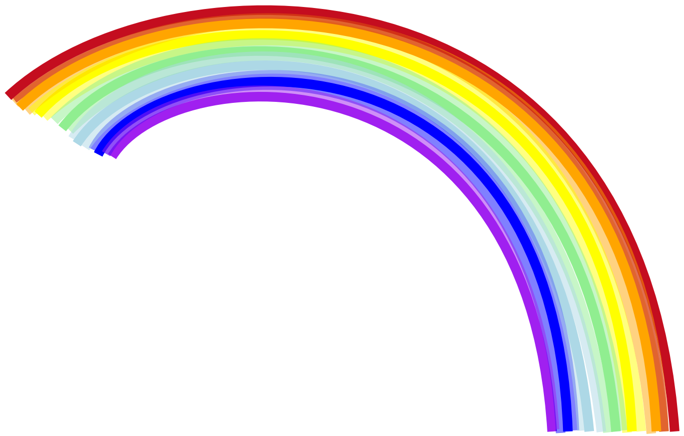


Rainbow Cycles in Flip Graphs

Linda Kleist

joint work with

Stefan Felsner, Torsten Mütze, Leon Sering

Technische Universität Berlin

Flip graph of triangulations

flip graph G_n^T

vertices: triangulations of a convex n-gon

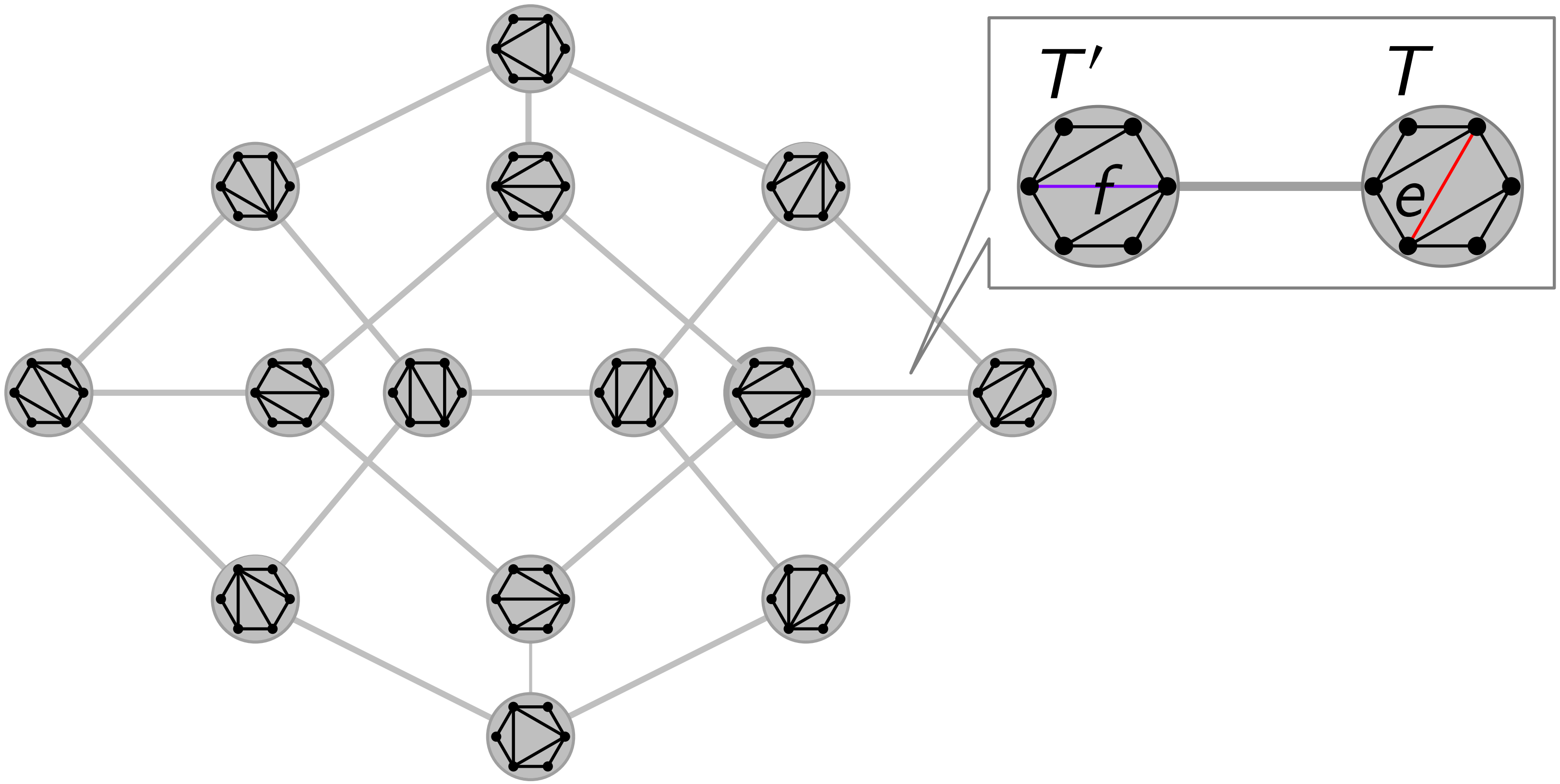
edges: flip a diagonal of the triangulation

associahedron

Flip graph of triangulations

flip graph G_n^T

- vertices: triangulations of a convex n-gon
- edges: flip a diagonal of the triangulation
- associahedron



Properties of G_n^T

- diameter

  $2n - 10$ for sufficiently large n [Sleater, Tarjan, Thurston 88]

  $2n - 10$ for $n > 12$, combinatorial [Pournin 14]

- many realizations as a convex polytope
  [Ceballos, Santos, Ziegler 15]
- vertex-connectivity, chromatic number, ...
- Hamiltonicity/Gray Codes
  [Lucas 87, Hurtado & Noy 99]

Hamilton cycle:
cyclic listing of objects such that each object appears 1 time

Rainbow cycles

Hamilton cycle:
cyclic listing of objects such that each object appears 1 time

dual

Rainbow cycle:
cyclic listing of objects such that each flip type appears 1 time

Rainbow cycles

Rainbow cycle: c o l o r

cyclic listing of objects such that each flip type appears 1 time

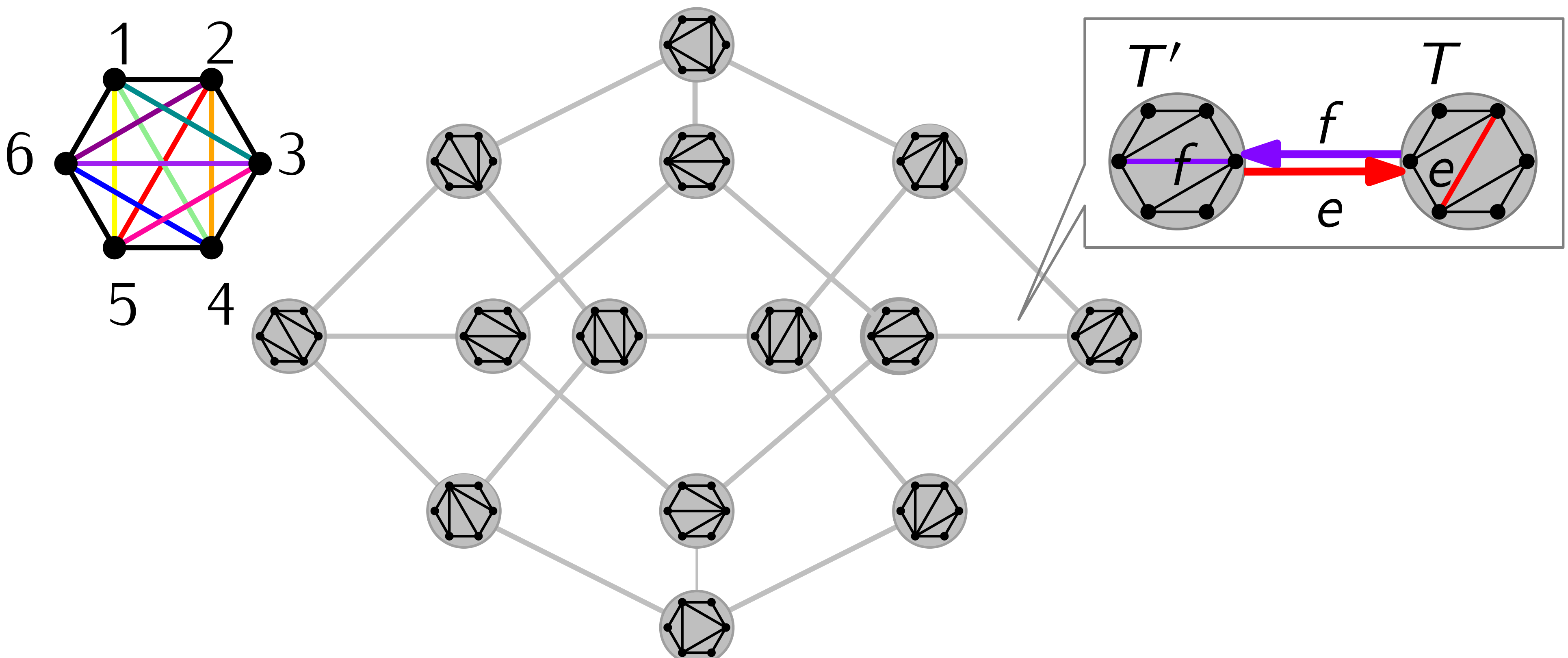
Rainbow cycles

Rainbow cycle: c o l o r
cyclic listing of objects such that each flip type appears 1 time

vertices: triangulations of a convex n-gon

arcs: flip a diagonal of the triangulation

arc color: the new edge



Rainbow cycles

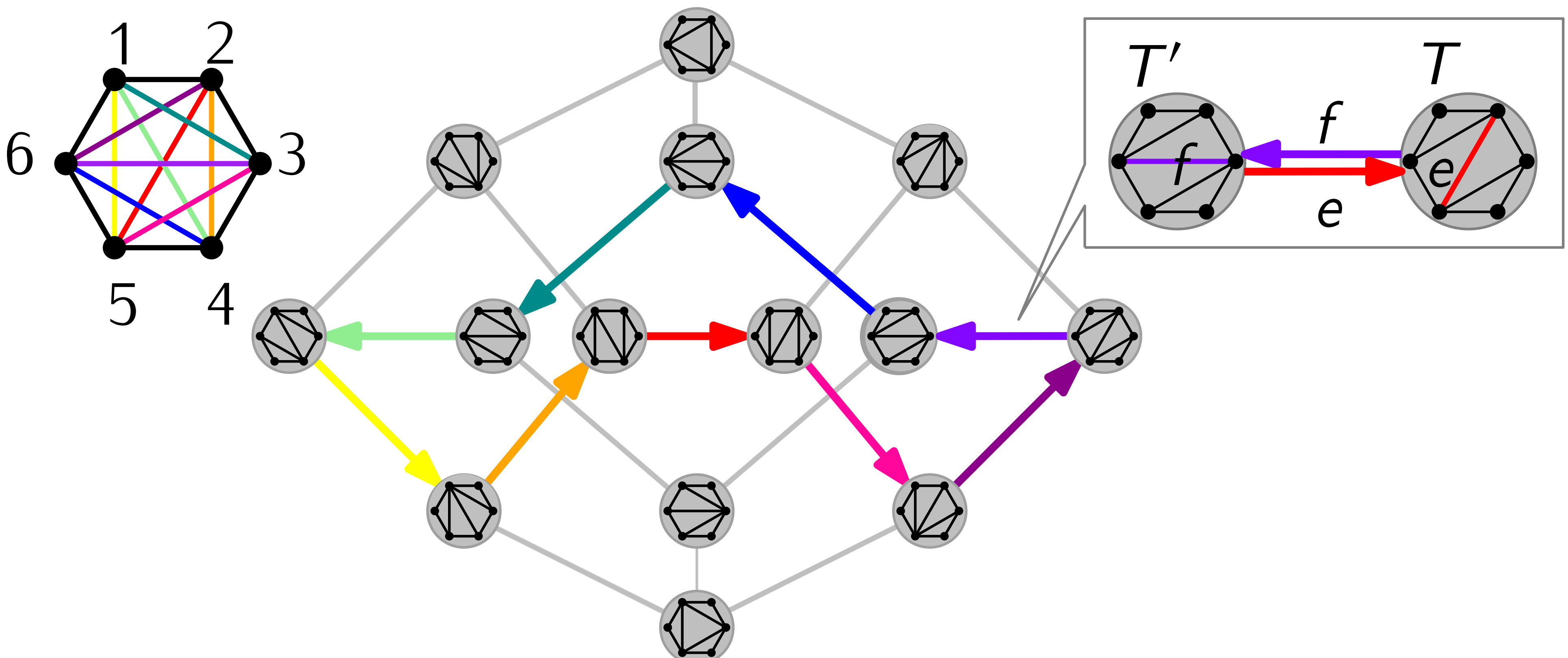
Rainbow cycle: c o l o r

cyclic listing of objects such that each flip type appears 1 time

vertices: triangulations of a convex n-gon

arcs: flip a diagonal of the triangulation

arc color: the new edge



Rainbow cycles

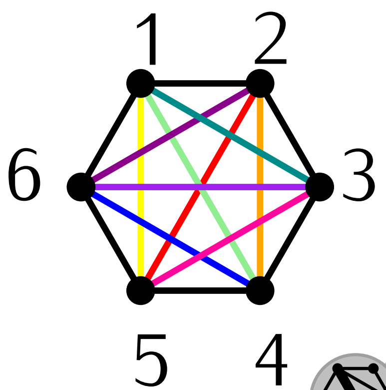
Rainbow cycle: c o l o r

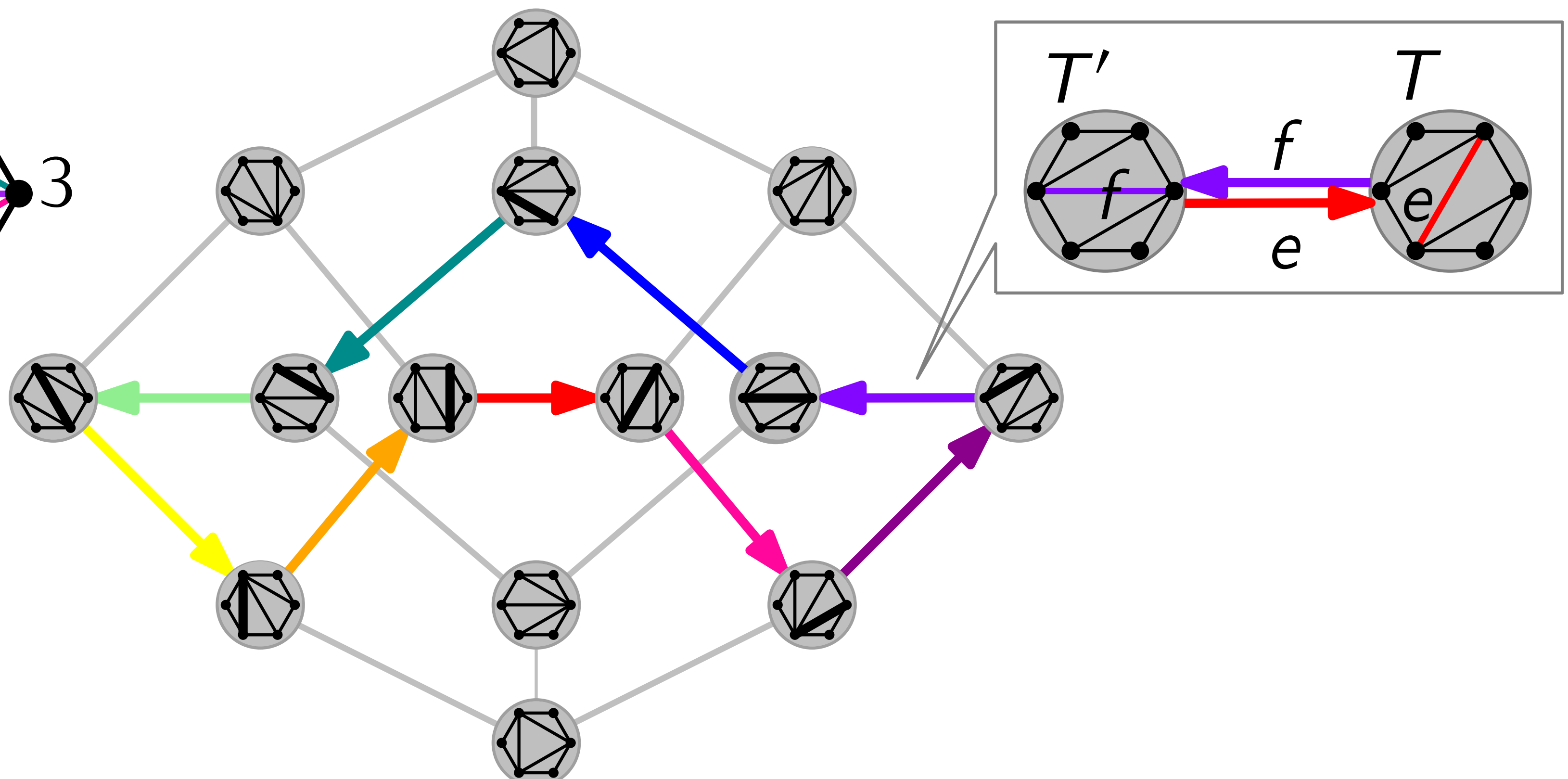
cyclic listing of objects such that each flip type appears 1 time

vertices: triangulations of a convex n-gon

arcs: flip a diagonal of the triangulation

arc color: the new edge





Rainbow cycles

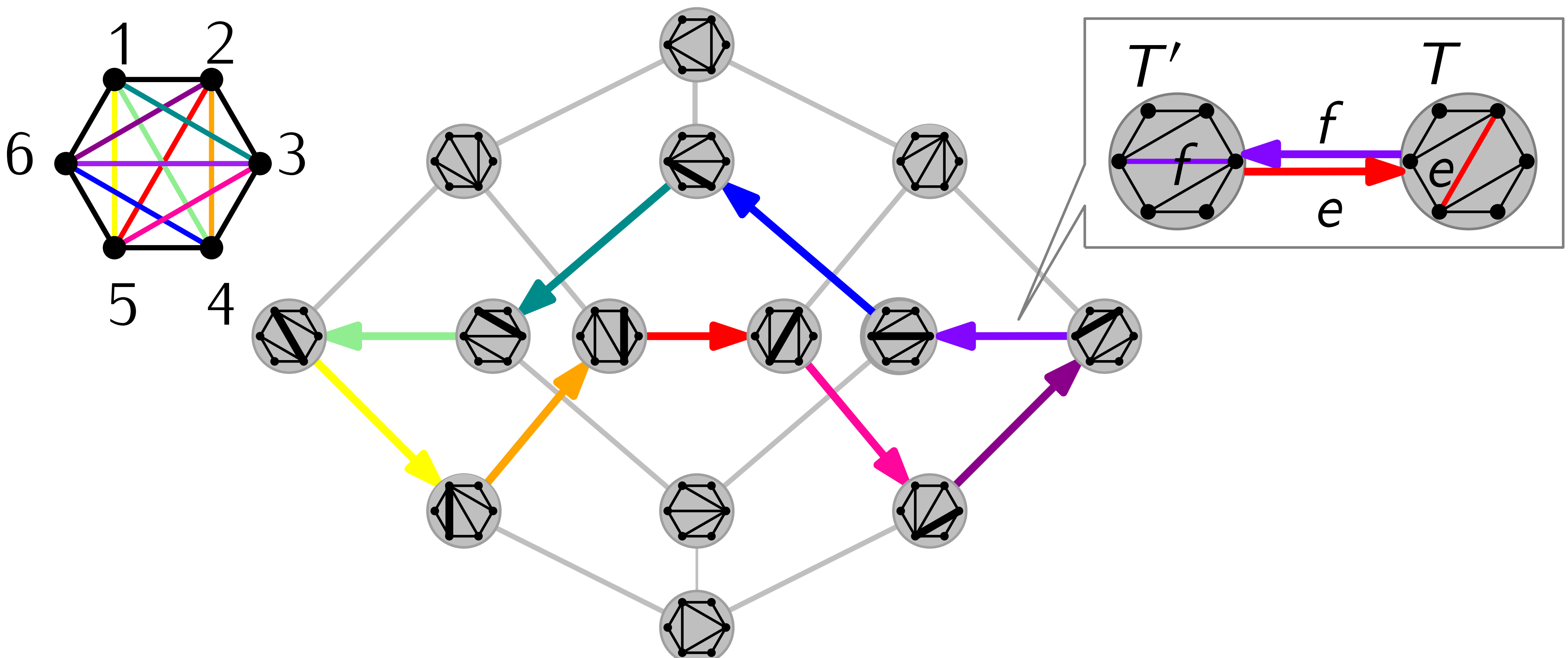
r-Rainbow cycle: c o l o r *r* times

cyclic listing of objects such that each flip type appears 1 time

vertices: triangulations of a convex *n*-gon

arcs: flip a diagonal of the triangulation

arc color: the new edge



Motivation

- binary Gray codes [Frank Gray 53]
  generate all 2^n bitstrings of length n by flippling a single bit per step

000
001
011
010
110
111
101
100

224

Motivation

- binary Gray codes [Frank Gray 53]

  generate all 2^n bitstrings of length n by flippling a single bit per step

  balanced Gray code:
  - each bit is flipped equally often ($2^n/n$ times) [Tootill 35, Bhat &Savage 96]
  - is a r-rainbow cycle for $r = 2^n/n$

000
001
011
010
110
111
101
100

224

Motivation

- binary Gray codes [Frank Gray 53]

  generate all 2^n bitstrings of length n by flippling a single bit per step

  balanced Gray code:
  - each bit is flipped equally often ($2^n/n$ times) [Tootill 35, Bhat &Savage 96]
  - is a r-rainbow cycle for $r = 2^n/n$

  000
  001
  011
  010
  110
  111
  101
  100
  224

- Our work: step towards balanced Gray codes for other classes
- known Gray codes
  - plane spanning trees [Hernando, Hurtado, Noy 02]
  - non-crossing perfect matchings [Aichholzer et al. 07]
  - non-crossing partitions, dissections of convex polygons [Huemer et al. 09]

Settings & Results

		flip graph			existence of r-rainbow cycle	
		vertices	arcs/edges	r	Yes	No
GEOMETRIC	G_n^{T}	triangulations of convex n-gon	edge flip	1	$n \geq 4$	
				2	$n \geq 7$	
	G_X^{S}	plane spanning trees on point set X in general position	edge flip	$1, \ldots, \lvert X\rvert - 2$	$\lvert X\rvert \geq 3$	
	G_m^{M}	non-crossing perfect matchings on $2m$ points in convex position	two edge flip	1	$m \in \{2, 4\}$	odd m, $m \in \{6, 8, 10\}$
				2	$m \in \{6, 8\}$	
ABSTRACT	G_n^{P}	permutations of $[n]$	transposition	1	$\lfloor n/2 \rfloor$ even	$\lfloor n/2 \rfloor$ odd
	$G_{n,k}^{\mathsf{C}}$	k-subsets of $[n]$, $2 \leq k \leq \lfloor n/2 \rfloor$	element exchange	1	odd n and $k < n/3$	even n
		2-subsets of $[n]$ for odd n		1	two edge-disjoint 1-rainbow Ham. cycles	

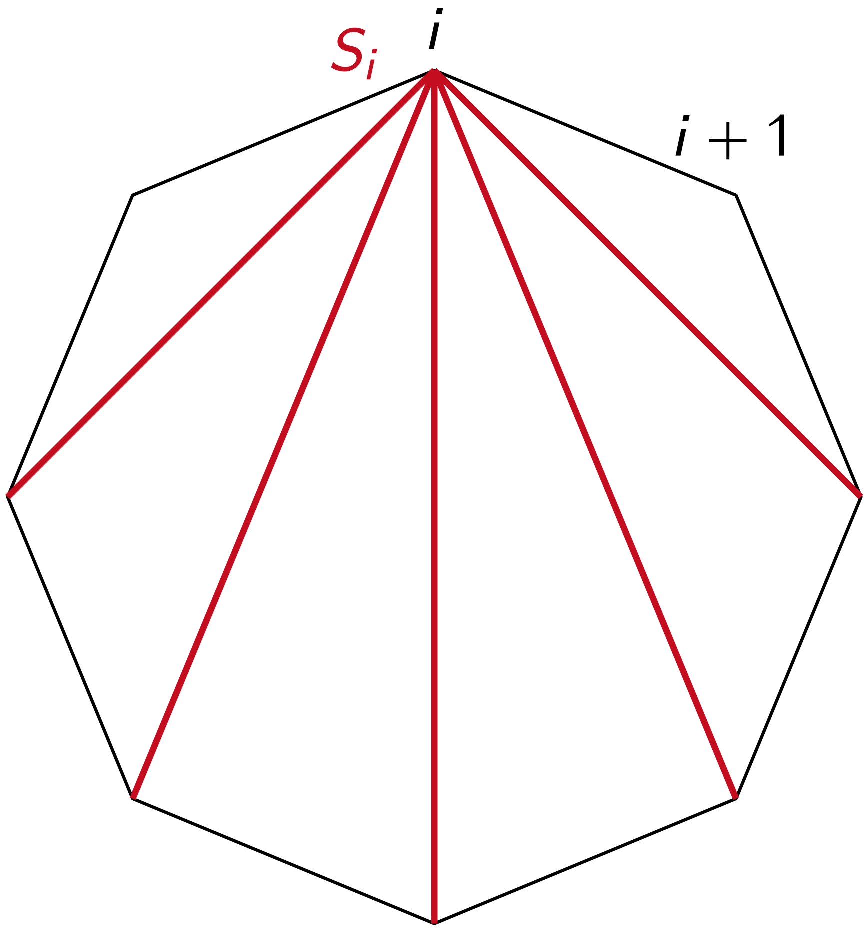
Settings & Results

		flip graph			existence of r-rainbow cycle	
		vertices	arcs/edges	r	Yes	No
GEOMETRIC	G_n^{T}	triangulations of convex n-gon	edge flip	1	$n \geq 4$	
				2	$n \geq 7$	
	G_X^{S}	plane spanning trees on point set X in general position	edge flip	$1,\ldots,\lvert X\rvert-2$	$\lvert X\rvert \geq 3$	
	G_m^{M}	non-crossing perfect matchings on $2m$ points in convex position	two edge flip	1	$m \in \{2,4\}$	odd m, $m \in \{6,8,10\}$
				2	$m \in \{6,8\}$	
ABSTRACT	G_n^{P}	permutations of $[n]$	transposition	1	$\lfloor n/2 \rfloor$ even	$\lfloor n/2 \rfloor$ odd
	$G_{n,k}^{\mathsf{C}}$	k-subsets of $[n]$, $2 \leq k \leq \lfloor n/2 \rfloor$	element exchange	1	odd n and $k < n/3$	even n
		2-subsets of $[n]$ for odd n		1	two edge-disjoint 1-rainbow Ham. cycles	

Triangulations

Thm: For $n \geq 7$, G_n^T has a 2-rainbow cycle.

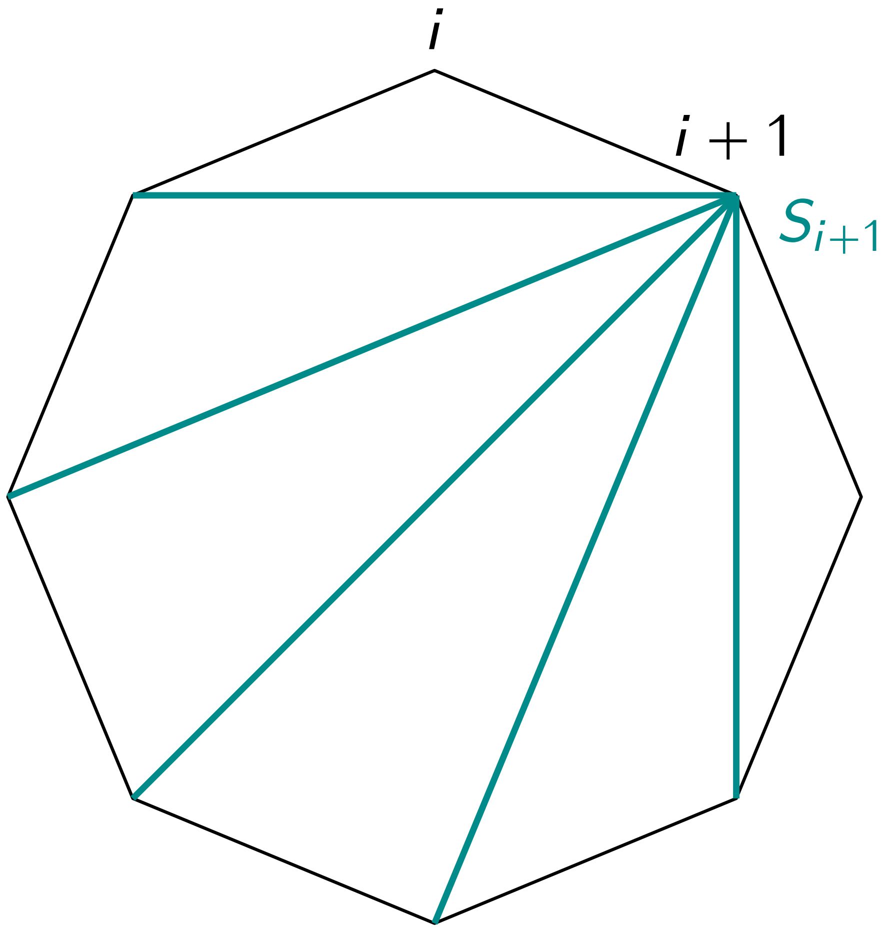
Proof:



Triangulations

Thm: For $n \geq 7$, G_n^T has a 2-rainbow cycle.

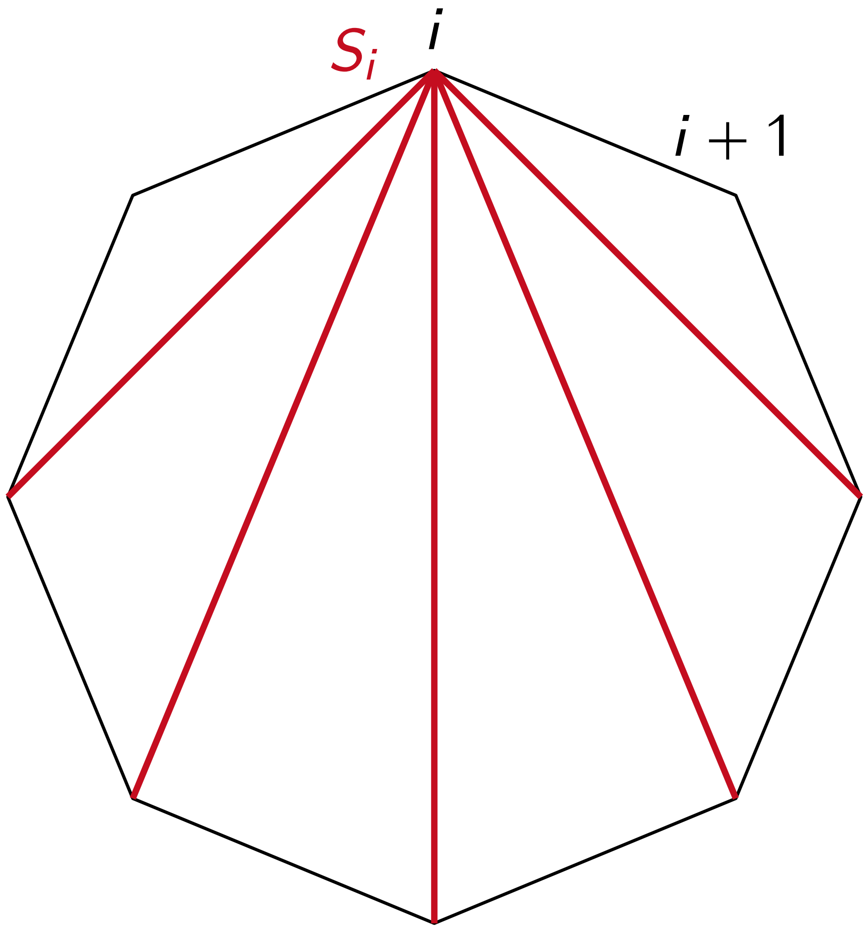
Proof:



Triangulations

Thm: For $n \geq 7$, G_n^T has a 2-rainbow cycle.

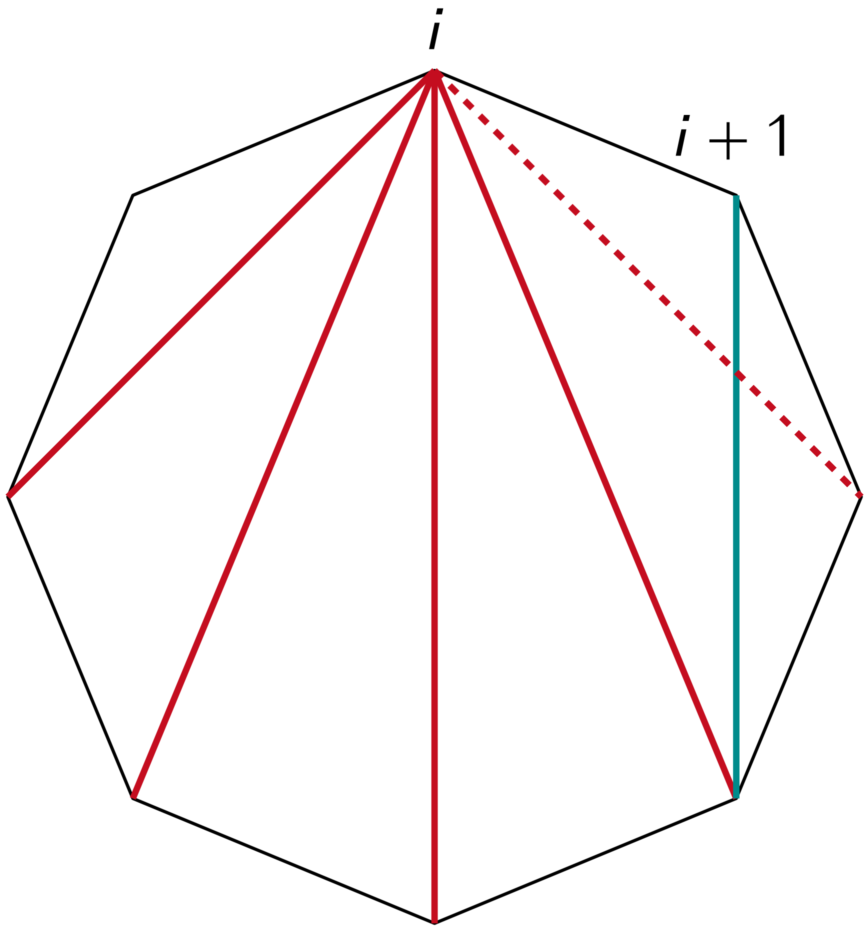
Proof:



Triangulations

Thm: For $n \geq 7$, G_n^T has a 2-rainbow cycle.

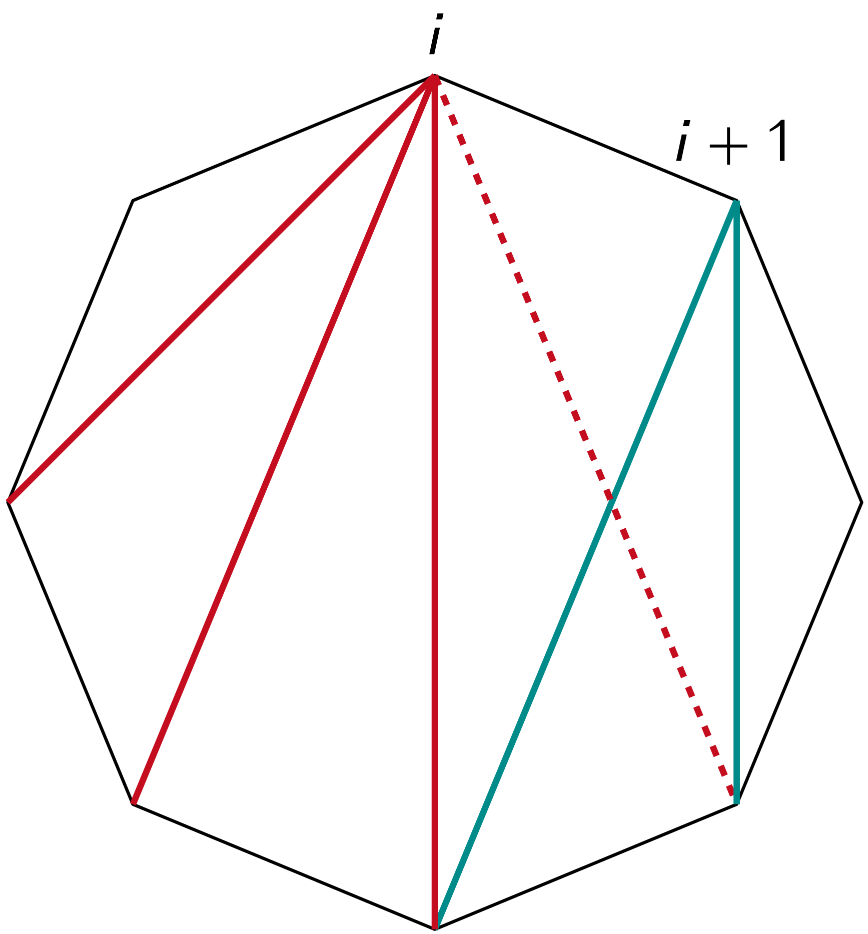
Proof:



Triangulations

Thm: For $n \geq 7$, G_n^T has a 2-rainbow cycle.

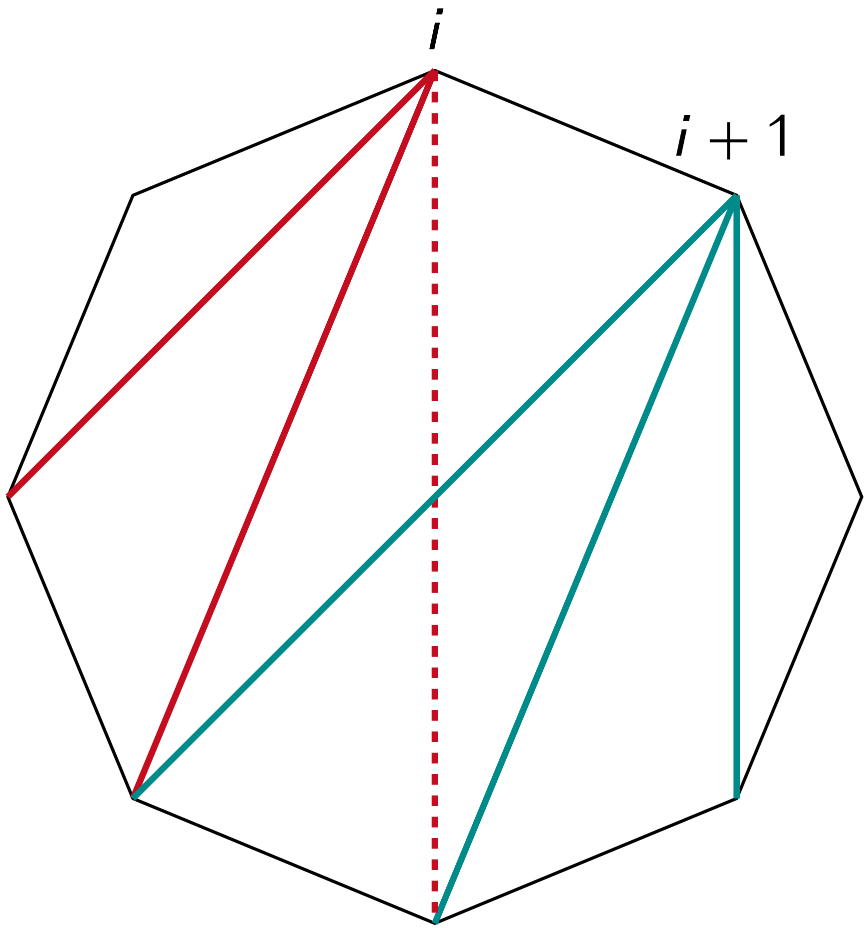
Proof:



Triangulations

Thm: For $n \geq 7$, G_n^T has a 2-rainbow cycle.

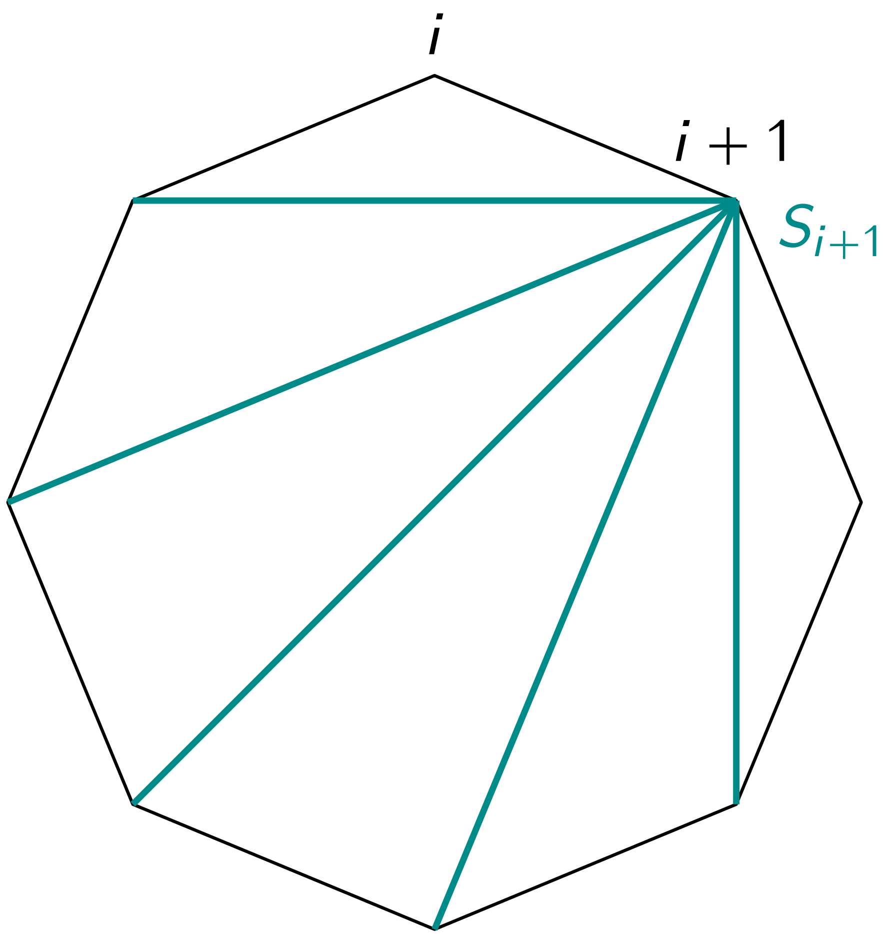
Proof:



Triangulations

Thm: For $n \geq 7$, G_n^T has a 2-rainbow cycle.

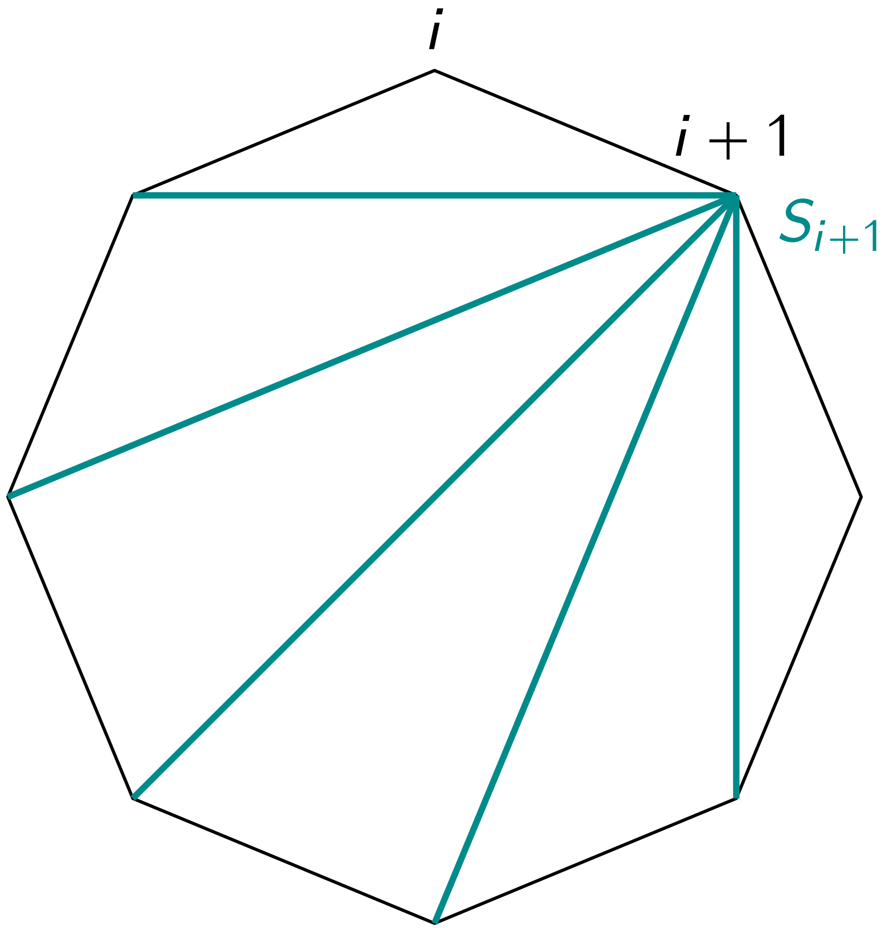
Proof:



Triangulations

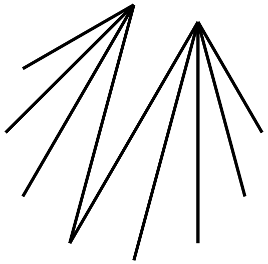
Thm: For $n \geq 7$, G_n^T has a 2-rainbow cycle.

Proof:



Claim:
$S_1 \rightarrow S_2 \rightarrow \ldots \rightarrow S_n \rightarrow S_1$
is a 2-rainbow cycle.

- $\{i, j\}$ appears twice
- triangulations are unique



□

Settings & Results

		flip graph			existence of r-rainbow cycle	
		vertices	arcs/edges	r	Yes	No
GEOMETRIC	G_n^{T}	triangulations of convex n-gon	edge flip	1	$n \geq 4$	
				2	$n \geq 7$	
	G_X^{S}	plane spanning trees on point set X in general position	edge flip	$1, \ldots, \lvert X\rvert - 2$	$\lvert X\rvert \geq 3$	
	G_m^{M}	non-crossing perfect matchings on $2m$ points in convex position	two edge flip	1	$m \in \{2, 4\}$	odd m, $m \in \{6, 8, 10\}$
				2	$m \in \{6, 8\}$	
ABSTRACT	G_n^{P}	permutations of $[n]$	transposition	1	$\lfloor n/2 \rfloor$ even	$\lfloor n/2 \rfloor$ odd
	$G_{n,k}^{\mathsf{C}}$	k-subsets of $[n]$, $2 \leq k \leq \lfloor n/2 \rfloor$	element exchange	1	odd n and $k < n/3$	even n
		2-subsets of $[n]$ for odd n		1	two edge-disjoint 1-rainbow Ham. cycles	

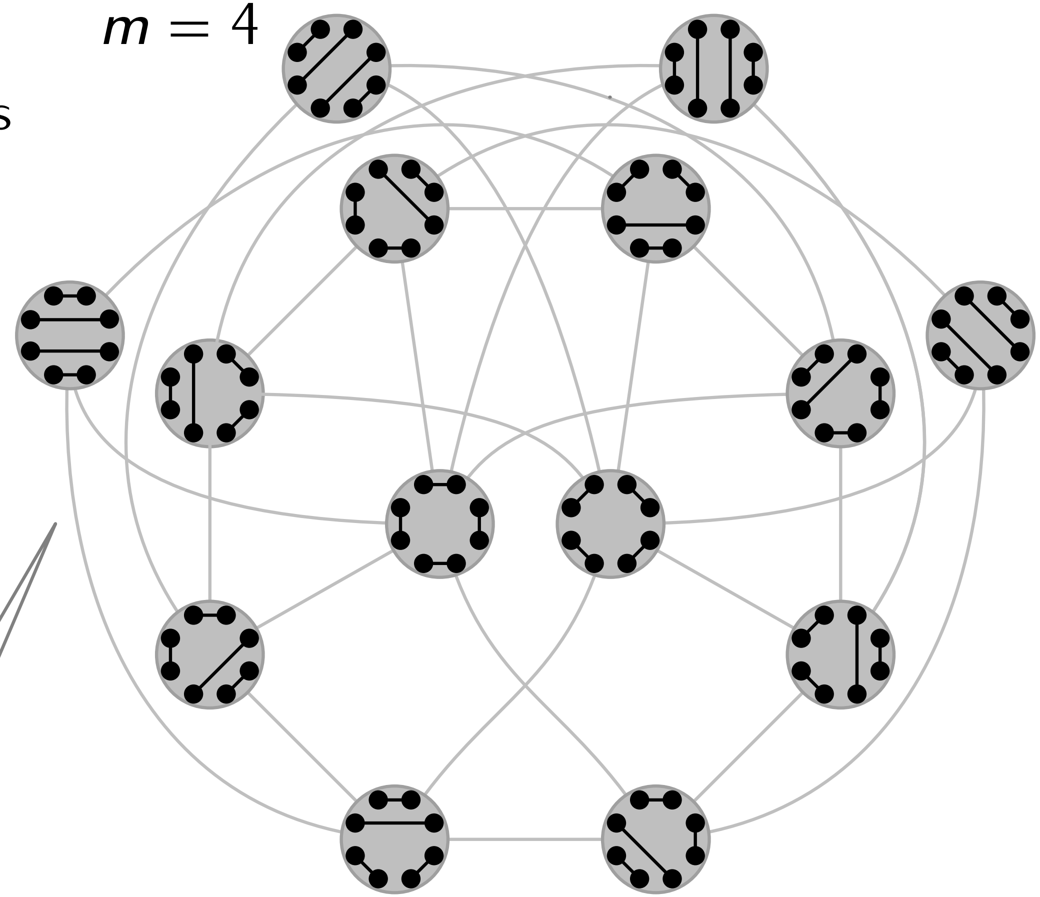
Matchings

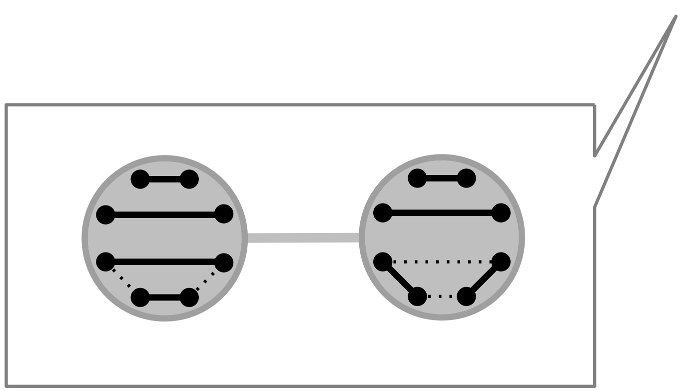
Thm: For $m \in \{2,4\}$, G_m^M has a 1-rainbow cycle.
For $m \in \{6,8,10\}$, G_m^M has no 1-rainbow cycle.

flip graph G_m^M

- vertices: crossing-free matchings
- arcs: 2-edge exchange
- arc color: the 2 new edges

$m = 4$



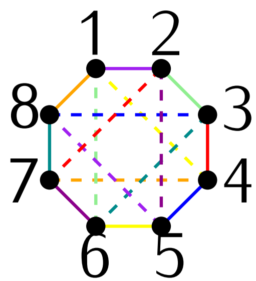


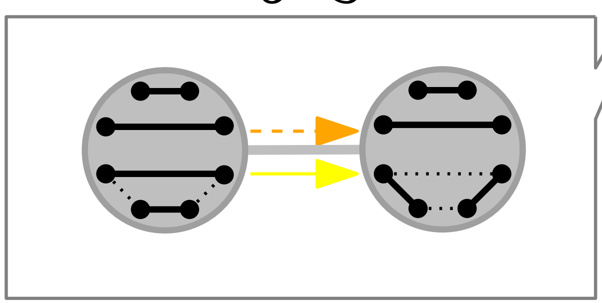
Matchings

Thm: For $m \in \{2, 4\}$, G_m^M has a 1-rainbow cycle.
For $m \in \{6, 8, 10\}$, G_m^M has no 1-rainbow cycle.

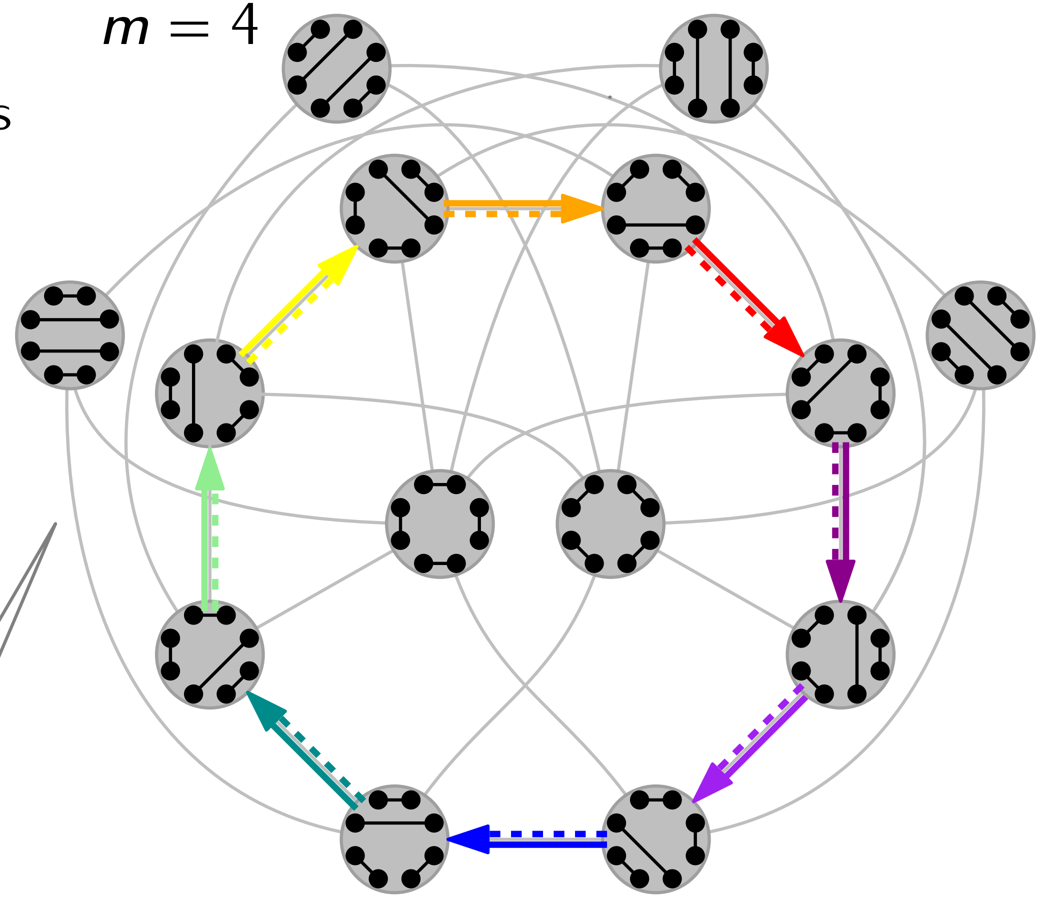
flip graph G_m^M

- vertices: crossing-free matchings
- arcs: 2-edge exchange
- arc color: the 2 new edges





$m = 4$



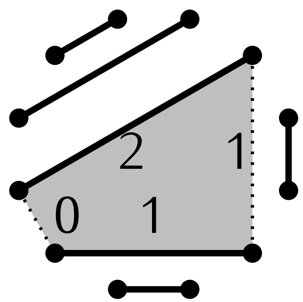
Matchings

Thm: For $m \in \{2, 4\}$, G_m^M has a 1-rainbow cycle.
For $m \in \{6, 8, 10\}$, G_m^M has no 1-rainbow cycle.

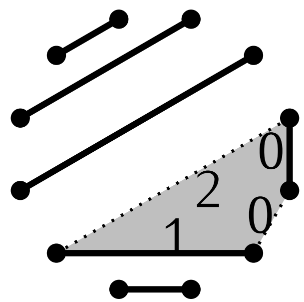
length of an edge= min # points on either side,
divided by 2

centered flip:= edge length of quadrilateral is $m - 2$

$m = 6$



centered



not centered

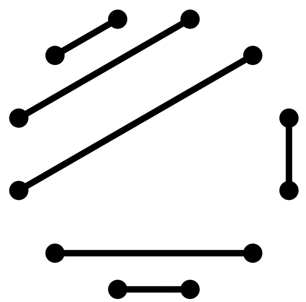
Obs: A flip is centered iff the quadrilateral contains origin.

Lemma: rainbow cycles use only centered flips.

Matchings

Thm: For $m \in \{2, 4\}$, G_m^M has a 1-rainbow cycle.
For $m \in \{6, 8, 10\}$, G_m^M has no 1-rainbow cycle.

Lemma: rainbow cycles use only centered flips.

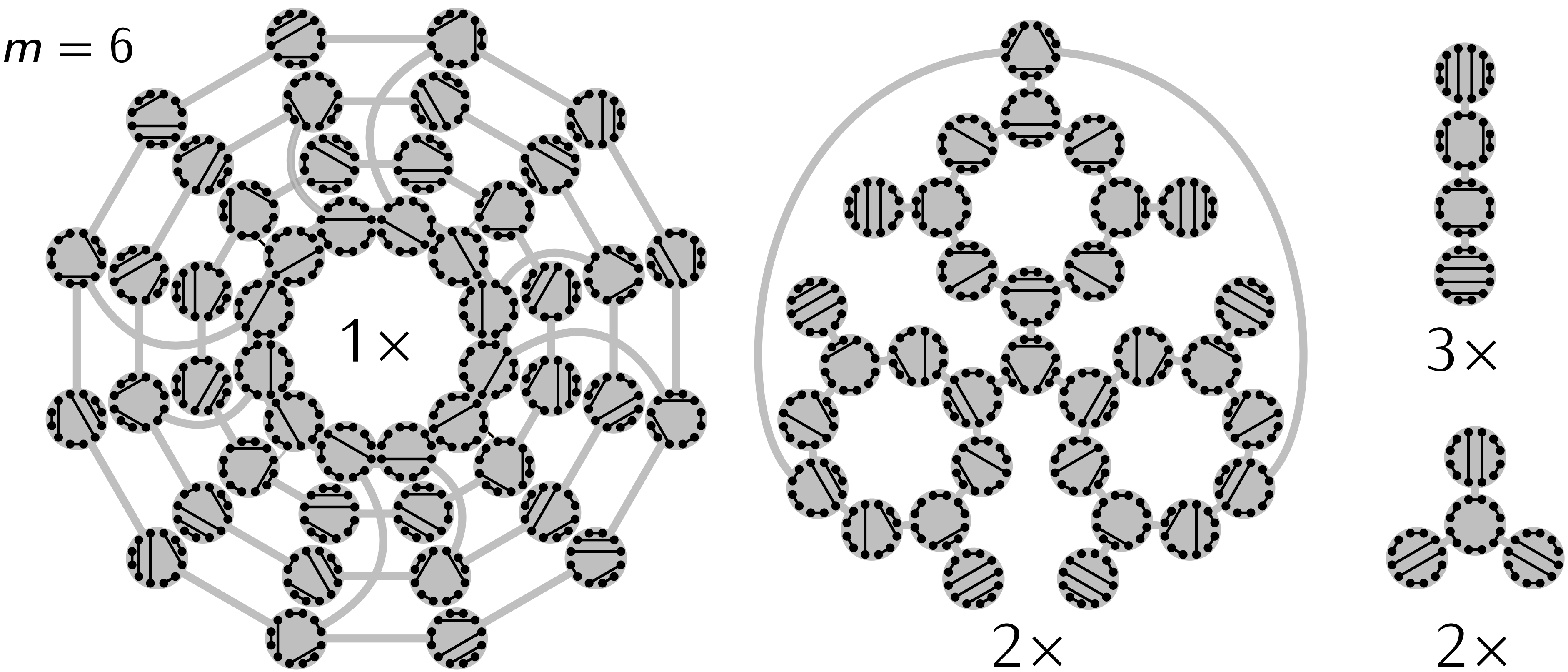


- the length of matching edges are equi-distributed
- ...and range from 0 to $\frac{m-2}{2}$
- average in rainbow cycle $= \frac{m-2}{4}$
- average of centered flips $= \frac{m-2}{4}$
- average of non-centered flips $< \frac{m-2}{4}$

Matchings

Thm: For $m \in \{2,4\}$, G_m^M has a 1-rainbow cycle.
For $m \in \{6,8,10\}$, G_m^M has no 1-rainbow cycle.

Lemma: rainbow cycles use only centered flips. → restricted flip graph



Settings & Results

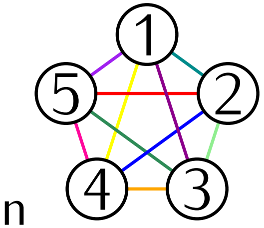
		flip graph			existence of r-rainbow cycle	
		vertices	arcs/edges	r	Yes	No
GEOMETRIC	G_n^{T}	triangulations of convex n-gon	edge flip	1	$n \geq 4$	
				2	$n \geq 7$	
	G_X^{S}	plane spanning trees on point set X in general position	edge flip	$1, \ldots, \lvert X\rvert - 2$	$\lvert X\rvert \geq 3$	
	G_m^{M}	non-crossing perfect matchings on $2m$ points in convex position	two edge flip	1	$m \in \{2, 4\}$	odd m, $m \in \{6, 8, 10\}$
				2	$m \in \{6, 8\}$	
ABSTRACT	G_n^{P}	permutations of $[n]$	transposition	1	$\lfloor n/2 \rfloor$ even	$\lfloor n/2 \rfloor$ odd
	$G_{n,k}^{\mathsf{C}}$	k-subsets of $[n]$, $2 \leq k \leq \lfloor n/2 \rfloor$	element exchange	1	odd n and $k < n/3$	even n
		2-subsets of $[n]$ for odd n		1	two edge-disjoint 1-rainbow Ham. cycles	

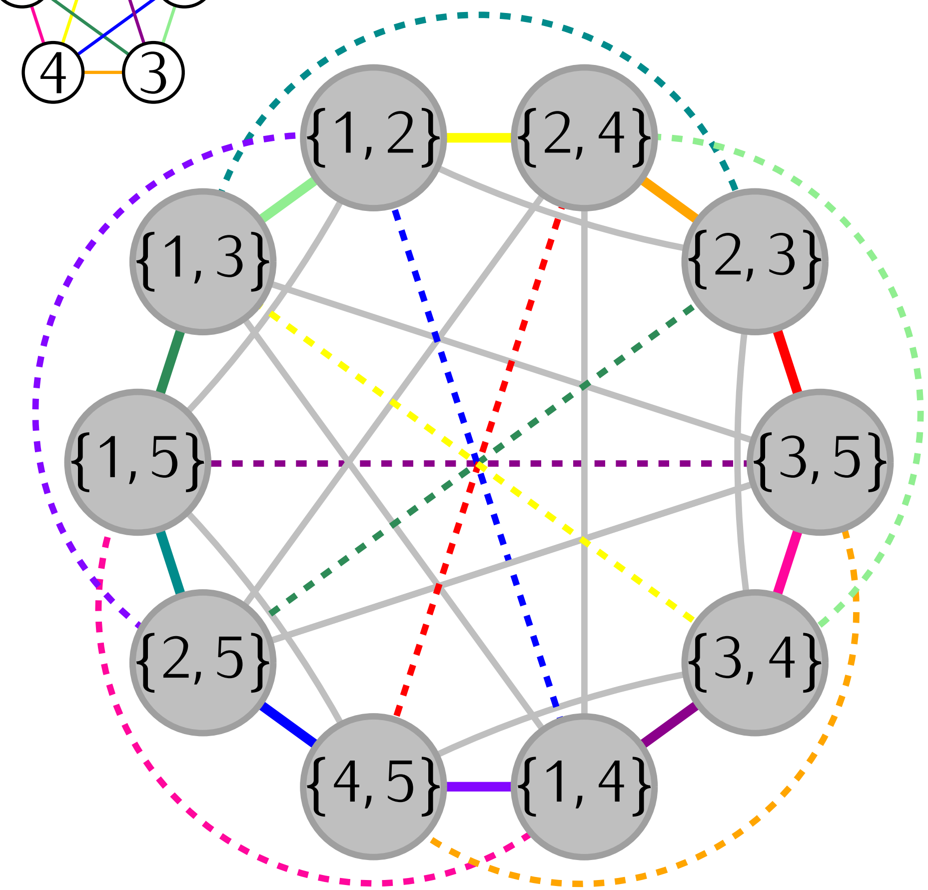
Subsets

Thm: For odd n and $k = 2$, $G^C_{n,k}$ has a 1-rainbow Hamilton cycle.

flip graph $G^C_{n,2}$

- vertices: 2-subsets of $[n]$
- edges: element exchange/ transposition
- edge color: the transposition





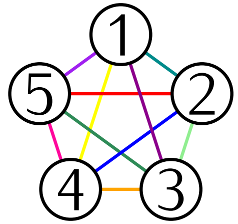
Subsets

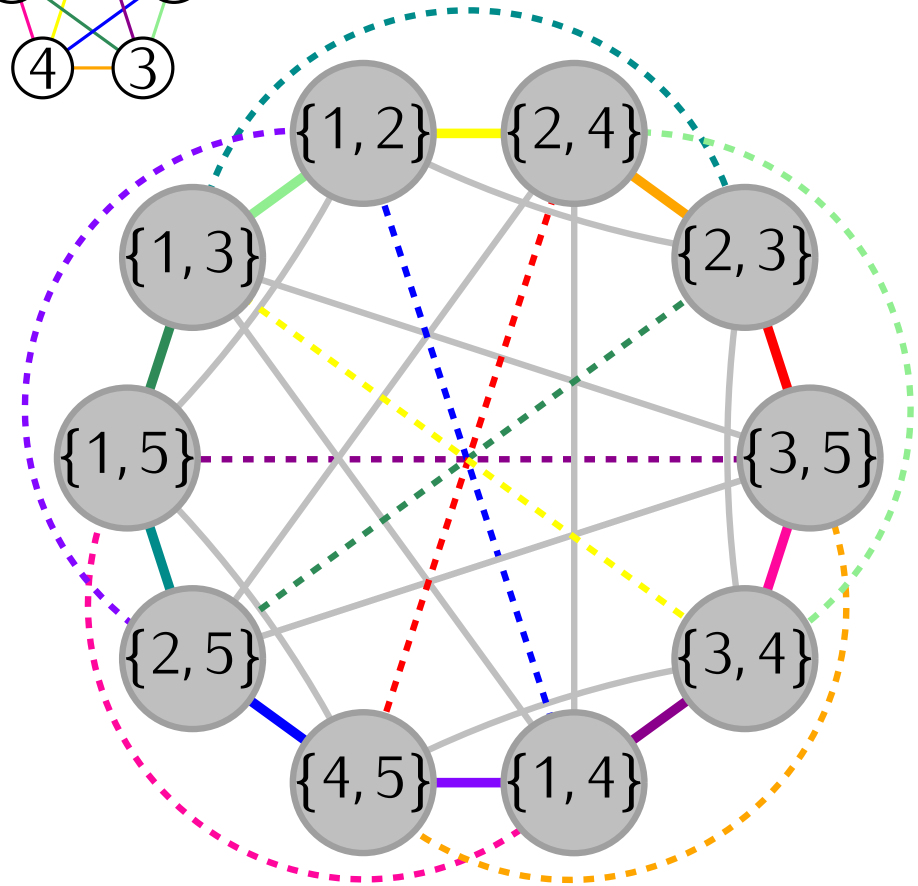
Thm: For odd n and $k = 2$, $G^{C}_{n,k}$ has a 1-rainbow Hamilton cycle.

rainbow blocks $\quad n = 2\ell + 1$

$B = (B_1, B_2, \ldots, B_\ell)$ with $B_i \in C_{n,k}$ is a rainbow block if $C(B) := (B, \sigma(B), \ldots, \sigma^{2\ell}(B))$ is a rainbow cycle

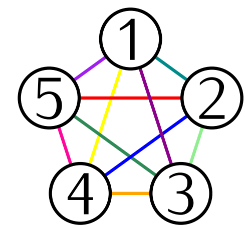
B_i	1	2	3	4	5
B_1	×				×
B_2	×		×		
$\sigma(B_1)$	×	×			
$\sigma(B_2)$		×		×	
⋮			⋮		
$\sigma^4(B_1)$				×	×
$\sigma^4(B_2)$		×			×





Subsets

Thm: For odd n and $k = 2$, $G^C_{n,k}$ has a 1-rainbow Hamilton cycle.



Proof:

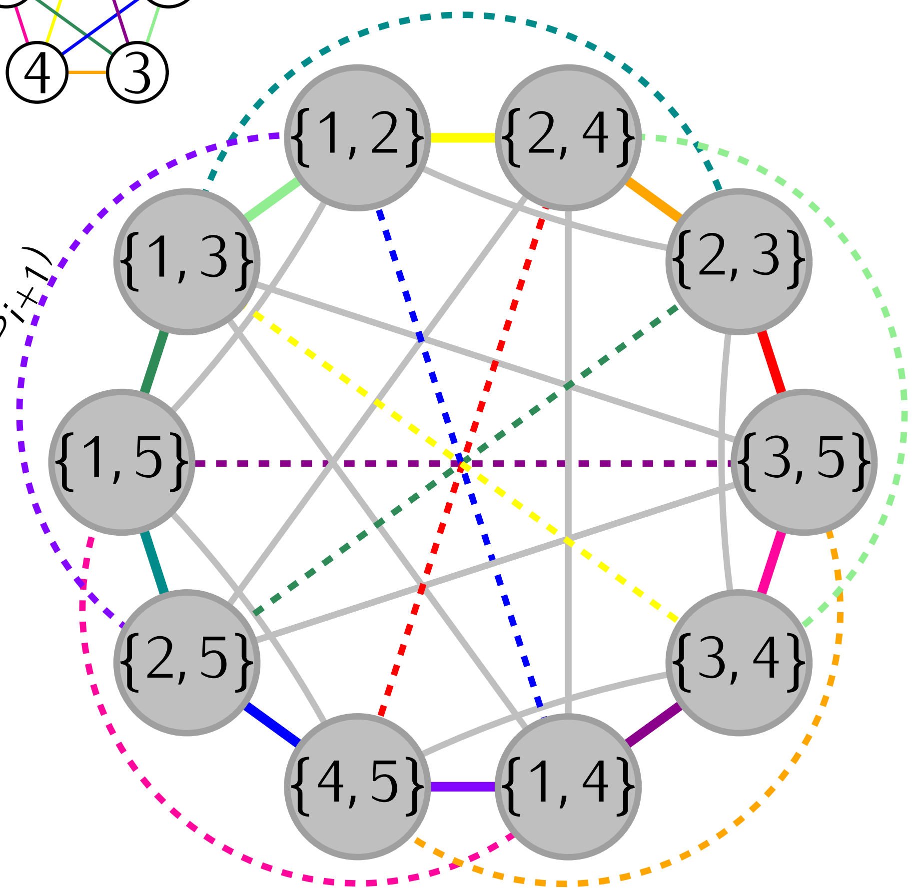
Use special rainbow blocks:

(a) $B_i = \{1, b_i\}$ for $i \in [\ell]$ with $3 \leq b_i \leq n$ and $b_1 = n$,

(b) $\{\text{dist}(B_i) \mid i \in [\ell]\} = [\ell]$

(c) $\{\text{dist}(B_i \triangle B_{i+1}) \mid i \in [\ell - 1]\} \cup \{\text{dist}(B_\ell \triangle B_1\} = [\ell]$

B_i	1	2	3	4	5	$\text{dist}(B_i)$	$\text{dist}(B_i \triangle B_{i+1})$
B_1	×				×	1	2
B_2	×		×			2	1
$\sigma(B_1)$	×	×				1	2
$\sigma(B_2)$		×		×		2	1
⋮			⋮				
$\sigma^4(B_1)$				×	×	1	2
$\sigma^4(B_2)$		×			×	2	1



Subsets

Thm: For odd n and $k = 2$, $G^C_{n,k}$ has a 1-rainbow Hamilton cycle.

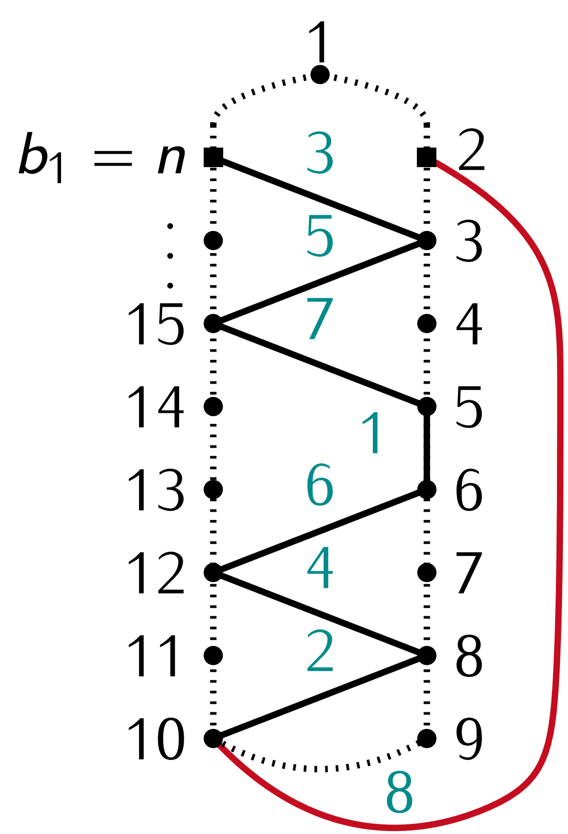
Proof:

Use special rainbow blocks:

(a) $B_i = \{1, b_i\}$ for $i \in [\ell]$ with
$3 \leq b_i \leq n$ and $b_1 = n$,
(b) $\{\mathrm{dist}(B_i) \mid i \in [\ell]\} = [\ell]$
(c) $\{\mathrm{dist}(B_i \triangle B_{i+1}) \mid i \in [\ell - 1]\}$
$\cup \{\mathrm{dist}(B_\ell \triangle B_1\} = [\ell]$

(a) start at vertex n, skip vertex 1
(b) visit each level once
(c) use all edge length once

Definition of b_i



Subsets

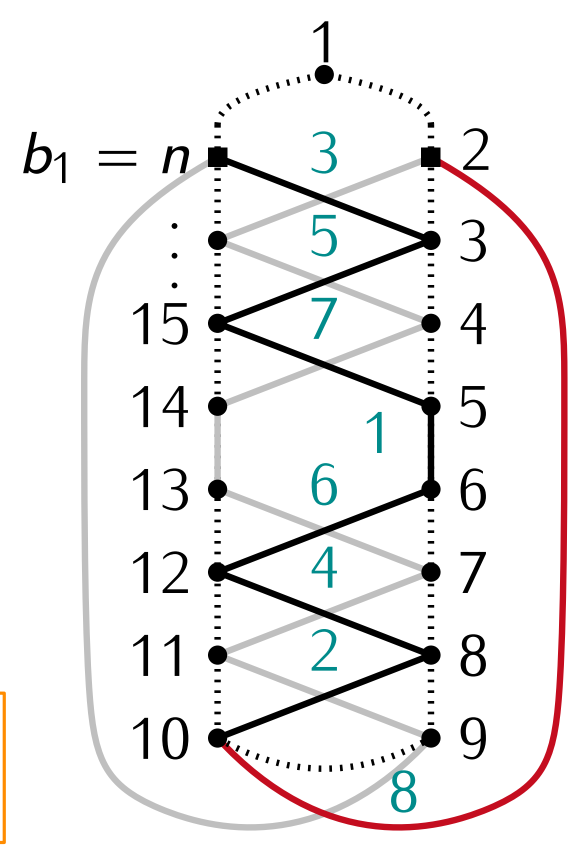
Thm: For odd n and $k = 2$, $G^{C}_{n,k}$ has a 1-rainbow Hamilton cycle.

Proof:

Use special rainbow blocks:

(a) $B_i = \{1, b_i\}$ for $i \in [\ell]$ with $3 \leq b_i \leq n$ and $b_1 = n$,
(b) $\{\mathrm{dist}(B_i) \mid i \in [\ell]\} = [\ell]$
(c) $\{\mathrm{dist}(B_i \triangle B_{i+1}) \mid i \in [\ell - 1]\} \cup \{\mathrm{dist}(B_\ell \triangle B_1\} = [\ell]$

Definition of b_i

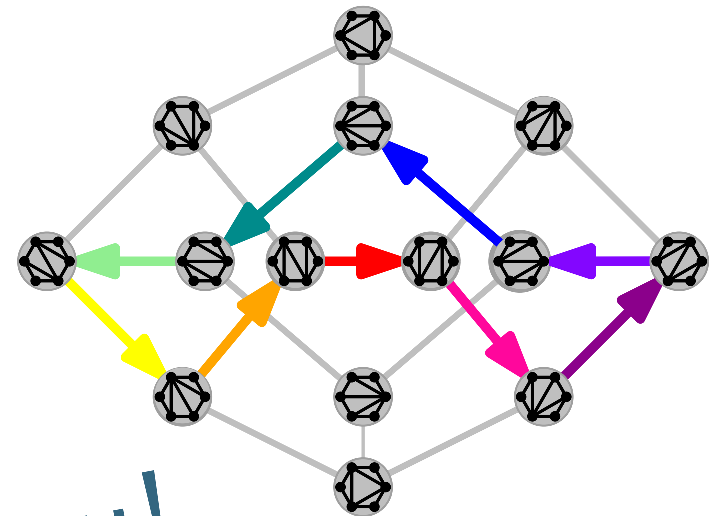


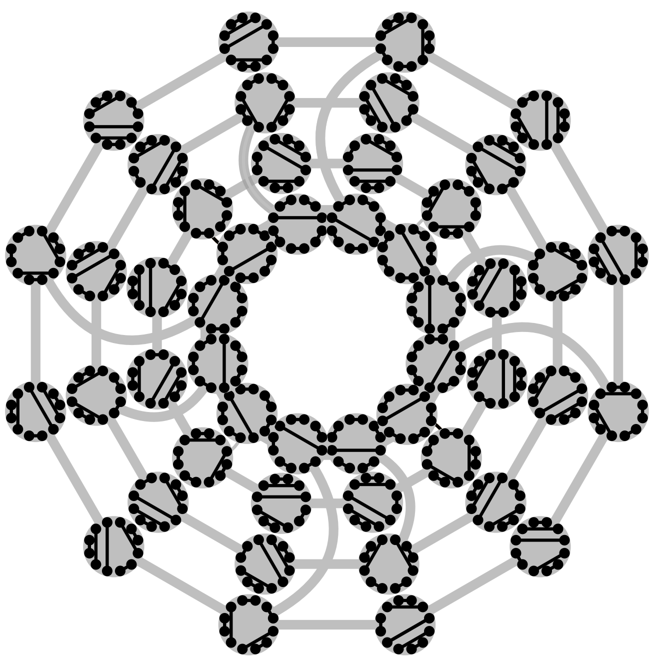
(a) start at vertex n, skip vertex 1
(b) visit each level once
(c) use all edge length once

A rainbow block and its partner ("mirror image") yield two edge disjoint rainbow Hamilton cycles.

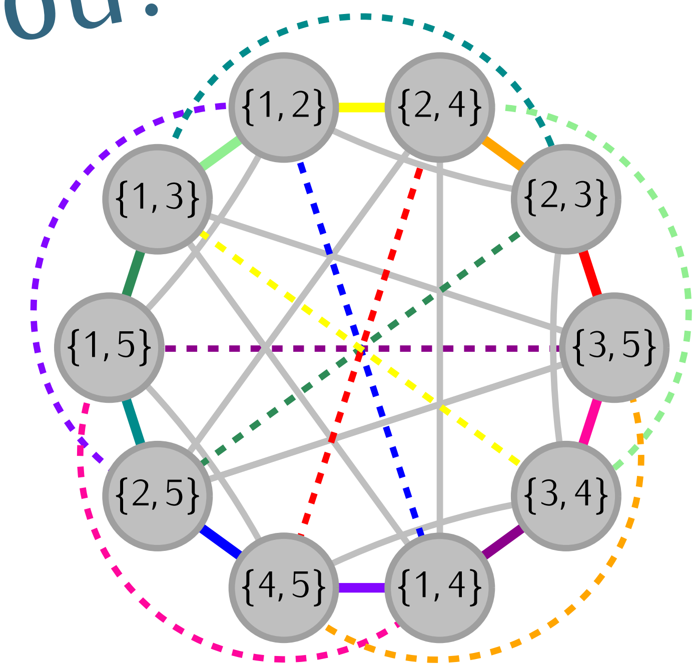
Open Problems

- r-rainbow cycles for larger r?
- other classes?
- matchings: non-existence of 1-rainbow cycles for $m > 4$
- subsets: 1-rainbow cycles for all k $(1 < k < n)$



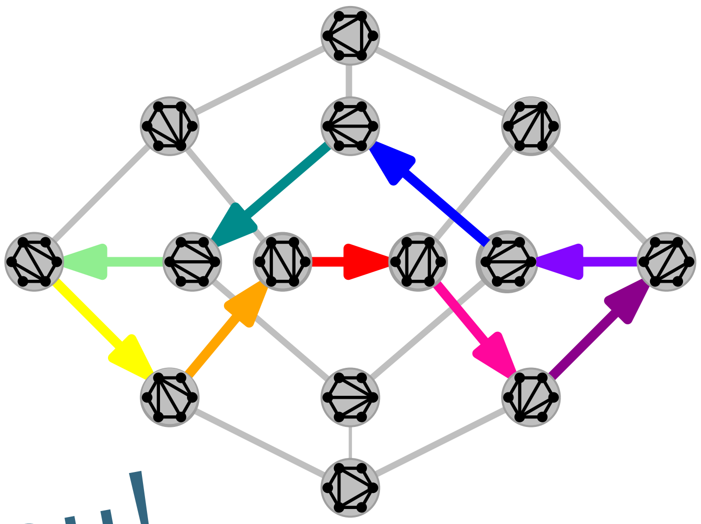


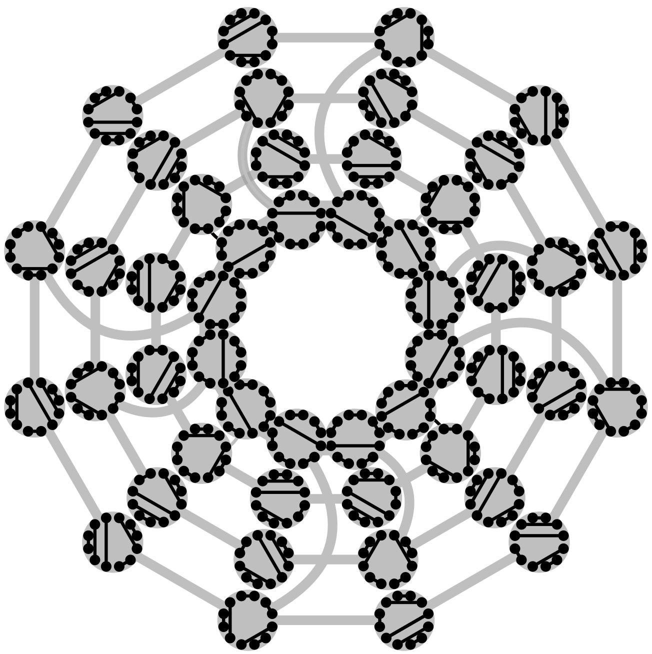
Thank you!



Open Problems

- r-rainbow cycles for larger r?
- other classes?
- matchings: non-existence of 1-rainbow cycles for $m > 4$
- subsets: 1-rainbow cycles for all k $(1 < k < n)$





Thank you!

connection to polytopes?