \max\left\{m^\circ, M - |E| + \max\left\{m^\circ, d_i^\circ + M\right\}\right\} \\
&= |\mathcal{D}_{i+1}| + M - |E| + \max\left\{m^\circ, d_i^\circ + M\right\} \\
&= |\mathcal{D}_{i+1}| + M + d_{i+1}^\circ
\end{aligned}$$

By definition of M, $M - |E|$ is non-negative and hence, the outer-maximum in line 2 is attained for the second term. The last term in the third line is exactly d_{i+1}°. Hence the numerator degree $|\mathcal{N}_{i+2}^\circ|$ is of the claimed form.

C.3   Proof of Theorem 2

Theorem 2. *The accordion graph $\mathcal{K}_\ell$ is area-universal if and only if ℓ is odd.*

To complete the proof of Theorem 2, we have to show Lemma 10 and Lemma 11.

Lemma 10. *For all $i \geq 5$, it holds that $|\mathcal{D}_5| = 1$ and*

$$|\mathcal{N}_{i+1}^x| = |\mathcal{D}_{i+1}| = |\mathcal{N}_{i+1}^y| + 1 = |\mathcal{D}_i| + 1.$$

Proof. We denote the two face areas incident to v_i and its predecessors by a_i and b_i for $i \geq 5$, see Figure 12 for an illustration.

(a) A p-order of an accordion graph.

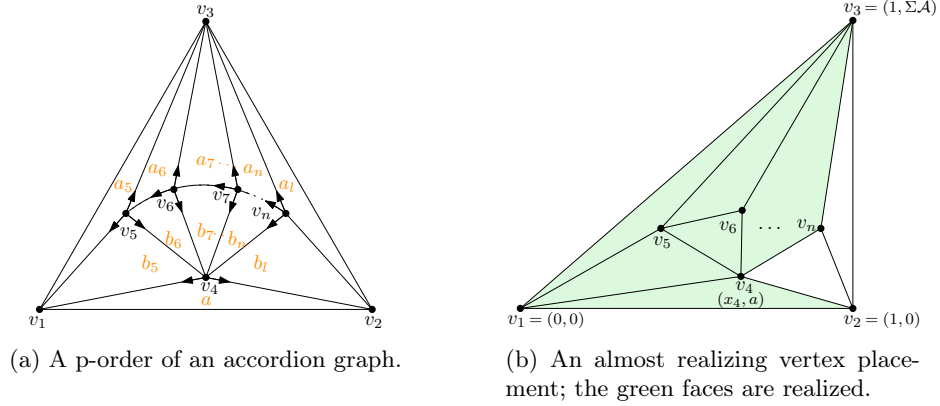(b) An almost realizing vertex placement; the green faces are realized.

Fig. 12: Illustration of Theorem 2 and its proof.

We show this claim by induction. For v_5, we have already evaluated the crr-free vertex coordinates in Equation (5). Thus, it holds that

$$|\mathcal{N}_5^x| = |\mathcal{D}_5| = |\mathcal{N}_5^y| + 1 = 1.$$

For $\circ \in \{x, y\}$, we define $d_i^\circ := |\mathcal{N}_i^\circ| - |\mathcal{D}_i|$. Consequently, it holds that $d_5^x = 0$ and $d_5^y = -1$. Moreover, since $\mathcal{D}_M = 1$, it holds that $F_5 = 1$ and $\mathcal{D}_5 = E_5 = \tilde{\mathcal{D}}_5$.

Recall that all vertices v_i with $i > 5$, the three predecessors of v_i are $p_F = v_3$, $p_M = v_{i-1}$ and $p_L = v_4$. In other words, all vertices v_i with $i > 4$ are stacked on the same angle. Consequently, for all $i > 5$, it holds that $|\mathcal{N}_F^\circ| = |\mathcal{D}_F| = |\mathcal{D}_L| = 0$ and $d_F^x = d_F^y = 0$, and $d_L^x = 1, d_L^y = 0$. Defining $M := \max\{|E_5|, |\mathcal{D}_F| + |\mathcal{D}_L| + |F_5|\} = \max\{1, 0\} = 1$, we obtain for v_i, $i > 5$, with the help of Lemma 16:

$$|\mathcal{N}_{i+1}^x| = |\mathcal{D}_i| + M + d_5^x = |\mathcal{D}_i| + 1$$
$$|\mathcal{N}_{i+1}^y| = |\mathcal{D}_i| + M + d_5^y = |\mathcal{D}_i|$$
$$|\mathcal{D}_{i+1}| = |\mathcal{D}_i| + M = |\mathcal{D}_i| + 1.$$

Lemma 11. *For all $i \geq 5$ and $\circ \in \{x, y\}$, it holds that $\mathcal{N}_i^\circ$ and $\mathcal{D}_i$ are crr-free.*

Proof. We show this by induction on i. The induction base is settled for $i = 5$ since the polynomials in Equation (5) are crr-free.

Suppose, for a contradiction, that $\mathcal{N}_i^\circ$ and $\mathcal{D}_i$, share a common real root. Then, we are either in case (i) or (ii) of Lemma 14. The fact that $\mathcal{D}_\mathrm{F} = \mathcal{D}_\mathrm{L} = 1$ excludes case (i).

Thus, we are in case (ii). To arrive at the final contradiction, we work a little harder. Since $\mathcal{D}_\mathrm{F} = \mathcal{D}_\mathrm{L} = 1$, there exists z which is a zero of both $\{\mathcal{N}_\mathrm{L}^\circ \mathcal{D}_\mathrm{M} - \mathcal{N}_\mathrm{M}^\circ\}, \{\mathcal{N}_\mathrm{F}^\circ \mathcal{D}_\mathrm{M} - \mathcal{N}_\mathrm{M}^\circ\}$. This implies that either $\mathcal{D}_\mathrm{M}(z) = 0$ or $\mathcal{N}_\mathrm{L}^\circ(z) = \mathcal{N}_\mathrm{F}^\circ(z)$. In the first case, $\mathcal{D}_\mathrm{M}(z) = 0$, it follows that $\mathcal{N}_\mathrm{M}^\circ[z] = 0$. This is an immediate contradiction to the fact that $\mathcal{D}_\mathrm{M}$ and $\mathcal{N}_\mathrm{M}^\circ$ are crr-free.

Thus it remains to consider the latter case, namely that $\mathcal{N}_\mathrm{L}^\circ(z) = \mathcal{N}_\mathrm{F}^\circ(z)$. For $\circ = y$, we immediately obtain a contradiction since $\mathcal{N}_\mathrm{L}^y = a < \Sigma\mathcal{A} = \mathcal{N}_\mathrm{F}^y$.

For $\circ = x$ it follows that $z = 1$ since $\mathcal{N}_\mathrm{L}^x = x$, $\mathcal{N}_\mathrm{F}^x = 1$. Moreover, by Lemma 14, z is a zero of E_i. Consequently, it suffices to show that $E_i[1] \neq 0$. In order to analyze the zeros of E_i, we define for $i \in \{5, \dots, n\}$

$$\alpha_i := a + \sum_{j=5}^{i-1} (a_j + b_j).$$

Recall that $E_5[x] = (\Sigma\mathcal{A})x - a$ by Equation (5). Thus, by Lemma 15, it holds for $i \in \{5, \dots, n-1\}$ that

$$E_{i+1}[x] = E_i[x] - (a_i + b_i) = E_5[x] - \sum_{j=5}^{i} (a_j + b_j) = (\Sigma\mathcal{A})x - \alpha_{i+1}. \qquad (12)$$

Since $\alpha_i < \Sigma\mathcal{A}$, it follows $E_i[1] \neq 0$ that for all $i \geq 5$. Consequently, $\mathcal{N}_i^\circ$ and $\mathcal{D}_i$ are crr-free.

C.4   Proof of Theorem 3

Theorem 3. *A plane graph in $[\mathcal{H}_{\ell,k}]$ is area-universal if and only if $\ell \cdot k$ is even.*

Proof. We start to consider $\mathcal{H}_{\ell,k}$. Note that the degree of all but four vertices is exactly four; namely, the degree of B and ℓ is $k+3$, the degree of v and C is $\ell+3$. Thus, if both ℓ and k are odd, then $\mathcal{H}_{\ell,k}$ is Eulerian and thus not area-universal as shown in [15, Theorem 1]. Since the degree depends on the planar graph, all plane graphs in $[\mathcal{H}_{\ell,k}]$ are Eulerian and not area-universal if $\ell \cdot k$ is odd.

Assume that $\ell \cdot k$ is even. In order to show the area-universality of $\mathcal{H}_{\ell,k}$, we consider the p-order $(A, B, C, v, 1, \dots, \ell, 1' \dots, k')$ in which $k'C$ is the unique undirected edge. For an algebraically independent area assignment $\mathcal{A}$, we define $a := \mathcal{A}(ABv)$ and place v_3 at $(1, \Sigma\mathcal{A})$ and v_4 at (x_4, a). Observe that the vertices $1, 2, \dots \ell$ have the predecessors C and v and are locally identical to an accordion

graph. Consequently, by Lemma 10 and Lemma 11, the coordinates of vertex ℓ can be expressed by crr-free polynomials $\mathcal{N}_\ell^\circ, \mathcal{D}_\ell$ with the degrees

$$|\mathcal{N}_\ell^x| = |\mathcal{N}_\ell^y| + 1 = |\mathcal{D}_\ell| = \ell.$$

Since $\mathcal{D}_v = 1$ and $C_{1'} = E_{1'}\mathcal{D}_v = \tilde{D}_{1'}F_{1'}$ by definition, see Equation (9), it follows that $E_{1'} = \tilde{D}_{1'}$ and $F_{1'} = 1$. As we will see it holds that $|E_{1'}| = |\tilde{D}_{1'}| = \ell$; this implies that

$$\max\{|E_{1'}| - |\mathcal{D}_\ell| - |\mathcal{D}_B| - |F_{1'}|, 0\} = \max\{\ell - \ell - 0 - 0, 0\} = 0.$$

(a) A p-order of a ℓk-graph. (b) An almost realizing vertex placement.
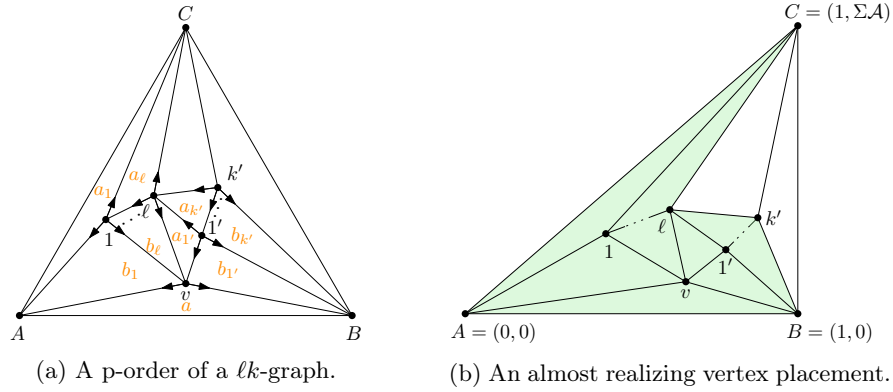
Fig. 13: Illustration of Theorem 3 and its proof.

Note that $d_B^x = 0$, $d_B^y = -\infty$, $d_v^x = 1$, $d_v^y = 0$, $d_\ell^x = 0$, and $d_\ell^y = -1$. Lemma 13 yields the following degrees:

$$|\mathcal{N}_{1'}^x| = \ell + 1 \quad \text{and} \quad |\mathcal{N}_{1'}^y| = |\mathcal{D}_{1'}| = |\tilde{\mathcal{D}}_{1'}| = \ell$$

We will later show that these polynomials are crr-free. Now, we proceed to compute the degrees of the vertex coordinates. Defining $M := \max\{|E_{1'}|, |\mathcal{D}_\ell| + |\mathcal{D}_B| + |F_{1'}|)\} = \max\{\ell, \ell + 0 + 0)\} = \ell$ and by Lemma 16, it follows for $j > 1$

$$|\mathcal{N}_{j'}^x| = |\mathcal{D}_{1'}| + (j-1) \cdot M + d_{1'}^x = j \cdot \ell + 1$$
$$|\mathcal{N}_{j'}^y| = |\mathcal{D}_{j'}| = |\mathcal{D}_{1'}| + (j-1) \cdot M = j \cdot \ell.$$

Assume for now, that the resulting polynomials are crr-free. As our last face we choose the triangle $k'BC$. Then the last face function $\mathfrak{f}$ evaluates to

$$\mathfrak{f}(x) := \det(k', B, C) = 1 - x_{k'} = 1 - \frac{\mathcal{N}_{k'}^x}{\mathcal{D}_{k'}}.$$

For the last vertex k', the degree of the numerator, namely $k \cdot \ell + 1$, exceeds the degree of the denominator $k \cdot \ell$ and is odd since $\ell \cdot k$ is even. Consequently, $\mathfrak{f}$ has odd max-degree and is almost surjective by Lemma 9. Consequently, Corollary 1 shows that every plane graph in $[\mathcal{H}_{\ell,k}]$ is area-universal.

It remains to guarantee that the polynomials are crr-free.

Lemma 17. *For all $j \geq 1$, it holds that $\mathcal{N}_{j'}^\circ$ and $\mathcal{D}_{j'}$ are crr-free.*

We prove this claim by induction and start with settling the base for $j = 1$ using Lemma 14. Suppose by contradiction that $\mathcal{N}_{1'}^\circ$ and $\mathcal{D}_{1'}$ share a common zero z. The fact that $\mathcal{D}_\mathrm{L} = \mathcal{D}_\mathrm{M} = 1$ excludes case (i). Thus, case (ii) holds and, since $\mathcal{D}_\mathrm{L} = \mathcal{D}_\mathrm{M} = 1$, z is a zero of the simplified polynomials $(\mathcal{N}_\mathrm{L}^\circ - \mathcal{N}_\mathrm{M}^\circ)$ and $(\mathcal{N}_\mathrm{F}^\circ - \mathcal{N}_\mathrm{M}^\circ \mathcal{D}_\mathrm{F})$. Recall that $\mathcal{N}_\mathrm{L}^x = 1$ and $\mathcal{N}_\mathrm{L}^y = 0$. Thus for $\circ = y$, it follows that z is a zero of $\mathcal{N}_\mathrm{L}^y$ and thus also of $\mathcal{N}_\mathrm{M}^y$. However, $\mathcal{N}_\mathrm{M}^y = a > 0$ yields a contradiction. For $\circ = x$, $\mathcal{N}_\mathrm{L}^x = 1$ and $\mathcal{N}_\mathrm{M}^x = x$ imply that $z = 1$. Consequently, it holds that $\mathcal{N}_\mathrm{F}^x[1] - \mathcal{D}_\mathrm{F}[1] = 0$. Recall that in our case $\mathrm{F} = \ell$. By appendix C.1, $\mathcal{N}_\mathrm{F}^x = \mathcal{N}_\ell^x$ depends on a_ℓ while $\mathcal{D}_\mathrm{F}$ does not. Consequently, $\mathcal{N}_\mathrm{F}^x[1]$ and $\mathcal{D}_\mathrm{F}[1]$ are polynomials in $\mathcal{A}$; due to the algebraic independence they cannot coincide.

Now, we come to the induction step and suppose, for a contradiction, that $\mathcal{N}_{j'+1}^\circ$ and $\mathcal{D}_{j'+1}$ share a common real root z. By Lemma 14 we distinguish two cases. In all cases z is zero of $E_{j'+1}$. By Lemma 15, we know that for $j \in [k-1]$ it holds that $E_{j'+1} = E_{j'} - (a_{j'} + b_{j'})\mathcal{D}_\ell$. Together with $E_{1'} = a(\mathcal{N}_\ell^x - \mathcal{D}_\ell) + (1-x)\mathcal{N}_\ell^y$, we obtain

$$E_{j'+1} = a\mathcal{N}_\ell^x + (1-x)\mathcal{N}_\ell^y - \left(a + \sum_{k=1}^{j}(a_{k'} + b_{k'})\right)\mathcal{D}_\ell.$$

We claim that z does not depend on $a_{j'}$ and $b_{j'}$. Then, it follows from the algebraic independence, that z is a zero of both $a\mathcal{N}_\ell^x + (1-x)\mathcal{N}_\ell^y$ and $\mathcal{D}_\ell$.

To prove this claim we distinguish the cases suggested by Lemma 14. Recall that the predecessor indices $\mathrm{F},\mathrm{M},\mathrm{L}$ of $j'+1$ are given by $\ell, j', 2$. If case (i) of Lemma 14 holds, then z is a zero of $\mathcal{D}_\ell$ and $\mathcal{D}_{j'}$ since $\mathcal{D}_2 = 1$. Then clearly z does not depend on $a_{j'}$ and $b_{j'}$ since $\mathcal{D}_\ell$ does not.

If case (ii) of Lemma 14 holds, then z is a zero of $\mathcal{N}_2^\circ \mathcal{D}_{j'} - \mathcal{N}_{j'}^\circ \mathcal{D}_2$ and $\mathcal{N}_\ell^\circ \mathcal{D}_{j'} - \mathcal{N}_{j'}^\circ \mathcal{D}_\ell$. We distinguish two cases for $\circ \in \{x, y\}$. For $\circ = y$, it holds that $\mathcal{N}_2^y = 0$ and $\mathcal{D}_2 = 1$. Thus, it follows that $\mathcal{N}_{j'}^y[z] = 0$ and $(\mathcal{N}_\ell^y \mathcal{D}_{j'})[z] = 0$. Since $\mathcal{N}_{j'}^y$ and $\mathcal{D}_{j'}$ are crr-free by the induction hypothesis, it holds that $\mathcal{N}_\ell^y[z] = 0$. Then as a zero of $\mathcal{N}_\ell^y$, z does neither depend on $a_{j'}$ nor on $b_{j'}$.

For $\circ = x$, $\mathcal{N}_2^x = 1$ and $\mathcal{D}_2 = 1$ imply that $\mathcal{N}_{j'}^x[z] = \mathcal{D}_{j'}[z]$ and $\mathcal{D}_{j'}[z](\mathcal{N}_\ell^x - \mathcal{D}_\ell)[z] = 0$. Since $\mathcal{D}_{j'}[z] \neq 0$, as otherwise $\mathcal{N}_{j'}^x$ and $\mathcal{D}_{j'}$ are not crr-free, it holds that $\mathcal{N}_\ell^x[z] = \mathcal{D}_\ell[z]$. Using the last fact, $E_{j'+1}[z]$ simplifies to $(1-z)\mathcal{N}_\ell^y[z] - \left(\sum_{k=1}^{j}(a_{k'} + b_{k'})\right)\mathcal{D}_\ell[z] = 0$. This implies that

$$\mathcal{N}_\ell^x[z] = \mathcal{D}_\ell[z] = \frac{1}{\sum_{k=1}^{j}(a_{k'} + b_{k'})}(1-z)\mathcal{N}_\ell^y[z].$$

Since neither $\mathcal{N}_\ell^x$ nor $\mathcal{D}_\ell$ depend on $a_{k'}$ and $b_{k'}$, these three polynomial do not coincide at z for small variations of $a_{k'}$. Thus for a dense set of algebraically independent area assignments, these three polynomials share no common real root. Consequently, we can assume that z does not depend on $a_{j'}$ and $b_{j'}$ and thus z is a zero of both $(a\mathcal{N}_\ell^x + (1-x)\mathcal{N}_\ell^y)$ and $\mathcal{D}_\ell$.

However, we show that this is not the case.

Claim 4. $\mathcal{D}_\ell$ *and* $(a\mathcal{N}_\ell^x + (1-x)\mathcal{N}_\ell^y)$ *are crr-free.*

Suppose z is a zero of $\mathcal{D}_\ell$ and $a\mathcal{N}_\ell^x + (1-x)\mathcal{N}_\ell^y$. Recall that by Equation (12) and since $\mathcal{D}_{i+1} = E_{i+1}D_i$ it follows for $i \in [\ell]$ for $i \in [\ell]$ it holds that

$$E_i = x - \alpha_i \quad \text{and} \quad \mathcal{D}_i = \prod_{j=1}^{i} E_j = \prod_{j=1}^{i}(x - \alpha_j).$$

Therefore, the zero set of $\mathcal{D}_i$ is given by $\{\alpha_i \mid i \in [\ell]\}$. We define

$$G_j[x] := a\mathcal{N}_j^x[x] + (1-x)\mathcal{N}_j^y[x]$$

and aim to show by induction on $j \in [\ell]$ that for all $i \leq j$: $G_j[\alpha_i] \neq 0$. Note that the claim is equivalent to $G_\ell[\alpha_i] \neq 0$ for all $i \leq \ell$. For the induction base, Equation (5) shows that $\mathcal{N}_1^x = a_1 x + b_1$ and $\mathcal{N}_1^y = a_1 a + b_1$. Consequently, it holds that $G_1[\alpha_1] = G_1[a] = a_1 a + b_1 \neq 0$. By Equation (10), for $i \in [\ell-1]$, the numerator polynomials can be expressed by $\mathcal{N}_j^\circ = \mathcal{D}_j(a_j\mathcal{N}_v^\circ + b_j) + \mathcal{N}_j^\circ E_{j+1}$. This yields

$$G_j[\alpha_i] = a\big(\mathcal{D}_j(a_j\mathcal{N}_v^x + b_j) + \mathcal{N}_j^x E_{j+1}\big)[\alpha_i] + (1-\alpha_i)\big(\mathcal{D}_j(a_j\mathcal{N}_v^y + b_j) + \mathcal{N}_j^y E_{j+1}\big)[\alpha_i]$$
$$= \mathcal{D}_j[\alpha_i] \cdot (aa_j + b_j(1 + a - \alpha_i)) + E_{j+1}[\alpha_i] \cdot \big(a\mathcal{N}_j^x[\alpha_i] + (1-\alpha_i)\mathcal{N}_j^y[\alpha_i]\big)$$

If $i \leq j$, then the first summand vanishes since $\mathcal{D}_j[\alpha_i] = 0$. The second summand does not vanish by induction and since $E_{j+1}[\alpha_i] = \alpha_i - \alpha_{j+1} < 0$. For $i = j+1$, the second term vanishes since $E_{j+1}[\alpha_{j+1}] = 0$ and the first term does not vanish since both factors do not. Consequently, it holds that $G_\ell[\alpha_i] \neq 0$. This finishes both, the proof of the claim and the theorem.