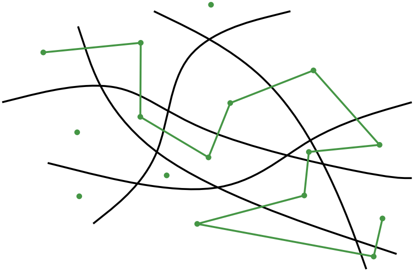
Upper and Lower Bounds of Long Paths in Line Arrangements



Udo Hoffmann
TU Berlin

Linda Kleist
TU Berlin

Tillmann Miltzow
FU Berlin

UPPER AND LOWER BOUNDS OF LONG PATHS IN LINE ARRANGEMENTS

UDO HOFFMANN
TU Berlin

LINDA KLEIST
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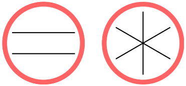
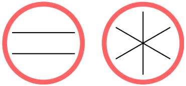
TILLMANN MILTZOW
FU Berlin

Pseudoline Arrangements

Simple



SIMPLE



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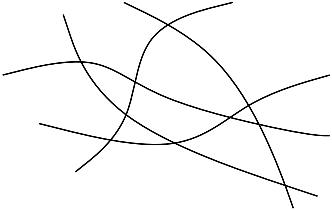
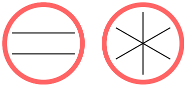


Pseudoline Arrangements

SIMPLE



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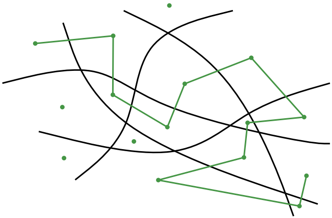
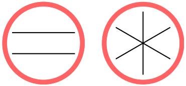


path

PSEUDOLINE ARRANGEMENTS

SIMPLE



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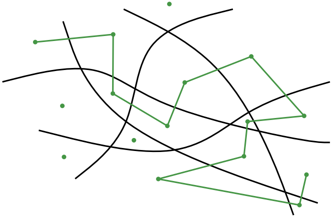
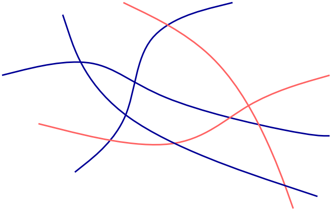


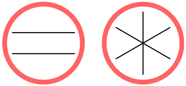
path

Bichromatic

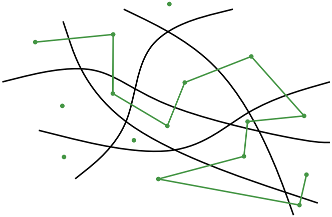


PSEUDOLINE ARRANGEMENTS

SIMPLE

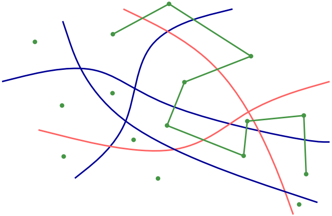


Monochromatic



path

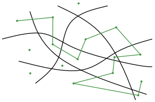
Bichromatic



alternating path

Related Work

Monochromatic

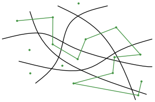


Aichholzer, Cardinal, Hackl, Hurtado, Korman, Pilz, Silveira, Uehara, Valtr, Vogtenhuber, Welzl [2014]

- $\forall \mathcal{A}$ with n lines, longest path $\geq \frac{n^2}{4} + O(n)$
- $\exists\ \mathcal{A}$ with n lines, longest path $\leq \frac{n^2}{3} + O(n)$

Related Work

Monochromatic



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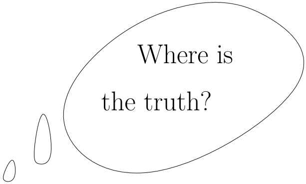
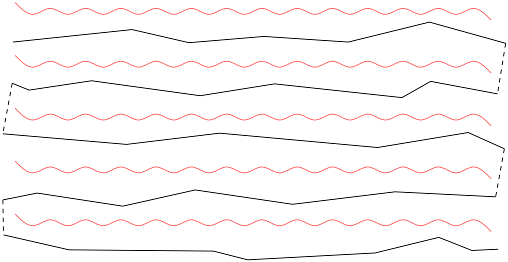


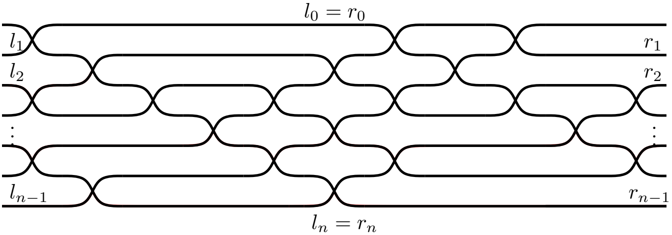
Theorem

In every simple arrangement $\mathcal{A}$ of n pseudolines, there exists a path of length $\frac{1}{3}n^2 - O(n)$.



Monochromatic

Level and Tunnel



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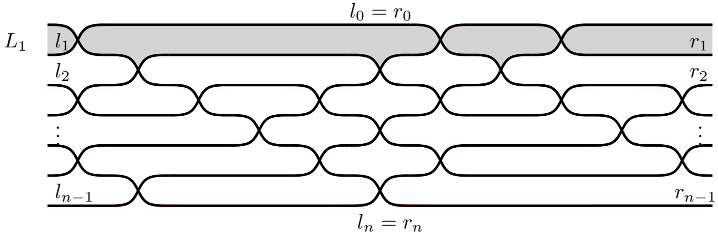

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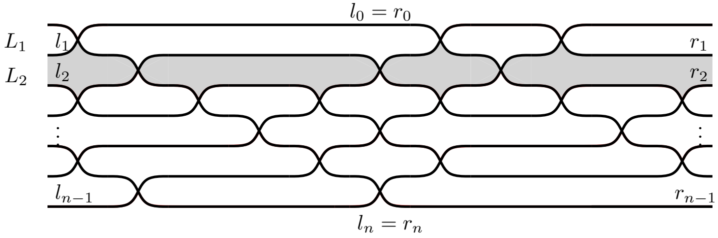

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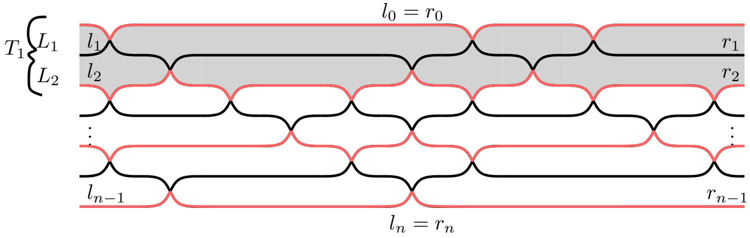
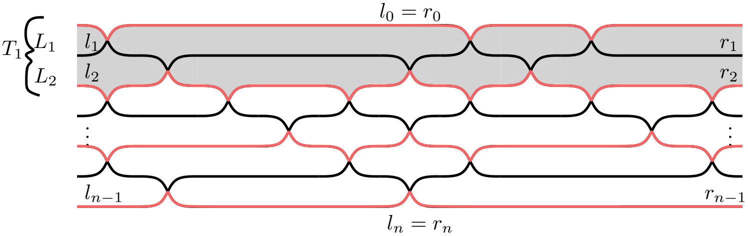
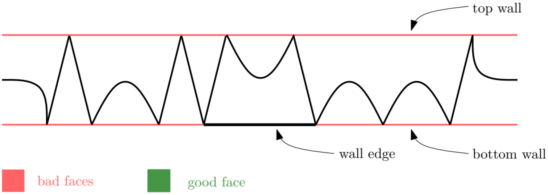


MONOCHROMATIC

LEVEL AND TUNNEL





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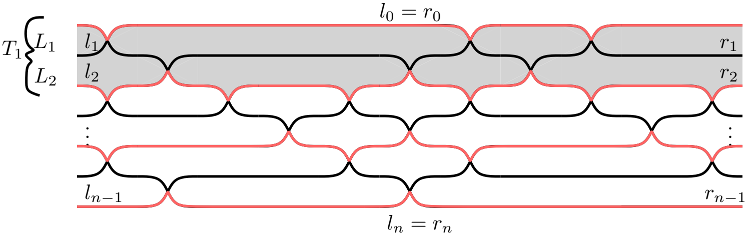
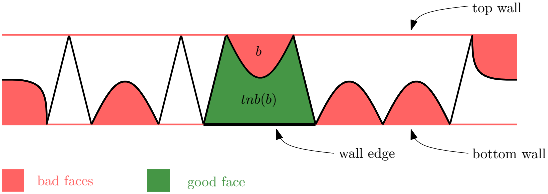
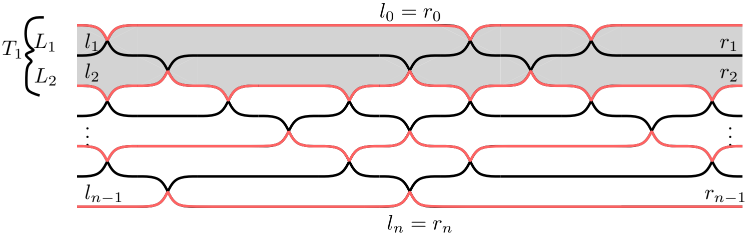
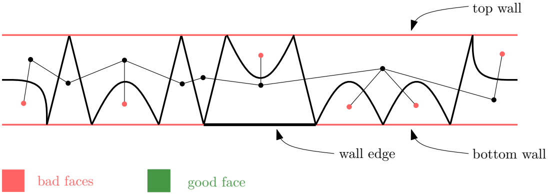


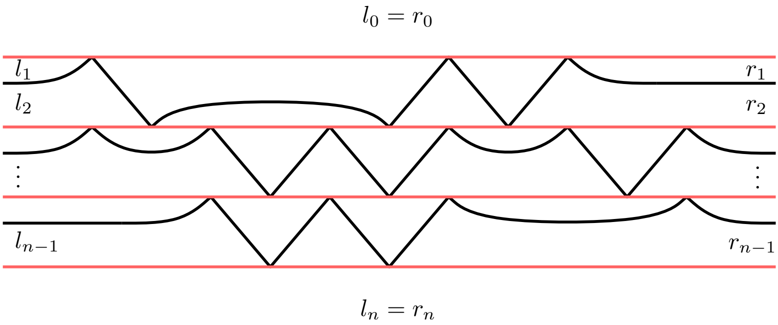


LEVEL AND TUNNEL



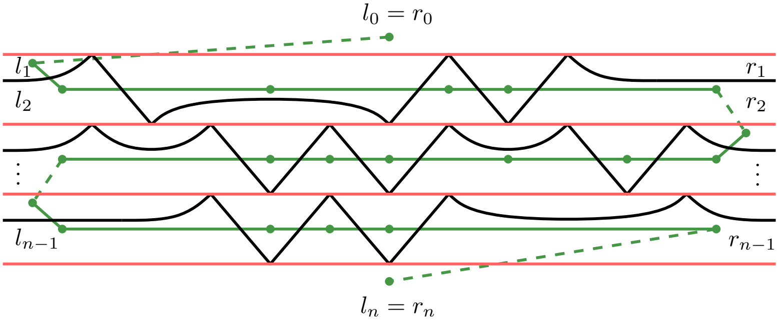


LEVEL AND TUNNEL



$$P_i := \text{path in tunnel } T_i \begin{cases} \text{from } l_{2i-1} \text{ to } r_{2i} & i \text{ odd,} \\ \text{from } l_{2i} \text{ to } r_{2i-1} & i \text{ even.} \end{cases}$$

$$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} |P_i| \geq 1/4\, n^2 - n$$



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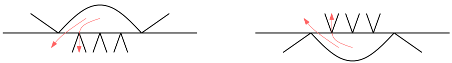
$$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} |P_i| \geq 1/4\, n^2 - n$$

$F = U + T + N$

$$ch(f) = \begin{cases} 2 & f \in N, \\ 0 & \text{else.} \end{cases}$$



Find path P and charge function ch such that

- $ch(f) = 0$ for all $f \in N$.
- $ch(f) \leq 1$ for all $f \in T$.
- $ch(f) \leq 2$ for all $f \in U$.
- $\sum_f ch(f) = 2|N|$

ratio

2 : 1

traversed : not traversed

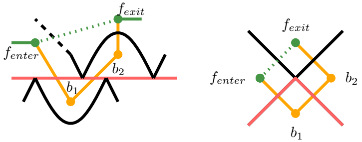
STEP 1: CHARGE-FREENESS FOR N

$p = b_1, b_2, \ldots$ maximal path of not traversed faces with $|p| \geq 2$

Step 1a) If $\deg_{\mathcal{A}}(b_1) \geq 3$

$\ldots, f_{\text{enter}}, f_{\text{exit}}, \ldots \rightsquigarrow \ldots, f_{\text{enter}}, b_1, b_2, f_{\text{exit}}, \ldots$

reroute,
delete charge,
repeat



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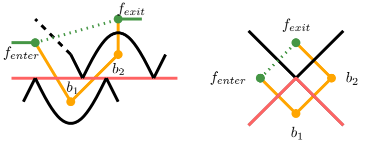

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reroute,
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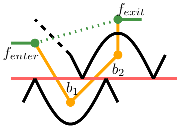
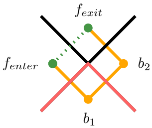
repeat





Step 1b) If $\deg(b_1) = 2$ and $|p| \geq 5$.

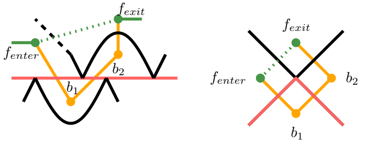
STEP 1: CHARGE-FREENESS FOR N

$p = b_1, b_2, \ldots$ maximal path of not traversed faces with $|p| \geq 2$

Step 1a) If $\deg_{\mathcal{A}}(b_1) \geq 3$

$\ldots, f_{\text{enter}}, f_{\text{exit}}, \ldots \rightsquigarrow \ldots, f_{\text{enter}}, b_1, b_2, f_{\text{exit}}, \ldots$

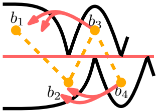
reroute,
delete charge,
repeat



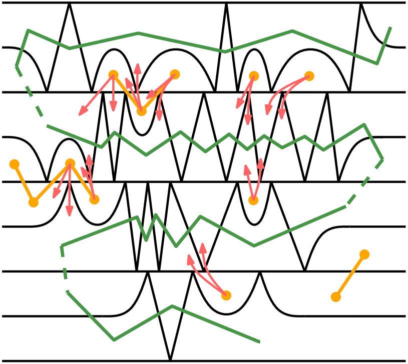
Step 1b) If $\deg(b_1) = 2$ and $|p| \geq 5$.

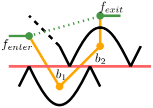
Step 1c) If $\deg(b_1) = 2$ and $|p| \leq 4$.

resend charge



STEP 1: CHARGE-FREENESS FOR N





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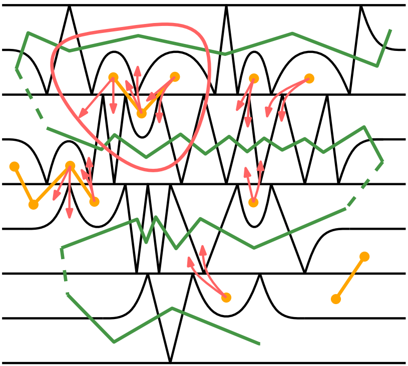
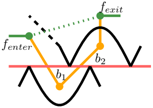
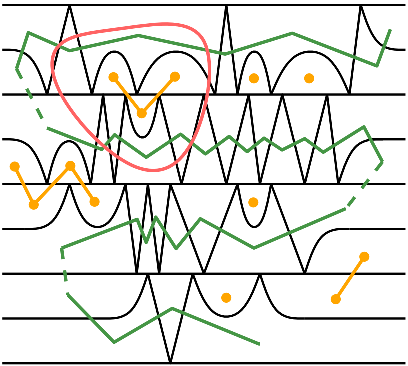
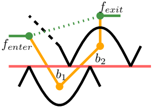




MONOCHROMATIC

STEP 1: CHARGE-FREENESS FOR N





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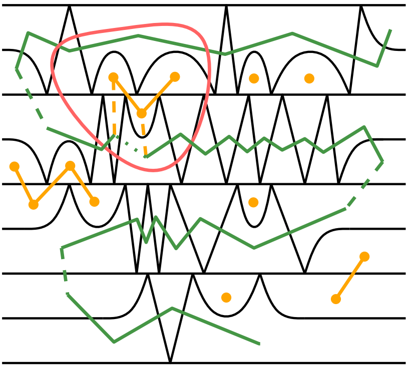
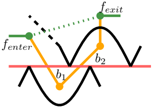
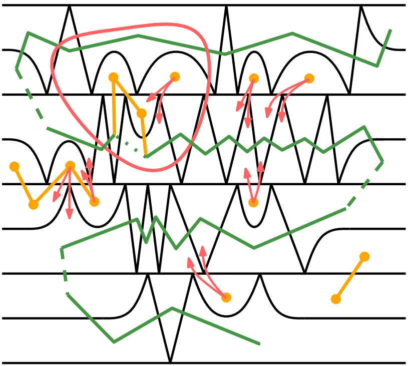
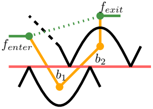




Monochromatic

Step 1: Charge-Freeness for N





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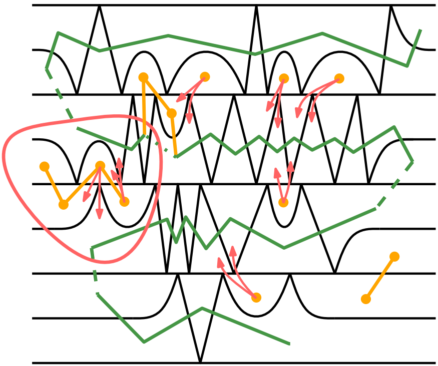

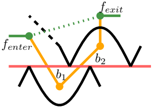

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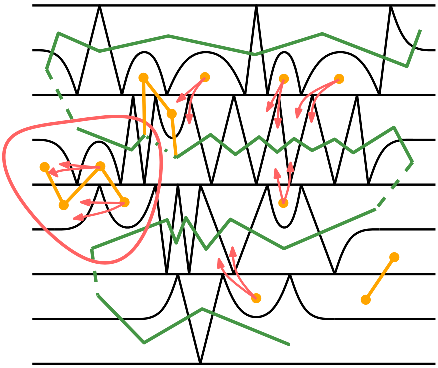

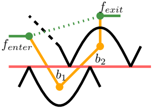

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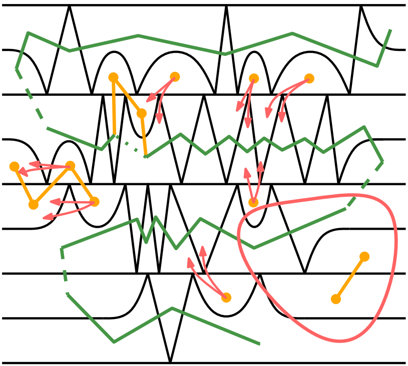

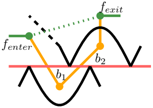

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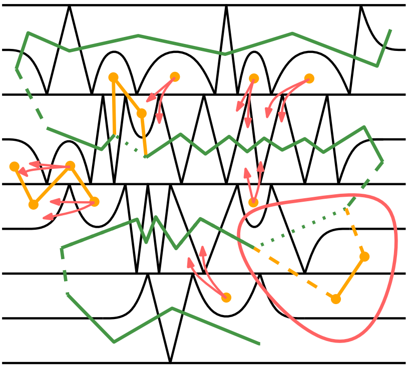

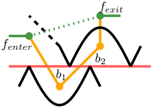

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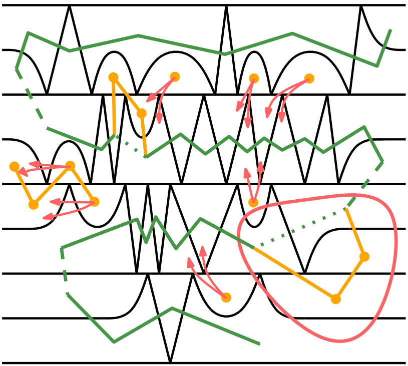

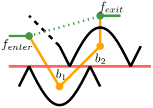

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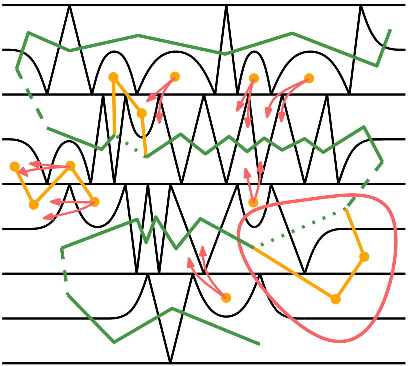
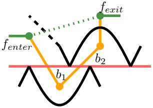




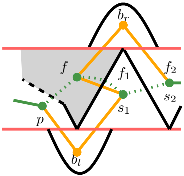
- Not traversed faces are free of charge!
- Only rightmost faces remain sending charge!

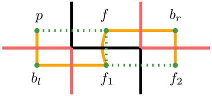
STEP 2: ONE UNIT FOR T

f bounded traversed face with charge ≥ 1

Step 2a) If (f_1, f_2) is an edge in P_i $\quad \ldots, p, f, f_1, f_2, \ldots \rightsquigarrow \ldots, p, b_l, f_1, f, b_r, f_2, \ldots$

reroute,
delete charge





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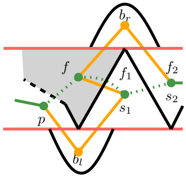
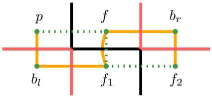
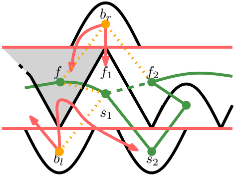
delete charge





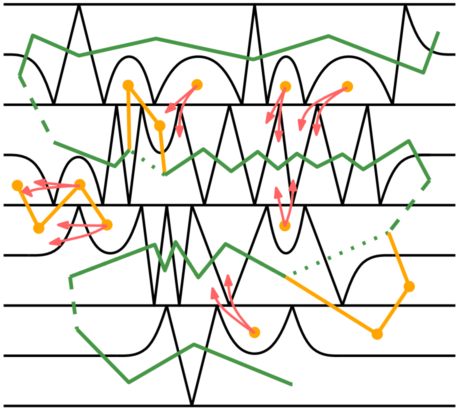
Step 2b) If (f_1, f_2) was replaced in Step 1

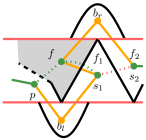
resend charge



Monochromatic

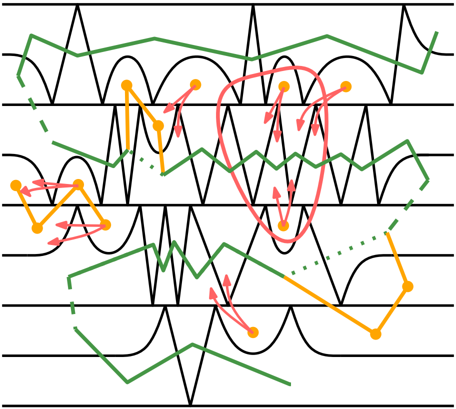
Step 2: One Unit for T

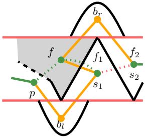




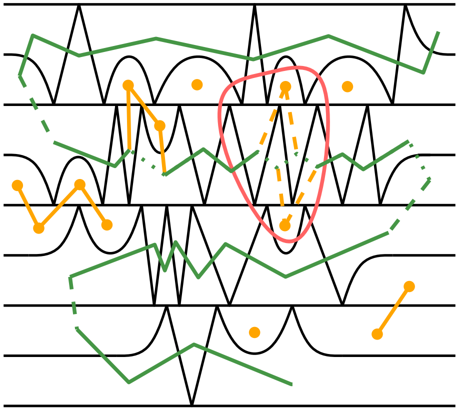
MONOCHROMATIC

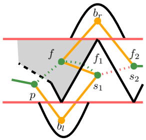
STEP 2: ONE UNIT FOR T





STEP 2: ONE UNIT FOR T





MONOCHROMATIC

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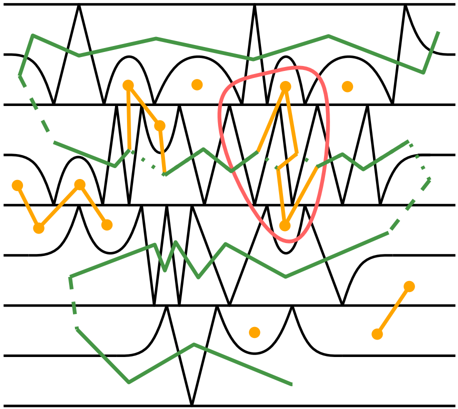
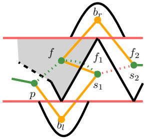
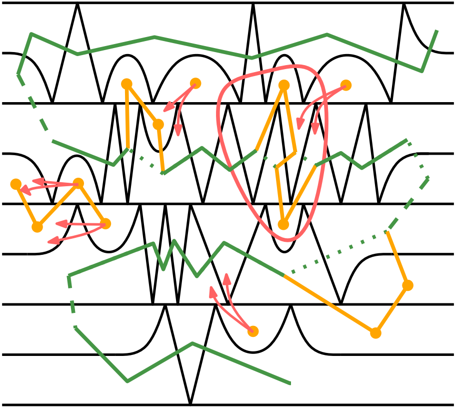
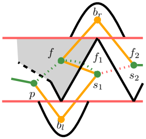




MONOCHROMATIC

STEP 2: ONE UNIT FOR T





We found path P and charge function ch such that

- $ch(f) = 0$ for all $f \in N$.
- $ch(f) \leq 1$ for all $f \in T$.
- $ch(f) \leq 2$ for all $f \in U$.
- $\sum_f ch(f) = 2|N|$

□

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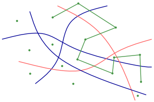
☺

□

Related Work

Bichromatic

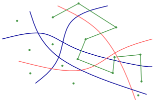


Aichholzer, Cardinal, Hackl, Hurtado, Korman, Pilz, Silveira,
Uehara, Valtr, Vogtenhuber, Welzl [2014]

- $\forall$ bicolored $\mathcal{A}$, longest alternating paths $\geq n$
- $\exists$ bicolored $\mathcal{A}$ with longest path $\leq 2n+1$

Related Work

Bichromatic



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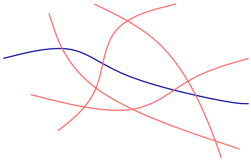
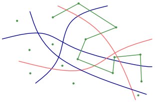


BICHROMATIC



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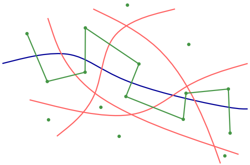
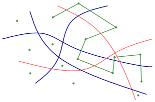


BICHROMATIC



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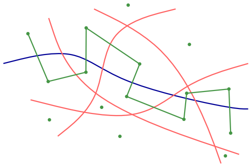
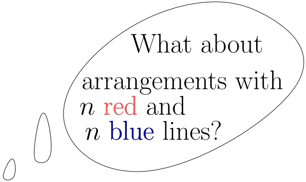




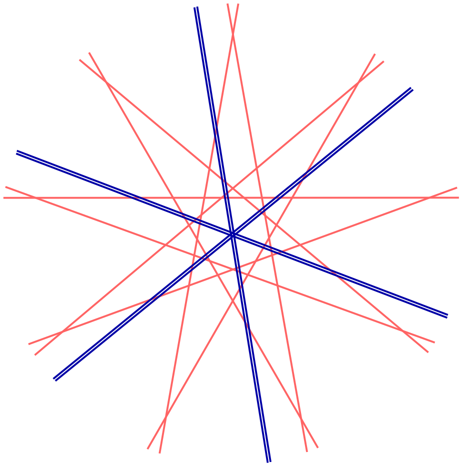
Does every arrangement of n red and n blue lines has an alternating path of quadratic length?

Does every arrangement of n red and n blue lines has an alternating path of quadratic length?

Theorem

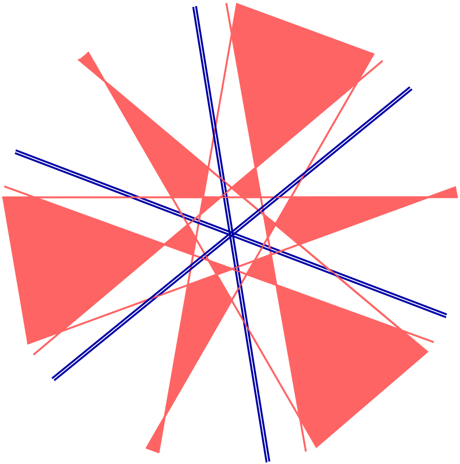
There exists a simple arrangement $\mathcal{A}$ of $3k$ red and $2k$ blue lines where any alternating dual path goes through at most $14k$ faces, for every odd k.

UPPER BOUND EXAMPLE



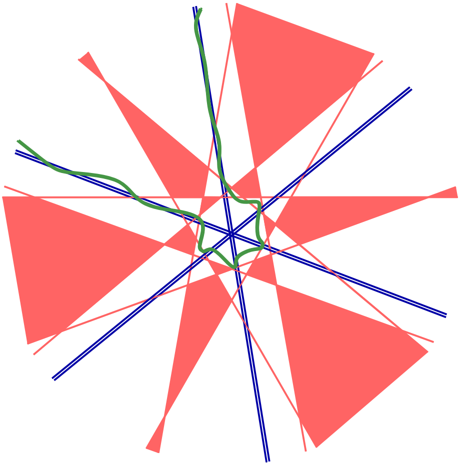
Bichromatic

Upper bound example



BICHROMATIC

UPPER BOUND EXAMPLE



Given an arrangment $\mathcal{A}$ of n lines, is there a coloring such that there exist an alternating path of length $\Omega(n^2)$?

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Theorem

In a random bicoloring of an arrangement of n pseudolines, there exists an alternating path of length $\Omega\left(n^2/\log n\right)$ with high probability.

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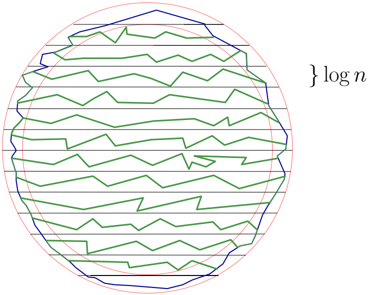
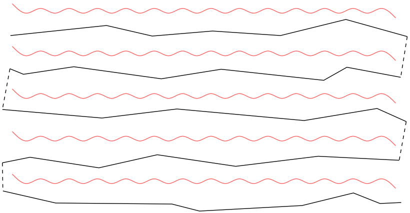
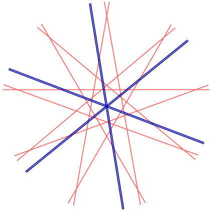


1. Bicolored arrangements:
   Gap between lower and upper bound
   $$\left[n \leq \min_{\mathcal{A}:|\mathcal{A}|=n} \max_{P \in \mathcal{A}} |P| \leq 2n+1 \right]$$
2. Balanced Bicolored Arrangements:
   In an arrangement of n red and n blue pseudolines, is there an alternating path of quadratic length?

THANKS ☺





- tight lower bound for monochromatic path $[n^2/3 - O(n)]$
- upper bound for almost balanced bichromatic arrangement $[O(n)$ for $3n$ red and $2n$ blue lines]
- $\Omega(n^2/\log n)$ for a random coloring with high probability