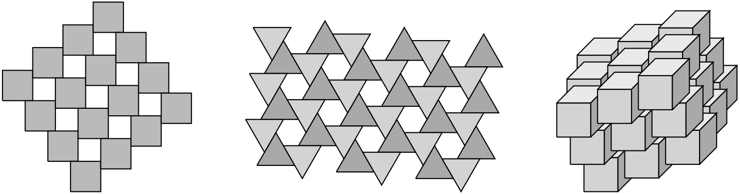
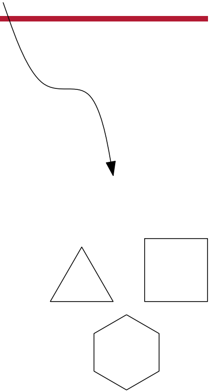
Unit Contact Representations of Grid Subgraphs with Regular Polytopes in 2D and 3D



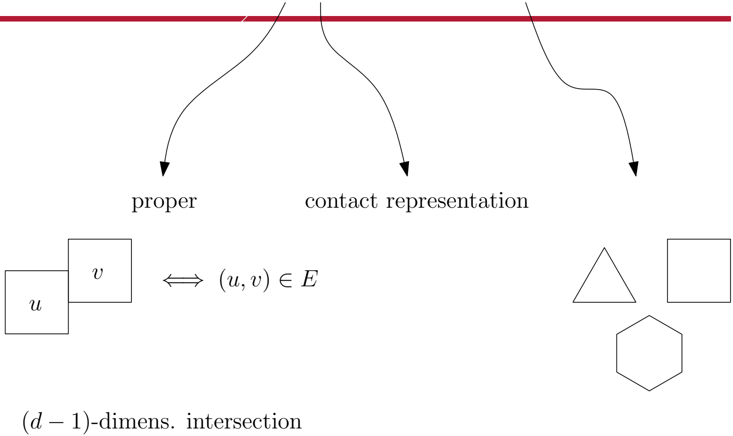
Linda Kleist & Benjamin Rahman
Technische Universität Berlin

UPCR with regular polygons



- vertices: congruent regular polygons, interiorly disjoint
- edges: $(d-1)$-dimensional intersections

UPCR with regular polygons



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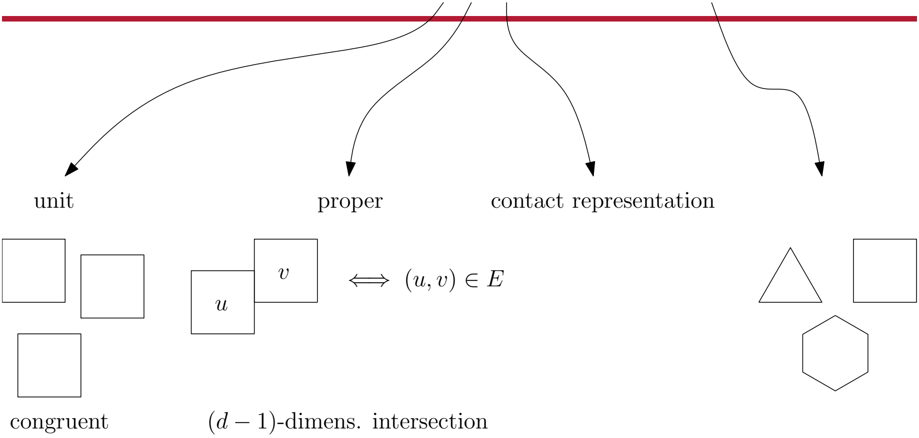

- vertices: congruent regular polygons, interiorly disjoint
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Basic properties of UPCR

- low maximal degree
- volume constraints

} $\Longrightarrow$ Let $\mathbb{G}$ be a grid. Does every subgraph $G \subseteq \mathbb{G}$ has a UPCR (with a particular object type)?

- NP-hard recognition
  - unit disks [Breu, Kirkpatrick, 1996]
  - unit cubes [Bremner, Evans, Frati, Heyer, Kobourov, Lenhart, Liotta, Rappaport, Whitesides, 2013]
  - squares, (triangles, hexagons, $6k$-gons, ...) [K., Rahman, 2014]

Previous Work and Questions

Theorem (Alam, Chaplick, Fijav, Kaufmann, Kobourov, Pupyrev, 2013)

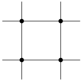
Every subgraph of the square grid allows for a UPCR with cubes.

Open:

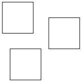
- Do subgraphs of the triangular grid allow for UPCR with cubes?

Results with strategy

Every subgraph of has a UPCR with



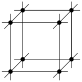
square grid



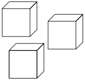
squares

pseudo-squares

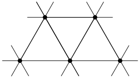
$4k$-gons



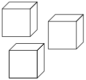
d-dimen. grid



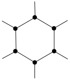
d-cubes



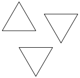
triangular grid



cubes

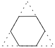


hexagonal grid



triangles

pseudo-triangles

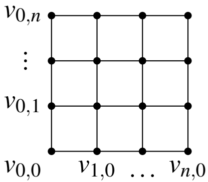


$3k$-gons

Strategy

- start with UPCR $\hat{\phi}$ of the grid $\mathbb{G}$
- remove unwanted contacts one by one
  - moving set
  - direction vector

Square grid $\mathbb{S}_n$



Theorem

Let G be a subgraph of $\mathbb{S}_n$. Then G has a USqPCR.

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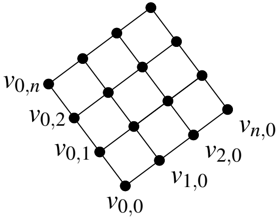
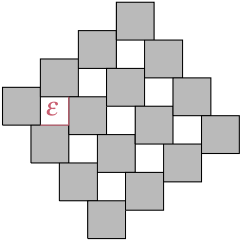
USqPCR $\hat{\phi}$ of $\mathbb{S}_n$ with $\varepsilon \in (0,1)$





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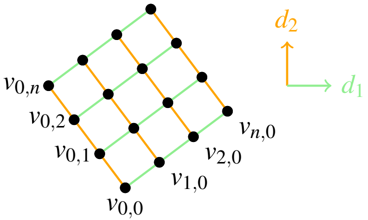

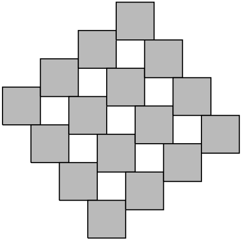

$E = E_1 \cup E_2$ (column and row edges)
direction vectors $d(e)$

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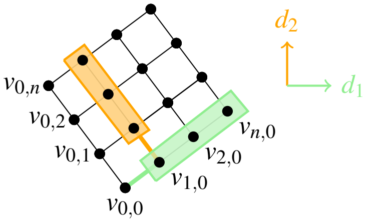
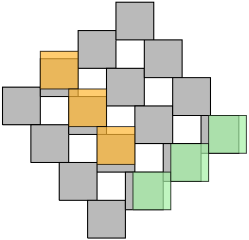




$E = E_1 \cup E_2$ (column and row edges)
moving sets $M(e)$

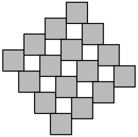
Construction- more formal

$\varepsilon \in (0,1), \quad \delta < \frac{1}{n}\min\{\varepsilon, 1-\varepsilon\}$

$$\phi : V \to \mathcal{P}(\mathbb{R}^2)$$

$$\phi(v) = \hat{\phi}(v) + \sum_i r_i(v) \cdot \delta d_i$$

$\hat{\phi}$



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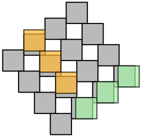

Properties

- $cs(\hat{\phi}) = 1 - \varepsilon$
- $sp_{\hat{\phi}}\big(M(e), d(e)\big) \geq \varepsilon$

- $cs(\phi) \geq 1 - \varepsilon - n\delta$
- $r_i(u) = r_i(v) \iff (u,v) \in E \cap E_i$

- interiorly disjoint (–> space)
- correct contacts (–> contact size)
- correct non-contacts (–> translation)

Generalization to all dimensions

Theorem

Let G be a subgraph of $\mathbb{S}_n^d$. Then G has a UPCR with d-cubes.

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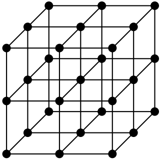
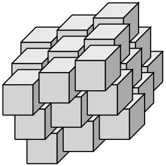
$$\hat{\phi}: V \to \mathcal{P}(\mathbb{R}^d)$$
$$\hat{\phi}(v_x) = Q(A \cdot x)$$

$$A := \begin{pmatrix} 1 & \varepsilon & \dots & \varepsilon \\ -\varepsilon & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \varepsilon \\ -\varepsilon & \dots & -\varepsilon & 1 \end{pmatrix}$$





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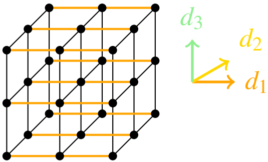
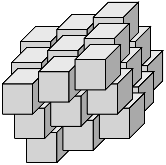
d types of edges: $E = E_1 \cup \dots \cup E_d$





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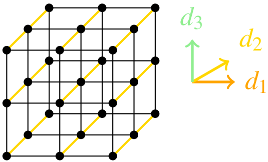

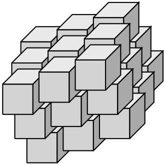

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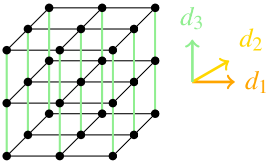

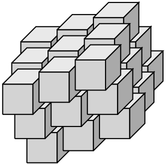

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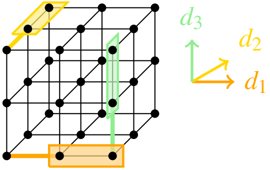

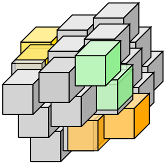

$$\phi : V \to \mathcal{P}(\mathbb{R}^d)$$

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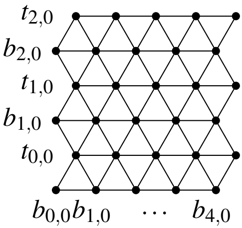
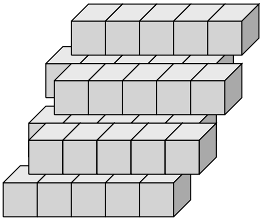
Theorem

Let G be a subgraph of $\mathbb{T}_{n,m}$. Then G has a UCuPCR.





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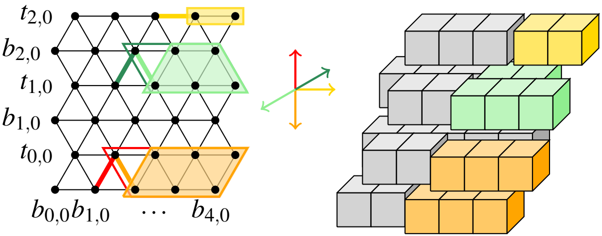
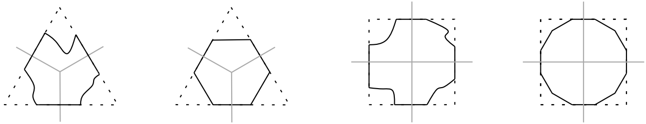


Pseudo-polygons



Lemma

Let G be a graph with a UPCR ϕ with regular k-gons and $cs(\phi) > 1 - s$. Then, G has a UPCR with pseudo k-gons with side length $\geq s$.

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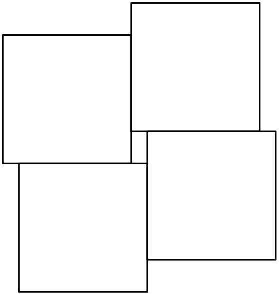

More regular polygons

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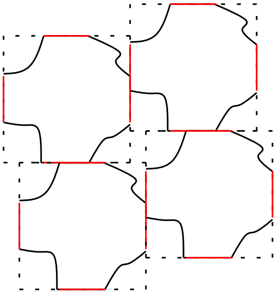

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Corollary

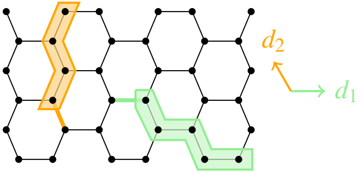
Let G be a subgraph of $\mathbb{S}_n$. Then G has a UPCR with $4k$-gons (pseudo-squares).

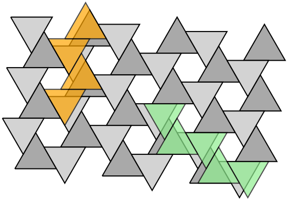
More regular polygons: $3k$-gons

Theorem

Let G be a subgraph of $\mathbb{H}_{n,m}$. Then G has a UPCR with $3k$-gons (pseudo-triangles).

Triangles+Lemma





Open problems

1. Characterization of graphs with USqPCRs?
2. Or with other polygons?
3. Is it NP-hard to recognize graphs admitting UPCRs with regular $(2k+1)$-gons?
4. Is it NP-hard to recognize graphs admitting UPCRs with d-cubes?
5. USqPCR for trihexagonal and truncated trihexagonal grid?
6. USqPCR for dual of snubsquare grid?
7. UCuPCR for duals of Archimedean grids not containing $K_{1,9}$?

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Thanks! ☺

USqPCR of Archimeadian grids